Constructibility and duality for simple holonomic modules on complex symplectic manifolds

Masaki Kashiwara and Pierre Schapira

June 9, 2007

Abstract
Consider a complex symplectic manifold $X$ and the algebroid stack $\mathcal{W}_X$ of deformation-quantization. For two regular holonomic $\mathcal{W}_X$-modules $L_i$ ($i = 0, 1$) supported by smooth Lagrangian submanifolds, we prove that the complex $R\mathcal{H}om_{\mathcal{W}_X}(L_1, L_0)$ is perverse over the field $\mathcal{W}_{pt}$ and dual to the complex $R\mathcal{H}om_{\mathcal{W}_X}(L_0, L_1)$.

Introduction
Consider a complex symplectic manifold $X$. A local model for $X$ is an open subset of the cotangent bundle $T^*X$ to a complex manifold $X$, and $T^*X$ is endowed with the filtered sheaf of rings $\mathcal{W}_{T^*X}$ of deformation-quantization. This sheaf of rings is similar to the sheaf $\mathcal{E}_{T^*X}$ of microdifferential operators of Sato-Kawai-Kashiwara [18], but with an extra central parameter $\tau$, a substitute to the lack of homogeneity (see §1). This is an algebra over the field $k = \mathcal{W}_{pt}$, a subfield of the field of formal Laurent series $\mathbb{C}[[\tau^{-1}, \tau]]$.

It would be tempting to glue the locally defined sheaves of algebras $\mathcal{W}_{T^*X}$ to give rise to a sheaf on $X$, but the procedure fails, and one is lead to replace the notion of a sheaf of algebras by that of an “algebroid stack”. A canonical algebroid stack $\mathcal{W}_X$ on $X$, locally equivalent to $\mathcal{W}_{T^*X}$, has been constructed by Polesello-Schapira [16] after Kontsevich [15] had treated the formal case (in the general setting of Poisson manifolds) by a different method.

In this paper, we study regular holonomic $\mathcal{W}_X$-modules supported by smooth Lagrangian submanifolds of $X$. For example, a regular holonomic module along the zero-section $T^*_X X$ of $T^*X$ is locally isomorphic to a finite sum of copies of the sheaf $O_X$ whose sections are series $\sum_{-\infty < j \leq m} f_j \tau^j$.

Mathematics Subject Classification: 46L65, 14A20, 32C38
(m ∈ ℤ), where the f_j's are sections of O_Χ and satisfy certain growth conditions.

Denote by D^{b}_{rh}(W_{Χ}) the full subcategory of the bounded derived category of W_{Χ}-modules on Χ consisting of objects with regular holonomic cohomologies. The main theorem of this paper asserts that if Ł_i (i = 0, 1) are objects of D^{b}_{rh}(W_{Χ}) supported by smooth Lagrangian manifolds Λ_i and if one sets F := RHom_{W_{Χ}}(Ł_1, Ł_0), then F is C-constructible over the field k, its microsupport is contained in the normal cone C(Λ_0, Λ_1) and F is dual over k to the object RHom_{W_{Χ}}(Ł_0, Ł_1). If Ł_0 and Ł_1 are concentrated in degree 0, then F is perverse. We make the conjecture that the hypothesis on the smoothness of the Λ_i's may be removed.

The strategy of our proof is as follows.

First, assuming only that Ł_0 and Ł_1 are coherent, we construct the canonical morphism

\[ RHom_{W_{Χ}}(Ł_1, Ł_0) \to \mathcal{D}_{Χ}(RHom_{W_{Χ}}(Ł_0, Ł_1 [\dim C_{Χ}])) \]  

(0.1)

where \( \mathcal{D}_{Χ} \) is the duality functor for sheaves of \( k_Χ \)-modules. By using a kind of Serre's duality for the sheaf \( O_Χ^τ \), we prove that (0.1) is an isomorphism as soon as \( RHom_{W_{Χ}}(Ł_0, Ł_1) \) is constructible. For that purpose, we need to develop some functional analysis over nuclear algebras, in the line of Houzel [6].

In order to prove the constructibility result for \( F = RHom_{W_{Χ}}(Ł_1, Ł_0) \), we may assume that \( Χ = T^*Χ, \ Λ_0 = T^*_ΧΧ, \ Λ_0 = O_Χ^τ \) and Λ_1 is the graph of the differential of a holomorphic function \( ϕ \) defined on \( Χ \). Consider the sheaf of rings \( D_{Χ}[τ^{-1}] = D_{Χ} \otimes C[τ^{-1}] \). We construct a coherent \( D_{Χ}[τ^{-1}] \)-module \( M \) which generates \( Ł_1 \) and, setting \( F_0 := RHom_{D_{Χ}[τ^{-1}]}(M, O_Χ^τ(0)) \), we prove that the microsupport of \( F_0 \) is a closed complex analytic Lagrangian subset of \( T^*Χ \) contained in \( C(Λ_0, Λ_1) \). Using a deformation argument as in [7] and some functional analysis (extracted from [6]) over the base ring \( k_0 := W_{pt}(0) \), we deduce that the fibers of the cohomology of \( F_0 \) are finitely generated, and the result follows from the isomorphism \( F \simeq F_0 \otimes_{k_0} k \).

In this paper, we also consider a complex compact contact manifold \( Υ \) and the algebroid stack \( ℋ \) of microdifferential operators on it. Denote by \( D^{b}_{rh}(ℋ) \) the C-triangulated category consisting of \( ℋ \)-modules with regular holonomic cohomology. Using results of Kashiwara-Kawai [10], we prove that this category has finite Ext, admits a Serre functor in the sense of Bondal-Kapranov [2] and this functor is nothing but a shift by \( \dim C Υ + 1 \).

In other words, \( D^{b}_{rh}(ℋ) \) is a Calabi-Yau category. Similar results hold over \( k \) for a complex compact symplectic manifold \( Χ \).
1 \( \mathcal{W} \)-modules on \( T^*X \)

Let \( X \) be a complex manifold, \( \pi: T^*X \to X \) its cotangent bundle. The homogeneous symplectic manifold \( T^*X \) is endowed with the \( \mathbb{C}^\times \)-conic \( \mathbb{Z} \)-filtered sheaf of rings \( \mathcal{E}_{T^*X} \) of finite-order microdifferential operators, and its subring \( \mathcal{E}_{T^*X}(0) \) of operators of order \( \leq 0 \) constructed in [18] (with other notations). We assume that the reader is familiar with the theory of \( \mathcal{E} \)-modules, referring to [9] or [19] for an exposition.

On the symplectic manifold \( T^*X \) there exists another (no more conic) sheaf of rings \( \mathcal{W}_{T^*X} \), called ring of deformation-quantization by many authors (see Remark 1.2 below). The study of the relations between the ring \( \mathcal{E}_{T^*X} \) on a complex homogeneous symplectic manifold and the sheaf \( \mathcal{W}_{T^*X} \) on a complex symplectic manifold is systematically performed in [16], where it is shown in particular how quantized symplectic transformations act on \( \mathcal{W}_{T^*X} \)-modules. We follow here their presentation.

Let \( t \in \mathbb{C} \) be the coordinate and set
\[
\mathcal{E}_{T^*(X \times \mathbb{C}),t} = \{ P \in \mathcal{E}_{T^*(X \times \mathbb{C})} ; [P, \partial_t] = 0 \}.
\]
Set \( T^*_{\tau \neq 0}(X \times \mathbb{C}) = \{(x, t; \xi, \tau) \in T^*(X \times \mathbb{C}) ; \tau \neq 0 \} \), and consider the map
\[
(1.1) \quad \rho: T^*_{\tau \neq 0}(X \times \mathbb{C}) \to T^*X
\]
given in local coordinates by \( \rho(x, t; \xi, \tau) = (x; \xi/\tau) \). The ring \( \mathcal{W}_{T^*X} \) on \( T^*X \) is given by
\[
\mathcal{W}_{T^*X} := \rho_*\left( \mathcal{E}_{X \times \mathbb{C},t}|_{T^*_{\tau \neq 0}(X \times \mathbb{C})} \right).
\]
In a local symplectic coordinate system \((x; u)\) on \( T^*X \), a section \( P \) of \( \mathcal{W}_{T^*X} \) on an open subset \( U \) is written as a formal series, called its total symbol:
\[
(1.2) \quad \sigma_{\text{tot}}(P) = \sum_{-\infty < j \leq m} p_j(x; u)\tau^j, \quad m \in \mathbb{Z} \quad p_j \in \mathcal{O}_{T^*X}(U),
\]
with the condition
\[
(1.3) \quad \text{for any compact subset } K \text{ of } U \text{ there exists a positive constant } C_K \text{ such that } \sup_K |p_j| \leq C_K^{-j}(-j)! \text{ for all } j < 0.
\]
The product is given by the Leibniz rule. If \( Q \) is an operator of total symbol \( \sigma_{\text{tot}}(Q) \), then
\[
\sigma_{\text{tot}}(P \circ Q) = \sum_{\alpha \in \mathbb{N}^n} \tau^{-|\alpha|} \frac{\partial^\alpha u}{\alpha!} \sigma_{\text{tot}}(P) \partial^\alpha_x \sigma_{\text{tot}}(Q).
\]
Denote by $\mathcal{W}_{T^*X}(m)$ the subsheaf of $\mathcal{W}_{T^*X}$ consisting of sections $P$ whose total symbol $\sigma_{\text{tot}}(P)$ satisfies: $p_j$ is zero for $j > m$. The ring $\mathcal{W}_{T^*X}$ is $\mathbb{Z}$-filtered by the $\mathcal{W}_{T^*X}(m)$’s, and $\mathcal{W}_{T^*X}(0)$ is a subring. If $P \in \mathcal{W}_{T^*X}(m)$ and $P \notin \mathcal{W}_{T^*X}(m-1)$, $P$ is said of order $m$. Hence 0 has order $-\infty$. Then there is a well-defined principal symbol morphism, which does not depend on the local coordinate system on $X$:

$$\sigma_m : \mathcal{W}_{T^*X}(m) \rightarrow \mathcal{O}_{T^*X} \cdot \tau^m. \quad (1.4)$$

If $P \in \mathcal{W}_{T^*X}$ has order $m$ on a connected open subset of $T^*X$, $\sigma_m(P)$ is called the principal symbol of $P$ and is denoted $\sigma(P)$. Note that a section $P$ in $\mathcal{W}_{T^*X}$ is invertible on an open subset $U$ of $T^*X$ if and only if its principal symbol is nowhere vanishing on $U$.

The principal symbol map induces an isomorphism of graded algebras

$$\text{gr} \mathcal{W}_{T^*X} \cong \mathcal{O}_{T^*X}[\tau, \tau^{-1}].$$

We set

$$k := \mathcal{W}_{\{\text{pt}\}}, \quad k(j) := \mathcal{W}_{\{\text{pt}\}}(j) (j \in \mathbb{Z}), \quad k_0 := k(0).$$

Hence, $k$ is a field and an element $a \in k$ is written as a formal series

$$a = \sum_{j \leq m} a_j \tau^j, \quad a_j \in \mathbb{C}, \quad m \in \mathbb{Z} \quad (1.5)$$

satisfying (1.3), that is,

$$\begin{cases} \text{there exists a positive constant } C \text{ such that } |a_j| \leq C^{-j}(-j)! \text{ for all } j < 0. \end{cases} \quad (1.6)$$

Note that $k_0$ is a discrete valuation ring.

**Convention 1.1.** *In the sequel, we shall identify $\partial_t$ and its total symbol $\tau$. Hence, we consider $\tau$ as a section of $\mathcal{W}_{T^*X}$.*

Note that

- $\mathcal{W}_{T^*X}$ is flat over $\mathcal{W}_{T^*X}(0)$ and in particular $k$ is flat over $k_0$,
- $k_0$ is faithfully flat over $\mathbb{C}[\tau^{-1}]$,
- the sheaves of rings $\mathcal{W}_{T^*X}$ and $\mathcal{W}_{T^*X}(0)$ are right and left Noetherian (see [9, Appendix]) and in particular coherent,
• If $\mathcal{M}$ is a coherent $\mathcal{W}_{T^*X}$-module, its support is a closed complex analytic involutive (by Gabber’s theorem) subset of $T^*X$,

• $\mathcal{W}_{T^*X}$ is a $k$-algebra and $\mathcal{W}_{T^*X}(0)$ is a $k_0$-algebra,

• There are natural monomorphisms of sheaves of $\mathbb{C}$-algebras

\[(1.7)\quad \pi^{-1}\mathcal{D}_X \hookrightarrow \mathcal{E}_{T^*X} \hookrightarrow \mathcal{W}_{T^*X}.\]

On an affine chart, morphism (1.7) is described on symbols as follows. To a section of $\mathcal{E}_{T^*X}$ of total symbol $\sum_{-\infty < j \leq m} p_j(x; \xi)$ (the $p_j$’s are homogeneous in $\xi$ of degree $j$) one associates the section of $\mathcal{W}_{T^*X}$ of total symbol $\sum_{-\infty < j \leq m} p_j(x; u)\tau^j$, with $u = \tau^{-1} \xi$.

**Remark 1.2.** (i) Many authors consider the filtered ring of formal operators, defined by

\[\hat{\mathcal{W}}_{T^*X}(m) = \lim_{\leftarrow j \leq m} \mathcal{W}_{T^*X}(m)/\mathcal{W}_{T^*X}(j) \quad m \in \mathbb{Z}, \quad \hat{\mathcal{W}}_{T^*X} = \bigcup_m \hat{\mathcal{W}}_{T^*X}(m).\]

Then

\[\hat{k} := \hat{W}_{(pt)} \simeq \mathbb{C}[[\tau^{-1}, \tau]], \quad \hat{k}_0 := \hat{W}_{(pt)}(0) \simeq \mathbb{C}[[\tau^{-1}]].\]

Note that $\hat{\mathcal{W}}_{T^*X}$ is faithfully flat over $\mathcal{W}_{T^*X}$ and $\hat{\mathcal{W}}_{T^*X}$ is flat over $\hat{\mathcal{W}}_{T^*X}(0)$.

(ii) Many authors also prefer to use the symbol $h = \tau^{-1}$ instead of $\tau$.

**Notation 1.3.** Let $\mathcal{I}_X$ be the left ideal of $\mathcal{W}_{T^*X}$ generated by the vector fields on $X$. We set

\[\mathcal{O}_X = \mathcal{W}_{T^*X}/\mathcal{I}_X, \quad \mathcal{O}_X(m) = \mathcal{W}_{T^*X}(m)/(\mathcal{I}_X \cap \mathcal{W}_{T^*X}(m)) \quad (m \in \mathbb{Z}).\]

Note that $\mathcal{O}_X$ is a coherent $\mathcal{W}_{T^*X}$-module supported by the zero-section $T^*_X$. A section $f(x, \tau)$ of this module may be written as a series:

\[(1.8)\quad f(x, \tau) = \sum_{-\infty < j \leq m} f_j(x)\tau^j, \quad m \in \mathbb{Z},\]

the $f_j$’s satisfying the condition (1.3).

Also note that $\mathcal{O}_X$ is a direct summand of $\mathcal{O}_{X^*}$ as a sheaf.

**Lemma 1.4.** After identifying $X$ and $X \times \{0\} \subset X \times \mathbb{C}$, there is an isomorphism of sheaves of $\mathbb{C}$-vector spaces (not of algebras)

\[(1.9)\quad \mathcal{O}_{X^*}(0) \simeq \mathcal{O}_{X \times \mathbb{C}}|_{X \times \{0\}}.\]
Proof. By its construction, $\mathcal{O}_X^\tau(0)$ is isomorphic to the sheaf of holomorphic microfunctions $\mathcal{C}_{X \times \{0\} | X \times \mathbb{C}(0)}$ of [18], and this last sheaf is isomorphic to $\mathcal{O}_{X \times \mathbb{C}}|_{X \times \{0\}}$ by loc. cit. q.e.d.

Note that isomorphism (1.9) corresponds to the map

$$\sum_{j \leq 0} f_j \tau^j \mapsto \sum_{j \geq 0} f_{-j} \frac{\tau^j}{j!} \in \mathcal{O}_{X \times \mathbb{C}}|_{X \times \{0\}}.$$ 

We shall use the following sheaves of rings. We set

$$D_X[\tau^{-1}] := D_X \otimes \mathbb{C}[\tau^{-1}], \quad D_X[\tau^{-1}, \tau] := D_X \otimes \mathbb{C}[\tau, \tau^{-1}].$$

Note that $\mathcal{O}_X^\tau(0)$ is a left $D_X[\tau^{-1}]$-module and $\mathcal{O}_X^\tau$ a left $D_X[\tau^{-1}, \tau]$-module.

Lemma 1.5. (i) $D_X[\tau^{-1}]$ is a right and left Noetherian sheaf of $\mathbb{C}$-algebras,

(ii) if $N \subset M$ are two coherent $D_X[\tau^{-1}]$-modules and $M_0$ a coherent $D_X$-submodule of $M$, then $N \cap M_0$ is $D_X$-coherent,

(iii) if $M$ is a coherent $D_X[\tau^{-1}]$-module and $M_0$ a $D_X$-submodule of $M$ of finite type, then $M_0$ is $D_X$-coherent.

Proof. (i) follows from [9, Th. A. 3].

(ii) Set $A := D_X[\tau^{-1}]$ and let $A_n \subset A$ be the $D_X$-submodule consisting of sections of order $\leq n$ with respect to $\tau^{-1}$. We may reduce to the case where $M = A^N$. By [9, Th. A.29, Lem. A21], $N \cap A_n$ is $D_X$-coherent. Since $\{N \cap A_n \cap M_0\}_n$ is an increasing sequence of coherent $D_X$-submodules of $M_0$ and this last module is finitely generated, the sequence is stationary.

(iii) Let us keep the notation in (ii). Here again, we may reduce to the case where $M = A^N$. Then $M_0 \cap A_n$ is $D_X$-coherent, and the proof goes as in (ii). q.e.d.

Lemma 1.6. (i) $D_X[\tau^{-1}, \tau]$ is flat over $D_X[\tau^{-1}]$,

(ii) if $I$ is a finitely generated left ideal of $D_X[\tau^{-1}, \tau]$, then $I \cap D_X[\tau^{-1}]$ is a locally finitely generated left ideal of $D_X[\tau^{-1}]$.

Proof. (i) $\mathbb{C}[\tau, \tau^{-1}]$ is flat over $\mathbb{C}[\tau^{-1}]$.

(ii) Let $I_0 \subset I$ be a coherent $D_X[\tau^{-1}]$-module which generates $I$. Set $J_n := \tau^n I_0 \cap D_X[\tau^{-1}]$. Then $I \cap D_X[\tau^{-1}] = \bigcup_n J_n$ and this increasing sequence of coherent ideals of $D_X[\tau^{-1}]$ is locally stationary. q.e.d.

Lemma 1.7. $W_{T^*X}$ is flat over $\pi^{-1}D_X[\tau^{-1}, \tau]$. 

6
Proof. One proves that $\mathcal{W}_{T^*X}$ is flat over $\pi^{-1}\mathcal{D}_X[\tau^{-1}]$ exactly as one proves that the ring of microdifferential operators $\mathcal{E}_{T^*X}$ is flat over $\pi^{-1}\mathcal{D}_X$. Since we have $\mathcal{D}_X[\tau^{-1}] \otimes_{\mathcal{D}_X[\tau^{-1}]} \mathcal{M} \simeq \mathcal{M}$ for any $\mathcal{D}_X[\tau^{-1}]$-module $\mathcal{M}$, the result follows. q.e.d.

2 Regular holonomic $\mathcal{W}$-modules

The following definition adapts a classical definition of [11] to $\mathcal{W}_{T^*X}$-modules (see also [4]).

**Definition 2.1.** Let $\Lambda$ be a smooth Lagrangian submanifold of $T^*X$.

(a) Let $\mathcal{L}(0)$ be a coherent $\mathcal{W}_{T^*X}(0)$-module supported by $\Lambda$. One says that $\mathcal{L}(0)$ is regular (resp. simple) along $\Lambda$ if $\mathcal{L}(0)/\mathcal{L}(-1)$ is a coherent $\mathcal{O}_\Lambda$-module (resp. an invertible $\mathcal{O}_\Lambda$-module). Here, $\mathcal{L}(-1) = \mathcal{W}_{T^*X}(-1)\mathcal{L}(0)$.

(b) Let $\mathcal{L}$ be a coherent $\mathcal{W}_{T^*X}$-module supported by $\Lambda$. One says that $\mathcal{L}$ is regular (resp. simple) along $\Lambda$ if there exists locally a coherent $\mathcal{W}_{T^*X}(0)$-submodule $\mathcal{L}(0)$ of $\mathcal{L}$ such that $\mathcal{L}(0)$ generates $\mathcal{L}$ over $\mathcal{W}_{T^*X}$ and is regular (resp. simple) along $\Lambda$.

Note that in (b), $\mathcal{L}(0)/\mathcal{L}(-1)$ is necessarily a locally free $\mathcal{O}_\Lambda$-module.

**Example 2.2.** The sheaf $\mathcal{O}_X^\tau$ is a simple $\mathcal{W}_{T^*X}$-module along the zero-section $T^*_X$.

The following result is easily proved (see [4]).

**Proposition 2.3.** Let $\Lambda$ be a smooth Lagrangian submanifold of $T^*X$ and let $\mathcal{M}$ be a coherent $\mathcal{W}_{T^*X}$-module.

(i) If $\mathcal{M}$ is regular, then it is locally a finite direct sum of simple modules.

(ii) Any two $\mathcal{W}_{T^*X}$-modules simple along $\Lambda$ are locally isomorphic. In particular, any simple module along $T^*_X$ is locally isomorphic to $\mathcal{O}_X^\tau$.

(iii) If $\mathcal{M}, \mathcal{N}$ are simple $\mathcal{W}_{T^*X}$-modules along $\Lambda$, then $R\mathcal{H}\mathcal{O}m_{\mathcal{W}_{T^*X}}(\mathcal{M}, \mathcal{N})$ is concentrated in degree 0 and is a $k$-local system of rank one on $\Lambda$.

**Definition 2.4.** Let $\mathcal{M}$ be a coherent $\mathcal{W}_{T^*X}$-module and let $\Lambda$ denote its support (a closed $\mathbb{C}$-analytic subset of $T^*X$).

(i) We say that $\mathcal{M}$ is holonomic if $\Lambda$ is Lagrangian.
Assume $\mathcal{M}$ is holonomic. We say that $\mathcal{M}$ is regular holonomic if there is an open subset $U \subset X$ such that $U \cap \Lambda$ is a dense subset of the regular locus $\Lambda_{\text{reg}}$ of $\Lambda$ and $\mathcal{M}|_U$ is regular along $U \cap \Lambda$.

In other words, a holonomic module $\mathcal{M}$ is regular if it is so at the generic points of its support. Note that when $\Lambda$ is smooth, Definition 2.4 is compatible with Definition 2.1. This follows from Gabber’s theorem. Indeed, we have the following theorem, analogous of [9, Th. 7.34]:

**Theorem 2.5.** Let $U$ be an open subset of $T^*X$, $\mathcal{M}$ a coherent $\mathcal{W}_{T^*X}|_U$-module and $\mathcal{N} \subset \mathcal{M}$ a sub-$\mathcal{W}_{T^*X}(0)$-module. Assume that $\mathcal{N}$ is a small filtrant inductive limit of coherent $\mathcal{W}_{T^*X}(0)$-modules. Let $V$ be the set of $p \in U$ in a neighborhood of which $\mathcal{N}$ is $\mathcal{W}_{T^*X}(0)$-coherent. Then $U \setminus V$ is an analytic involutive subset of $U$.

The fact that regularity of holonomic $\mathcal{W}$-modules is a generic property follows as in loc. cit. Prop. 8.28.

## 3 \mathcal{W}-modules on a complex symplectic manifold

We refer to [15] for the definition of an algebroid stack and to [3] for a more systematic study.

Let $\mathbb{K}$ be a commutative unital ring and let $X$ be a topological space. We denote by $\text{Mod}(\mathbb{K}_X)$ the abelian category of sheaves of $\mathbb{K}$-modules and by $\text{D}^b(\mathbb{K}_X)$ its bounded derived category.

If $\mathcal{A}$ is a $\mathbb{K}$-algebra, we denote by $\mathcal{A}^+$ the category with one object and having $\mathcal{A}$ as endomorphisms of this object. If $\mathcal{A}$ is a sheaf of $\mathbb{K}$-algebras on $X$, we denote by $\mathcal{A}^+$ the $\mathbb{K}$-linear stack associated with the prestack $U \mapsto \mathcal{A}(U)^+$ ($U$ open in $X$) and call it the $\mathbb{K}$-algebroid stack associated with $\mathcal{A}$. It is equivalent to the stack of right $\mathcal{A}$-modules locally isomorphic to $\mathcal{A}$, and $\mathcal{A}$-linear homomorphisms.

The projective cotangent bundle $P^*Y$ to a complex manifold $Y$ is endowed with the sheaf of rings $\mathcal{E}_{P^*Y}$ of microdifferential operators. (This sheaf is the direct image of the sheaf $\mathcal{E}_{T^*Y}$ of $\mathcal{E}^1$ by the map $T^*Y \setminus T^*_YY \to P^*Y$.) A complex contact manifold $\mathfrak{M}$ is locally isomorphic to an open subset of a projective cotangent bundle $P^*Y$ and on such a contact manifold, a canonical algebroid stack $\mathcal{E}_{\mathfrak{M}}$ locally equivalent to the stack associated with the sheaf of rings $\mathcal{E}_{P^*Y}$ has been constructed in [8].

This construction has been adapted to the symplectic case by [16]. A complex symplectic manifold $\mathfrak{X}$ is locally isomorphic to the cotangent bundle $T^*X$ to a complex manifold $X$ and a canonical algebroid stack $\mathcal{W}_X$ locally
equivalent to the stack associated with the sheaf of rings $\mathcal{W}_{T^*X}$ of § 1 is constructed in loc. cit., after Kontsevich [15] had treated the general case of complex Poisson manifolds in the formal setting by a different approach.

Denote by $\mathfrak{X}$ the complex manifold $\mathfrak{X}$ endowed with the symplectic form $-\omega$, where $\omega$ is the symplectic form on $\mathfrak{X}$. There is a natural equivalence of algebroid stacks $\mathcal{W}_{\mathfrak{X}a} \simeq (\mathcal{W}_{\mathfrak{X}})^{\text{op}}$.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two complex symplectic manifolds. There exist a natural $k$-algebroid stack $\mathcal{W}_{\mathfrak{X} \boxtimes \mathfrak{Y}}$ on $\mathfrak{X} \times \mathfrak{Y}$ and a natural functor of $k$-algebroid stacks $\mathcal{W}_{\mathfrak{X} \boxtimes \mathfrak{Y}} \rightarrow \mathcal{W}_{\mathfrak{X} \times \mathfrak{Y}}$ which locally corresponds to the morphism of sheaves of rings $\mathcal{W}_{T^*\mathfrak{X} \boxtimes \mathcal{W}_{T^*\mathfrak{Y}}} \rightarrow \mathcal{W}_{T^*(\mathfrak{X} \times \mathfrak{Y})}$.

One sets

$$\text{Mod}(\mathcal{W}_{\mathfrak{X}}) = \text{Fct}_k(\mathcal{W}_{\mathfrak{X}}, \text{Mod}(k_{\mathfrak{X}})),$$

where $\text{Fct}_k(\bullet, \bullet)$ denotes the $k$-linear category of $k$-linear functors of stacks and $\text{Mod}(k_{\mathfrak{X}})$ is the stack of sheaves of $k$-modules on $\mathfrak{X}$. We denote by $\text{Mod}(\mathcal{W}_{\mathfrak{X}})$ the stack on $\mathfrak{X}$ given by $U \mapsto \text{Mod}(\mathcal{W}_{\mathfrak{X}}|_U)$.

Then $\text{Mod}(\mathcal{W}_{\mathfrak{X}})$ is a Grothendieck abelian category. We denote by $D^b(\mathcal{W}_{\mathfrak{X}})$ its bounded derived category and call an object of this derived category a $\mathcal{W}_{\mathfrak{X}}$-module on $\mathfrak{X}$, for short. Objects of $\text{Mod}(\mathcal{W}_{\mathfrak{X}})$ are described with some details in [4].

We denote by $\text{Hom}_{\mathcal{W}_{\mathfrak{X}}}$ the hom-functor of the stack $\text{Mod}(\mathcal{W}_{\mathfrak{X}})$, a functor from $\text{Mod}(\mathcal{W}_{\mathfrak{X}})^{\text{op}} \times \text{Mod}(\mathcal{W}_{\mathfrak{X}})$ to $\text{Mod}(k_{\mathfrak{X}})$. The object $R\text{Hom}_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M}, \mathcal{N})$ is thus well defined in $D^b(k_{\mathfrak{X}})$ for two objects $\mathcal{M}$ and $\mathcal{N}$ of $D^b(\mathcal{W}_{\mathfrak{X}})$.

The natural functor $\mathcal{W}_{\mathfrak{X} \boxtimes \mathfrak{Y}} \rightarrow \mathcal{W}_{\mathfrak{X} \times \mathfrak{Y}}$ defines a functor of stacks

$$(3.1) \quad \text{for: } \text{Mod}(\mathcal{W}_{\mathfrak{X} \times \mathfrak{Y}}) \rightarrow \text{Mod}(\mathcal{W}_{\mathfrak{X} \boxtimes \mathfrak{Y}})$$

and this last functor admits an adjoint (since it does locally). For $\mathcal{M} \in D^b(\mathcal{W}_{\mathfrak{X}})$ and $\mathcal{N} \in D^b(\mathcal{W}_{\mathfrak{Y}})$, it thus exists a canonically defined object $\mathcal{M} \boxtimes \mathcal{N} \in D^b(\mathcal{W}_{\mathfrak{X} \boxtimes \mathfrak{Y}})$ such that, locally, $\mathcal{M} \boxtimes \mathcal{N} \simeq \mathcal{W}_{T^*(\mathfrak{X} \times \mathfrak{Y})} \boxtimes (\mathcal{W}_{T^*\mathfrak{X}} \boxtimes \mathcal{W}_{T^*\mathfrak{Y}})$ ($\mathcal{M} \boxtimes \mathcal{N}$) in $D^b(\mathcal{W}_{T^*(\mathfrak{X} \times \mathfrak{Y})})$.

Being local, the notions of coherent or holonomic, regular holonomic or simple object of $\text{Mod}(\mathcal{W}_{\mathfrak{X}})$ make sense. We denote by

- $D^b_{\text{coh}}(\mathcal{W}_{\mathfrak{X}})$ the full triangulated subcategory of $D^b(\mathcal{W}_{\mathfrak{X}})$ consisting of objects with coherent cohomologies,
- $D^b_{\text{hol}}(\mathcal{W}_{\mathfrak{X}})$ the full triangulated subcategory of $D^b_{\text{coh}}(\mathcal{W}_{\mathfrak{X}})$ consisting of objects with holonomic cohomologies, or equivalently, of objects with Lagrangian supports in $\mathfrak{X}$,
• $D^b_{\text{rh}}(\mathcal{W}_X)$ the full triangulated subcategory of $D^b_{\text{hol}}(\mathcal{W}_X)$ consisting of objects with regular holonomic cohomologies.

The support of an object $\mathcal{M}$ of $D^b_{\text{coh}}(\mathcal{W}_X)$ is denoted by $\text{supp}(\mathcal{M})$ and is also called its characteristic variety. This is a closed complex analytic involutive subset of $\mathcal{X}$.

In the sequel, we shall denote by $\Delta_X$ the diagonal of $X \times X^{\circ}$ and identify it with $\mathcal{X}$ by the first projection.

The next lemma follows from general considerations on stacks and its verification is left to the reader.

**Lemma 3.1.** There exists a canonical simple $\mathcal{W}_{X \times X^{\circ}}$-module $C_{\Delta_X}$ supported by the diagonal $\Delta_X$ such that if $U$ is open in $\mathcal{X}$ and isomorphic to an open subset $V$ of a cotangent bundle $T^*X$, then $C_{\Delta_X}|_U$ is isomorphic to $\mathcal{W}_{T^*X}|_V$ as a $\mathcal{W}_{T^*X} \otimes (\mathcal{W}_{T^*X})^{\text{op}}$-module.

**Definition 3.2.** Let $\mathcal{M} \in D^b(\mathcal{W}_X)$. Its dual $D_w'\mathcal{M} \in D^b(\mathcal{W}_{X^{\circ}})$ is given by

$$D_w'\mathcal{M} := R\text{Hom}_{\mathcal{W}_X}(\mathcal{M}, C_{\Delta_X}).$$

The main theorem of [4] asserts that simple $\mathcal{W}_{X}$-modules along $\Lambda$ are in one-to-one correspondence with twisted local systems of rank one on $\Lambda$ with twist $c_{1/2}$.

**Proposition 3.3.** Let $\mathcal{M}, \mathcal{N}$ be two objects of $D^b_{\text{coh}}(\mathcal{W}_X)$. There is a natural isomorphism in $D^b(\mathcal{k}_X)$:

$$R\text{Hom}_{\mathcal{W}_X}(\mathcal{M}, \mathcal{N}) \simeq R\text{Hom}_{\mathcal{W}_{X \times X^{\circ}}}(\mathcal{M} \boxtimes D_w'\mathcal{N}, C_{\Delta_X}).$$

**Proof.** We have the isomorphism in $D^b(\mathcal{W}_X)$:

$$\mathcal{N} \simeq R\text{Hom}_{\mathcal{W}_{X^{\circ}}}(D_w'\mathcal{N}, C_{\Delta_X}),$$

from which we deduce

$$R\text{Hom}_{\mathcal{W}_{X \times X^{\circ}}}(\mathcal{M} \boxtimes D_w'\mathcal{N}, C_{\Delta_X}) \simeq R\text{Hom}_{\mathcal{W}_X}(\mathcal{M}, R\text{Hom}_{\mathcal{W}_{X^{\circ}}}(D_w'\mathcal{N}, C_{\Delta_X}))$$

$$\simeq R\text{Hom}_{\mathcal{W}_X}(\mathcal{M}, \mathcal{N}).$$

q.e.d.
4 Functional analysis I

In this section, we will use techniques elaborated by Houzel [6] and will follow his terminology. (See also Kiel-V erdier [14] for related results.)

We call a bornological convex $C$-vector space (resp. $C$-algebra), a $bc$-space (resp. $bc$-algebra) and we denote by $\text{Mod}^{bc}(C)$ the category of $bc$-spaces and bounded linear maps. This additive category admits small inductive and projective limits, but is not abelian.

Let $A$ be a $bc$-algebra. We denote by $\text{Mod}^{bc}(A)$ the additive category of bornological $A$-modules and bounded $A$-linear maps. For $E, F \in \text{Mod}^{bc}(A)$, we set:

$$\text{Bhom}_A(E, F) = \text{Hom}_{\text{Mod}^{bc}(A)}(E, F),$$

$$E^\vee = \text{Bhom}_A(E, A).$$

Let $E \in \text{Mod}^{bc}(A)$ and let $B \subset E$ be a convex circled bounded subset of $E$. For $x \in E$, one sets

$$\|x\|_B = \inf_{x \in B, c \in C} |c|.$$

For $u \in \text{Bhom}_A(E, F)$, $B$ bounded in $E$ and $B'$ convex circled bounded in $F$, one sets

$$\|u\|_{BB'} = \sup_{x \in B} \|u(x)\|_{B'}.$$

One says that a sequence $\{u_n\}_n$ in $\text{Bhom}_A(E, F)$ is bounded if for any bounded subset $B \subset E$ there exists a convex circled bounded subset $B' \subset F$ such that $\sup_n \|u_n\|_{BB'} < \infty$.

One says that $u \in \text{Bhom}_A(E, F)$ is $A$-nuclear if there exist a bounded sequence $\{y_n\}_n$ in $F$, a bounded sequence $\{u_n\}_n$ in $\text{Bhom}_A(E, A)$ and a summable sequence $\{\lambda_n\}_n$ in $\mathbb{R}_{\geq 0}$ such that

$$u(x) = \sum_n \lambda_n u_n(x) y_n \text{ for all } x \in E.$$  

For $E, F \in \text{Mod}^{bc}(A)$, there is a natural structure of $bc$-space on $E \otimes_A F$ and one denotes by $E \widehat{\otimes}_A F$ the completion of $E \otimes_A F$. Assuming $F$ is complete, there is a natural linear map

$$E^\vee \widehat{\otimes}_A F \rightarrow \text{Bhom}_A(E, F).$$
An element \( u \in \text{Bhom}_A(E, F) \) is \( A \)-nuclear if and only if it is in the image of \( \hat{E} \hat{\otimes}_A F \).

Note that, if \( u: E \to F \) is a nuclear \( C \)-linear map, \( \hat{u} \circ 1: \hat{E} \hat{\otimes} A \to \hat{F} \hat{\otimes} A \) is \( A \)-nuclear.

Recall that a \( C \)-vector space \( E \) is called a DFN-space if it is an inductive limit \( \{ (E_n, u_n) \}_{n \in \mathbb{N}} \) of Banach spaces such that the maps \( u_n: E_n \to E_{n+1} \) are \( C \)-nuclear and injective. Note that any bounded subset of \( E \) is contained in \( E_n \) for some \( n \).

In the sequel, we will consider a DFN-algebra \( A \) and the full subcategory \( \text{Mod}^{\text{dfn}}(A) \) of \( \text{Mod}^{\text{bc}}(A) \) consisting of DFN-spaces. Note that any epimorphism \( u: E \to F \) in \( \text{Mod}^{\text{bc}}(A) \) is semi-strict, that is, any bounded sequence in \( F \) is the image by \( u \) of a bounded sequence in \( E \). Also note that for \( E \) and \( F \) in \( \text{Mod}^{\text{dfn}}(A) \), \( \text{Bhom}_A(E, F) \) is the subspace of \( \text{Hom}_A(E, F) \) consisting of continuous maps. Since the category \( \text{Mod}^{\text{dfn}}(A) \) is not abelian, we introduce the following definition, referring to [20] for a more systematic treatment of homological algebra in terms of quasi-abelian categories.

**Definition 4.1.** (i) Let \( A \) be a DFN-algebra. A complex \( 0 \to E' \to E \to E'' \to 0 \) in \( \text{Mod}^{\text{dfn}}(A) \) is called a short exact sequence if it is an exact sequence in \( \text{Mod}(A) \).

(ii) Let \( A \) and \( B \) be two DFN-algebra. An additive functor from \( \text{Mod}^{\text{dfn}}(A) \) to \( \text{Mod}^{\text{dfn}}(B) \) is called exact if it sends short exact sequences to short exact sequences.

The following result is well-known and follows from [5].

**Proposition 4.2.** Let \( A \) be a DFN-algebra. The functor \( \bullet \hat{\otimes} A: \text{Mod}^{\text{dfn}}(C) \to \text{Mod}^{\text{dfn}}(A) \) is exact.

Recall that a \( bc \)-algebra \( A \) is multiplicatively convex if for any bounded set \( B \subset A \), there exist a constant \( c > 0 \) and a convex circled bounded set \( B' \) such that \( B \subset c \cdot B' \) and \( B' \cdot B' \subset B' \).

As usual, for an additive category \( C \), we denote by \( \text{C}^b(C) \) the category of bounded complexes in \( C \) and, for \( a \leq b \) in \( \mathbb{Z} \), by \( \text{C}^{[a,b]}(C) \) the full subcategory consisting of complexes concentrated in degrees \( j \in [a,b] \). We denote by \( \text{K}^b(C) \) the homotopy category associated with \( \text{C}^b(C) \). Finally, we denote by \( \text{Ind}(C) \) and \( \text{Pro}(C) \) the categories of ind-objects and pro-objects of \( C \), respectively.

**Theorem 4.3.** Let \( A \) be a multiplicatively convex DFN-algebra and assume that \( A \) is a Noetherian ring (when forgetting the topology). Consider an inductive system \( \{(E_n^\bullet, u_n^\bullet)\}_{n \in \mathbb{N}} \) in \( \text{C}^{[a,b]}(\text{Mod}^{\text{dfn}}(A)) \) for \( a \leq b \in \mathbb{Z} \). Assume:
(i) \( u_n^\cdot: E_n^\cdot \to E_{n+1}^\cdot \) is a quasi-isomorphism for all \( n \geq 0 \),

(ii) \( u_i^j: E_n^j \to E_{n+1}^j \) is \( A \)-nuclear for all \( j \in \mathbb{Z} \) and all \( n \geq 0 \).

Then \( H^j(E_n^\cdot) \) is finitely generated over \( A \) for all \( j \in \mathbb{Z} \) and all \( n \geq 0 \).

This is a particular case of [6, §6 Th. 1, Prop. A.1].

**Theorem 4.4.** Let \( A \) be a DFN-algebra, and consider an inductive system
\( \{E_n^\cdot, u_n^\cdot\}_{n \in \mathbb{N}} \) in \( C^{[a,b]}(\text{Mod}^{\text{dfn}}(A)) \) for \( a \leq b \) in \( \mathbb{Z} \). Assume

(i) for each \( i \in \mathbb{Z} \) and \( n \in \mathbb{N} \), the map \( u_i^n: E_n^i \to E_{n+1}^i \) is \( A \)-nuclear,

(ii) for each \( i \in \mathbb{Z} \), \( \lim_{n \to} H^i(E_n^\cdot) \simeq 0 \) in \( \text{Ind}(\text{Mod}(\mathbb{C})) \).

Then \( \lim_{n \to} E_n^\cdot \simeq 0 \) in \( \text{Ind}(K^{[a,b]}(\text{Mod}^{\text{dfn}}(A))) \).

First, we need a lemma.

**Lemma 4.5.** Consider the solid diagram in \( \text{Mod}^{\text{dfn}}(A) \):

\[
\begin{array}{ccc}
E & \xrightarrow{u} & F \\
\downarrow{v} & & \downarrow{v'} \\
E' & \xrightarrow{u'} & F'.
\end{array}
\]

Assume that \( u \) is \( A \)-nuclear and \( \text{Im } v' \subset \text{Im } u' \). Then there exists a morphism \( v: E \to E' \) making the whole diagram commutative.

**Proof.** The morphism \( w: E' \times_{F'} F \to F \) is well defined in the category \( \text{Mod}^{\text{dfn}}(A) \) and is surjective by the hypothesis. We get a diagram

\[
\begin{array}{ccc}
E & \xrightarrow{u} & F \\
\downarrow{w} & & \\
E' \times_{F'} F' & \xrightarrow{w} & F.
\end{array}
\]

In this situation, the nuclear map \( u \) factors through \( E' \times_{F'} F' \) by [6, §4 Cor. 2].

q.e.d.
Proof of Theorem 4.4. We may assume that $H^i(E_n^•) \to H^i(E_{n+1}^•)$ is the zero morphism for all $i \in \mathbb{Z}$ and all $n \in \mathbb{N}$. Consider the solid diagram

![Diagram]

Since $H^b(E^•_p) \simeq \text{Coker} \ d^{b-1}_p$ for all $p$, Im $u_n^b \subset \text{Im} \ d^{b-1}_{n+1}$. Moreover, $u_n^{b-1}$ is $A$-nuclear by the hypothesis. Therefore we may apply Lemma 4.5 and we obtain a map $k_n^b : E_{n-1}^b \to E_{n+1}^b$ making the whole diagram commutative. Set $v_n^i = u_n^i \circ u_n^{i-1}$ and $h_n^b = d_{n+1}^{b-1} \circ k_n^b$. Consider the diagram

![Diagram]

The morphisms $v_n^i$'s define a morphism of complex $v_n : E_{n-1}^• \to E_{n+1}^•$. We define $h_n^i : E_{n-1}^i \to E_{n+1}^i$ by setting $h_n^i = 0$ for $i \neq b$. Now denote by $\sigma^{\leq b-1}E_n^•$ the stupid truncated complex obtained by replacing $E_n^b$ with 0. The morphism

$$v_n - h_n \circ d_{n-1} - d_{n+1} \circ h_n : E_{n-1}^• \to E_{n+1}^•$$

factorizes through $\sigma^{\leq b-1}E_n^•$. Hence, we get an isomorphism

$$\text{“lim”} \ E_n^• \simeq \text{“lim”} \ \sigma^{\leq b-1}E_n^•$$

in $\text{Ind}(K^{[a,b]}(\text{Mod}^{\text{dfn}}(A)))$. By repeating this argument, we find the isomorphism $\text{“lim”} \ E_n^• \simeq \text{“lim”} \ \sigma^{\leq b-p}E_n^•$ for any $p \in \mathbb{N}$. This completes the proof.

q.e.d.

Theorem 4.6. Let $A$ be a DFN-field, let $a \leq b$ in $\mathbb{Z}$, consider an inductive system $\{E_n^i, u_n^i\}_{n \in \mathbb{N}}$ in $C^{[a,b]}(\text{Mod}^{\text{dfn}}(A))$ and set $F^• = (E_n^•)^{\vee} = \text{Bhom}_A(E_n^•, A)$. Assume

(i) for each $i \in \mathbb{Z}$ and $n \in \mathbb{N}$, the map $u_n^i : E_n^i \to E_{n+1}^i$ is $A$-nuclear,
(ii) for each \( i \in \mathbb{Z} \), \( \lim_{\rightarrow n} H^i(E_n^\bullet) \) belongs to \( \text{Mod}^f(A) \) (the category of finite-dimensional \( A \)-vector spaces).

Then we have the isomorphism

\[
\lim_{\rightarrow n} F_n^\bullet \cong \lim_{\rightarrow n} \text{Hom}_A(E_n^\bullet, A) \quad \text{in} \quad \text{Pro}(K^b(\text{Mod}(A))).
\]

In particular, for each \( i \in \mathbb{Z} \), \( \lim_{\rightarrow n} H^{-i}(F_n^\bullet) \) belongs to \( \text{Mod}^f(A) \) and is dual of \( \lim_{\rightarrow n} H^i(E_n^\bullet) \).

**Proof.** Recall (see [13, Exe 15.1]) that for an abelian category \( \mathcal{C} \) and an inductive system \( \{X_j\}_{j \in J} \) in \( \text{D}^{[a,b]}(\mathcal{C}) \) indexed by a small filtrant category \( J \), if the object \( \lim_{\rightarrow j} H^i(X_j) \) of \( \text{Ind}(\mathcal{C}) \) is representable for all \( i \in \mathbb{Z} \), then \( \lim_{\rightarrow j} X_j \in \text{Ind}(\text{D}^b(\mathcal{C})) \) is representable.

Applying this result to our situation, we find that the object \( \lim_{\rightarrow n} E_n^\bullet \) of \( \text{Ind}(\text{D}^b(\text{Mod}(A))) \) is representable in \( \text{D}^b(\text{Mod}(A)) \).

Denote by \( L^\bullet \) the complex given by \( L^i = \lim_{\rightarrow n} H^i(E_n^\bullet) \) and zero differentials. Since \( A \) is a field, there exists an isomorphism \( L^\bullet \cong \lim_{\rightarrow n} E_n^\bullet \) in \( \text{D}^b(\text{Mod}(A)) \), hence a quasi-isomorphism \( u: L^\bullet \to \lim_{\rightarrow n} E_n^\bullet \) in \( \text{C}^b(\text{Mod}(A)) \).

There exists \( n \in \mathbb{N} \) such that \( u \) factorizes through \( L^\bullet \to E_n^\bullet \) for some \( n \) and we may assume \( n = 0 \). Since \( L^\bullet \) belongs to \( \text{C}^b(\text{Mod}^f(A)) \), \( L^\bullet \to E_0^\bullet \) is well defined in \( \text{C}^b(\text{Mod}^{dfn}(A)) \). For any \( n \), let \( u_n: L^\bullet \to E_n^\bullet \) be the induced morphism. Let \( G_n^\bullet \) be the mapping cone of \( u_n \). The morphism \( u_n \) is embedded in a distinguished triangle in \( \text{K}^b(\text{Mod}^{dfn}(A)) \)

\[
L^\bullet \xrightarrow{u_n} E_n^\bullet \to G_n^{\bullet +1} \to .
\]

By Theorem 4.4, \( \lim_{\rightarrow n} H^i(G_n^\bullet) \simeq 0 \) implies \( \lim_{\rightarrow n} G_n^\bullet \simeq 0 \) in \( \text{Ind}(\text{K}^b(\text{Mod}^{dfn}(A))) \).

Hence, for any \( K \in \text{K}^b(\text{Mod}^{dfn}(A)) \), the morphism

\[
\text{Hom}_{\text{K}^b(\text{Mod}^{dfn}(A))}(K, L^\bullet) \to \lim_{\rightarrow n} \text{Hom}_{\text{K}^b(\text{Mod}^{dfn}(A))}(K, E_n^\bullet)
\]

is an isomorphism. By the Yoneda lemma, we have thus obtained the isomorphism

\[
L^\bullet \cong \lim_{\rightarrow n} E_n^\bullet \quad \text{in} \quad \text{Ind}(\text{K}^b(\text{Mod}^{dfn}(A))).
\]
Since $L^\bullet$ belongs to $K^b(\text{Mod}^f(A))$, $(L^\bullet)^\vee \simeq \text{Hom}_A(L^\bullet, A)$ and we obtain

$$\lim_n \ F_n^\bullet \simeq (L^\bullet)^\vee \simeq \text{Hom}_A(L^\bullet, A) \simeq \lim_n \text{Hom}_A(E_n^\bullet, A)$$

in $\text{Pro}(K^b(\text{Mod}(A)))$. The isomorphisms for the cohomologies follow since $\lim_n$ commutes with $H^{-i}(\bullet)$ and $\text{Hom}_A(\bullet, A)$. q.e.d.

A similar result to Theorem 4.6 holds for projective system.

**Theorem 4.7.** Let $A$ be a DFN-field, let $a \leq b$ in $\mathbb{Z}$, consider a projective system $\{F_n^\bullet, v_n^\bullet\}_{n \in \mathbb{N}}$ in $C^{[a,b]}(\text{Mod}^{dfn}(A))$ and set $E_n^\bullet = \text{Bhom}_A(F_n^\bullet, A)$. Assume

(i) for each $i \in \mathbb{Z}$ and $n \in \mathbb{N}$, the map $v_n^i: F_{n+1}^i \to F_n^i$ is $A$-nuclear,

(ii) for each $i \in \mathbb{Z}$, $\lim_n H^i(F_n^\bullet)$ belongs to $\text{Mod}^f(A)$ (the category of finite-dimensional $A$-vector spaces).

Then we have the isomorphism

$$(4.2) \quad \lim_n \ E_n^\bullet \simeq \lim_n \text{Hom}_A(F_n^\bullet, A) \text{ in } \text{Ind}(K^b(\text{Mod}(A))).$$

In particular, for each $i \in \mathbb{Z}$, $\lim_n H^{-i}(E_n^\bullet)$ belongs to $\text{Mod}^f(A)$ and is dual of $\lim_n H^i(F_n^\bullet)$.

The proof being similar to the one of Theorem 4.6, we shall not repeat it.

In the course of § 5 below, we shall also need the next lemma.

**Lemma 4.8.** Let $A$ be a DFN-algebra and let $u: E_0 \to E_1$ be a $\mathbb{C}$-nuclear map of DFN-spaces. Recall that $(\bullet)^\vee = \text{Bhom}_A(\bullet, A)$ and set $(\bullet)^* = \text{Bhom}_C(\bullet, \mathbb{C})$. Then the solid commutative diagram below may be completed with the dotted arrow as a commutative diagram:

$$
\begin{array}{ccc}
E_1 \otimes A & \xrightarrow{u^* \otimes A} & E_0 \otimes A \\
\downarrow & & \downarrow \\
(E_1 \otimes A)^\vee & \xrightarrow{(u \otimes A)^\vee} & (E_0 \otimes A)^\vee.
\end{array}
$$
Proof. Consider the commutative diagram

\[
\begin{array}{ccc}
E_0^* \otimes E_1 & \longrightarrow & \text{Bhom}_C(E_0, E_1) \\
\downarrow & & \downarrow \\
\text{Bhom}_A((E_1 \otimes A)^\vee, E_0^* \otimes A) & \longrightarrow & \text{Bhom}_A((E_1 \otimes A)^\vee, (E_0 \otimes A)^\vee).
\end{array}
\]

Since \( u \) is nuclear, it is the image of an element of \( E_0^* \otimes E_1 \). q.e.d.

5 Functional analysis II

This section will provide the framework for applying Theorems 4.3 and 4.6.

Here, \( X \) will denote a complex manifold. For a locally free \( \mathcal{O}_X \)-module \( F \) of finite rank, we set

\[
\mathcal{F}^\tau := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^\tau, \quad \mathcal{F}^\tau(0) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^\tau(0).
\]

Lemma 5.1. (i) The \( \mathbb{C} \)-algebra \( k_0 \) is a multiplicatively convex DFN-algebra,

(ii) the \( \mathbb{C} \)-algebra \( k \) is a DFN-algebra,

(iii) the functor \( \bullet \otimes k_0 : \text{Mod}^{\text{dfn}}(\mathbb{C}) \rightarrow \text{Mod}^{\text{dfn}}(k_0) \) is well-defined and exact,

(iv) the functor \( \bullet \otimes k : \text{Mod}^{\text{dfn}}(\mathbb{C}) \rightarrow \text{Mod}^{\text{dfn}}(k) \) is well-defined and exact, and is isomorphic to the functor \( (\bullet \otimes k_0) \otimes_{k_0} k \).

Proof. (i) Define the subalgebra \( k_0(r) \) of \( k_0 \) by

\[
\begin{cases}
\ u = \sum_{j \leq 0} a_j \tau^j \ & \text{belongs to} \ k_0(r) \ \text{if and only if} \\
\ |u|_r := \sum_{j \leq 0} \frac{r^{-j}}{(-j)!} |a_j| < \infty.
\end{cases}
\]

Then, for \( u, v \) in \( k_0(r) \), we have

\[
|u \cdot v|_r \leq |u|_r \cdot |v|_r.
\]

Hence, \( (k_0(r), |\cdot|_r) \) is a Banach algebra and \( k_0 \) is multiplicatively convex since it is the inductive limit of the \( k_0(r') \)'s. Moreover, \( k_0 \) is a DFN-space because the linear maps \( k_0(r) \rightarrow k_0(r') \) are nuclear for \( 0 < r' < r \).

(ii)–(iv) are clear. q.e.d.
Note that $k$ is not multiplicatively convex.

Let $M$ be a real analytic manifold, $X$ a complexification of $M$. We denote as usual by $\mathcal{A}_M$ the sheaf on $M$ of real analytic functions, that is, $\mathcal{A}_M = \mathcal{O}_X|_M$. Recall that, for $K$ compact in $M$, $\Gamma(K; \mathcal{A}_M)$ is a DFN-space. We set

\begin{equation}
\mathcal{A}_M^\tau = \mathcal{O}_X^\tau|_M.
\end{equation}

**Lemma 5.2.** Let $M$ be a real analytic manifold and $K$ a compact subset of $M$. Then

(i) the sheaf $\mathcal{A}_M^\tau$ is $\Gamma(K; \bullet)$-acyclic,

(ii) $\Gamma(K; \mathcal{A}_M^\tau) \simeq \Gamma(K; \mathcal{A}_M) \hat{\otimes} k$,

(iii) the same result holds with $\mathcal{A}_M^\tau$ and $k$ replaced with $\mathcal{A}_M^\tau(0)$ and $k_0$, respectively.

**Proof.** Applying Lemma 1.4, we have isomorphisms for each holomorphically convex compact subset $K$ of $X$:

\[
\Gamma(K; \mathcal{O}_X^\tau(0)) \simeq \Gamma(K \times \{0\}; \mathcal{O}_X \times \mathbb{C}) \\
\simeq \Gamma(K; \mathcal{O}_X) \hat{\otimes} \mathcal{O}_{\mathbb{C},0} \\
\simeq \Gamma(K; \mathcal{O}_X) \hat{\otimes} k.
\]

This proves (i)–(ii) for $\mathcal{A}_M^\tau(0)$ and $k_0$. The other case follows since $\mathcal{A}_M^\tau \simeq \mathcal{A}_M^\tau(0) \otimes_{k_0} k$. q.e.d.

Let us denote, as usual, by $\mathcal{D}b_M$ the sheaf of Schwartz’s distribution on $M$. Recall that $\Gamma_c(M; \mathcal{D}b_M)$ is a DFN-space.

**Lemma 5.3.** Let $M$ be a real analytic manifold. There is a unique (up to unique isomorphism) sheaf of $k$-modules $\mathcal{D}b_M^\tau$ on $M$ which is soft and satisfies

\[
\Gamma_c(U; \mathcal{D}b_M^\tau) \simeq \Gamma_c(U; \mathcal{D}b_M) \hat{\otimes} k
\]

for each open subset $U$ of $M$. The same result holds with $k$ replaced with $k_0$. In this case, we denote by $\mathcal{D}b_M^\tau(0)$ the sheaf of $k_0$-modules so obtained.

**Proof.** For two open subsets $U_0$ and $U_1$, the sequence

\[
0 \rightarrow \Gamma_c(U_0 \cap U_1; \mathcal{D}b_M) \hat{\otimes} k \rightarrow (\Gamma_c(U_0; \mathcal{D}b_M) \hat{\otimes} k) \oplus (\Gamma_c(U_1; \mathcal{D}b_M) \hat{\otimes} k) \rightarrow \Gamma_c(U_0 \cup U_1; \mathcal{D}b_M) \hat{\otimes} k \rightarrow 0
\]

is exact, and similarly with $k$ replaced with $k_0$. The results then easily follow. q.e.d.
Denote by $\overline{X}$ the complex conjugate manifold to $X$ and by $X_{\mathbb{R}}$ the real underlying manifold, identified with the diagonal of $X \times \overline{X}$. We shall write for short $A_{X}^{\tau}$ and $D_{b}^{\tau}X$ instead of $A_{X_{\mathbb{R}}}^{\tau}$ and $D_{b}^{\tau}X_{\mathbb{R}}$, respectively. We set $A_{X}^{(p,q),\tau} = A_{X}^{(p,q)} \otimes_{A_{X}} A_{\overline{X}}$, and similarly with $D_{b}^{\tau}X$ instead of $A_{X}^{\tau}$.

Consider the Dolbeault-Grothendieck complexes of sheaves of $k$-modules

\begin{equation}
0 \rightarrow A_{X}^{(0,0),\tau} \xrightarrow{\partial} \cdots \xrightarrow{\partial} A_{X}^{(0,d),\tau} \rightarrow 0,
\end{equation}

\begin{equation}
0 \rightarrow D_{b}^{(0,0),\tau} \xrightarrow{\partial} \cdots \xrightarrow{\partial} D_{b}^{(0,d),\tau} \rightarrow 0.
\end{equation}

**Lemma 5.4.** Both complexes (5.3) and (5.4) are qis to $O_{\overline{X}}^{\tau}$. The same result holds when replacing $A_{X}^{\tau}$, $D_{b}^{\tau}X$ and $O_{\overline{X}}^{\tau}$ with $A_{X}^{\tau}(0)$, $D_{b}^{\tau}X(0)$ and $O_{\overline{X}}^{\tau}(0)$, respectively.

The easy proof is left to the reader.

**Lemma 5.5.** Let $X$ be a complex Stein manifold, $K$ a holomorphically convex compact subset and $\mathcal{F}$ a locally free $O_{X}$-module of finite rank. Then

(i) one has an isomorphism $\Gamma(K; \mathcal{F}^{\tau}) \simeq \Gamma(K; \mathcal{F}) \otimes k$,

(ii) the $\mathbb{C}$-vector space $\Gamma(K; \mathcal{F}^{\tau})$ is naturally endowed with a topology of DFN-space,

(iii) $R\Gamma(K; \mathcal{F}^{\tau})$ is concentrated in degree 0,

(iv) for $K_{0}$ and $K_{1}$ two compact subsets of $X$ such that $K_{0}$ is contained in the interior of $K_{1}$, the morphism $\Gamma(K_{1}; \mathcal{F}^{\tau}) \rightarrow \Gamma(K_{0}; \mathcal{F}^{\tau})$ is $k$-nuclear,

(v) the same results hold with $\mathcal{F}^{\tau}$ and $k$ replaced with $\mathcal{F}^{\tau}(0)$ and $k_{0}$, respectively.

**Proof.** (i)–(iii) By the hypothesis, the sequence

\begin{equation}
0 \rightarrow \Gamma(K; O_{X}) \rightarrow \Gamma(K; A_{X}^{(0,0)}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Gamma(K; A_{X}^{(0,d)}) \rightarrow 0
\end{equation}

is exact. It remains exact after applying the functor $\cdot \otimes k$. The result will follow when comparing the sequence so obtained with the complex

\begin{equation}
0 \rightarrow \Gamma(K; O_{X}^{\tau}) \rightarrow \Gamma(K; A_{X}^{(0,0),\tau}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Gamma(K; A_{X}^{(0,d),\tau}) \rightarrow 0,
\end{equation}

The case of $\mathcal{F}$ is treated similarly, replacing $A_{X}$ with $\mathcal{F} \otimes_{O_{X}} A_{X}$.

(iv) follows from (i) and the corresponding result for $O_{X}$.

(iv) The proof for $\mathcal{F}^{\tau}(0)$ and $k_{0}$ is similar. q.e.d.
Lemma 5.6. Let $X$ be a complex manifold of complex dimension $d$ and let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module of finite rank. Assume that $X$ is Stein. Then

(i) one has the isomorphism $H^d_c(X; \mathcal{F}^\tau) \simeq H^d_c(X; \mathcal{F}) \otimes k$,

(ii) the $\mathbb{C}$-vector space $H^d_c(X; \mathcal{F}^\tau)$ is naturally endowed with a topology of DFN-space,

(iii) $R \Gamma_c(X; \mathcal{F}^\tau_X)$ is concentrated in degree $d$,

(iv) for $U_0$ and $U_1$ two Stein open subset of $X$ with $U_0 \subset \subset U_1$, the map $H^d_c(U_0; \mathcal{F}^\tau_X) \to H^d_c(U_1; \mathcal{F}^\tau_X)$ is $k$-nuclear,

(v) the same results hold with $\mathcal{F}$ and $k$ replaced with $\mathcal{F}^\tau(0)$ and $k_0$, respectively.

Proof. The proof is similar to that of Lemma 5.5. By the hypothesis, the sequence

\[ 0 \to \Gamma_c(X; \mathcal{D}b^{(0,0)}_X) \to \cdots \to \Gamma_c(X; \mathcal{D}b^{(0,d)}_X) \to H^d_c(X; \mathcal{O}_X) \to 0 \]

is exact and $H^d_c(X; \mathcal{O}_X)$ is a DFN-space. This sequence will remain exact after applying the functor $\bullet \hat{\otimes} k$. The result will follow when comparing the obtained sequence with

\[ 0 \to \Gamma_c(X; \mathcal{D}b^{(0,0),\tau}_X) \to \cdots \to \Gamma_c(X; \mathcal{D}b^{(0,d),\tau}_X) \to H^d_c(X; \mathcal{O}_X^\tau) \to 0. \]

The case of $\mathcal{F}$ is treated similarly, replacing $\mathcal{D}b_X$ with $\mathcal{F} \otimes \mathcal{O}_X \mathcal{D}b_X$. q.e.d.

Proposition 5.7. Let $f: X \to Y$ be a morphism of complex manifolds of complex dimension $d_X$ and $d_Y$, respectively. There is a $k$-linear morphism

\[ \int_f : Rf_! \Omega^\tau_X [d_X] \to \Omega^\tau_Y [d_Y] \]

functorial with respect to $f$, which sends $Rf_! \Omega^\tau_X (0) [d_X]$ to $\Omega^\tau_Y (0) [d_Y]$ and which induces the classical integration morphism $\int_f : Rf_! \Omega_X [d_X] \to \Omega_Y [d_Y]$ when identifying $\Omega_X$ and $\Omega_Y$ with a direct summand of $\Omega^\tau_X$ and $\Omega^\tau_Y$, respectively.

In particular, there is a $k$-linear morphism

\[ \int_X : H^d_{\mathcal{O}_X} (X; \Omega^\tau_X) \to k \]

which induces the classical residues morphism $H^d_{\mathcal{O}_X} (X; \Omega_X) \to \mathbb{C}$ when identifying $H^d_{\mathcal{O}_X} (X; \Omega_X)$ (resp. $\mathbb{C}$) with a direct summand of $H^d_{\mathcal{O}_X} (X; \Omega^\tau_X)$ (resp. $k$).
In the particular case where $X$ is Stein and $Y = \text{pt}$, this follows easily from Lemma 5.6. Since we shall not use the general case, we leave the proof to the reader.

**Proposition 5.8.** Let $X$ be a Stein complex manifold of complex dimension $d$ and let $K$ be a holomorphically convex compact subset of $X$. Then the pairing for a Stein open subset $U$ of $X$

\[(5.11) \quad H^d_c(U; \Omega^*_X) \times \Gamma(U; \mathcal{O}^*_X) \to \mathbb{k}, \quad (fdx, g) \mapsto \int_U gfdx,\]

defines the isomorphisms \(\lim_{U \supset K}^{-\to} \text{Bhom}_k(\Gamma(U; \mathcal{O}^*_X), \mathbb{k}) \simeq \lim_{U \supset K}^{-\to} H^d_c(U; \Omega^*_X)\)
and \(\lim_{U \supset K}^{-\to} \text{Bhom}_k(H^d_c(U; \Omega^*_X), \mathbb{k}) \simeq \lim_{U \supset K}^{-\to} \Gamma(U; \mathcal{O}^*_X).\) Here, $U$ ranges over the family of Stein open neighborhoods of $K$.

**Proof.** We shall prove the first isomorphism, the other case being similar.

By Lemmas 5.5, 5.6 and 4.8, we have:

\[
\begin{align*}
\lim_{U \supset K}^{-\to} \text{Bhom}_k(\Gamma(U; \mathcal{O}^*_X), \mathbb{k}) & \simeq \lim_{U \supset K}^{-\to} \Gamma(U; \mathcal{O}^*_X)^\wedge \\
& \simeq \lim_{U \supset K}^{-\to} \Gamma(U; \mathcal{O}^*_X)\wedge \mathbb{k} \\
& \simeq \lim_{U \supset K}^{-\to} \Gamma(U; \mathcal{O}^*_X)\wedge \mathbb{k} \\
& \simeq \lim_{U \supset K}^{-\to} \Gamma(U; \mathcal{O}^*_X)\wedge \mathbb{k} \\
& \simeq \lim_{U \supset K}^{-\to} H^d_c(U; \Omega^*_X). \\
\end{align*}
\]

q.e.d.

In the proof of Corollary 7.6, we shall need the following result.

**Lemma 5.9.** Let $Y$ be a smooth closed submanifold of codimension $l$ of $X$. Then $H^j(R\Gamma_Y(\mathcal{O}^*_X))$ and $H^j(R\Gamma_Y(\mathcal{O}^*_X(0)))$ vanish for $j < l$.

**Proof.** Since the problem is local, we may assume that $X = Y \times \mathbb{C}^l$ where $Y \subset X$ is identified with $Y \times \{0\}$.

Let $U_n$ be the open ball of $\mathbb{C}^l$ centered at 0 with radius $1/n$. Using the Mittag-Leffler theorem (see [12, Prop. 2.7.1]), it is enough to prove that for any holomorphically convex compact subset $K$ of $Y$

\[H^j_c(K \times U_n; \mathcal{O}^*_X) = 0 \text{ for } j < l,\]

21
and similarly with $\mathcal{O}_X^*(0)$. It is enough to prove the result for the sheaf $\mathcal{O}_X^*(0)$. Then we may replace $\mathcal{O}_X^*(0)$ with the sheaf $\mathcal{O}_{X\times\mathbb{C}|X\times\{0\}}$ and we are reduced to the well-known result

$$H^j_l(K \times U_n \times \{0\}; \mathcal{O}_{X\times\mathbb{C}}) = 0 \text{ for } j < l.$$ 

q.e.d.

**Remark 5.10.** Related results have been obtained, in a slightly different framework, in [17].

## 6 Duality for $\mathcal{W}$-modules

We mainly follow the notations of [12]. Let $X$ be a real manifold and $\mathbb{K}$ a field. For $F \in \mathcal{D}^b(\mathbb{K}X)$, we denote by $SS(F)$ its microsupport, a closed $\mathbb{R}^+$-conic subset of $T^*X$. Recall that this set is involutive (see loc. cit. Def. 6.5.1).

We denote by $D_X$ the duality functor:

$$D_X : (\mathcal{D}^b(\mathbb{K}X))^{\text{op}} \rightarrow \mathcal{D}^b(\mathbb{K}X), \quad F \mapsto R\text{Hom}_{\mathbb{K}X}(F, \omega_X),$$

where $\omega_X$ is the dualizing complex.

Assume now that $X$ is a complex manifold. We denote by $\text{dim}_\mathbb{C} X$ its complex dimension. We identify the orientation sheaf on $X$ with the constant sheaf, and the dualizing complex $\omega_X$ with $\mathbb{K}X[2\text{dim}_\mathbb{C} X]$.

Recall that an object $F \in \mathcal{D}^b(\mathbb{K}X)$ is weakly-$\mathbb{C}$-constructible if there exists a complex analytic stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ such that $H^j(F)|_{X_\alpha}$ is locally constant for all $j \in \mathbb{Z}$ and all $\alpha \in A$. The object $F$ is $\mathbb{C}$-constructible if moreover $H^j(F)_x$ is finite-dimensional for all $x \in X$ and all $j \in \mathbb{Z}$. We denote by $\mathcal{D}_{\text{w-c-con}}^b(\mathbb{K}X)$ the full subcategory of $\mathcal{D}^b(\mathbb{K}X)$ consisting of weakly-$\mathbb{C}$-constructible objects and by $\mathcal{D}_{\text{c-con}}^b(\mathbb{K}X)$ the full subcategory consisting of $\mathbb{C}$-constructible objects.

Recall ([12]) that $F \in \mathcal{D}^b(\mathbb{K}X)$ is weakly-$\mathbb{C}$-constructible if and only if its microsupport is a closed $\mathbb{C}$-conic complex analytic Lagrangian subset of $T^*X$ or, equivalently, if it is contained in a closed $\mathbb{C}$-conic complex analytic isotropic subset of $T^*X$.

From now on, our base field is $\mathbb{k}$.

**Theorem 6.1.** Let $\mathcal{X}$ be a complex symplectic manifold and let $\mathcal{L}_0$ and $\mathcal{L}_1$ be two objects of $\mathcal{D}_{\text{coh}}^b(\mathcal{W}_X)$. 

22
(i) There is a natural morphism

$$R\text{Hom}_{W_X}(L_1, L_0) \to D_X(R\text{Hom}_{W_X}(L_0, L_1[\dim \mathcal{X}])).$$

(ii) Assume that $R\text{Hom}_{W_X}(L_0, L_1)$ belongs to $D^b_{\mathcal{C}-c}(k_X)$. Then the morphism (6.1) is an isomorphism. In particular, $R\text{Hom}_{W_X}(L_1, L_0)$ belongs to $D^b_{\mathcal{C}-c}(k_X)$.

Proof. By Proposition 3.3, we may assume that $L_0$ is a simple module along a smooth Lagrangian manifold $\Lambda_0$. In this case, $k_\Lambda_0 \to R\text{Hom}_{W_X}(L_0, L_0)$ is an isomorphism.

(i) The natural $k$-linear morphism

$$R\text{Hom}_{W_X}(L_1, L_0) \otimes_k R\text{Hom}_{W_X}(L_0, L_1) \to R\text{Hom}_{W_X}(L_0, L_0) \simeq k_\Lambda_0$$

defines the morphism

$$R\text{Hom}_{W_X}(L_0, L_1) \to R\text{Hom}_{k_\Lambda_0}(R\text{Hom}_{W_X}(L_1, L_0), k_\Lambda_0).$$

To conclude, remark that for an object $F \in D^b(k_X)$ supported by $\Lambda_0$, we have

$$D_X F \simeq R\text{Hom}_{k_\Lambda_0}(F, k_\Lambda_0)[\dim \mathcal{X}].$$

(ii) Let us prove that (6.2) is an isomorphism by adapting the proof of [7, Prop. 5.1].

Since this is a local problem, we may assume that $X = T^*X$, $X$ is an open subset of $\mathbb{C}^d$, $\Lambda_0 = T^*_X X$ and $L_0 = \mathcal{O}_X^r$. Since $R\text{Hom}_{W_X}(\mathcal{O}_X^r, L_1)$ is constructible, we are reduced to prove the isomorphisms for each $x \in X$

$$H^j(R\text{Hom}_{W_X}(L_1, \mathcal{O}_X^r)) \simeq \lim_{U \ni x} (H^j R\Gamma_c(U; R\text{Hom}_{W_X}(\mathcal{O}_X^r, L_1))[2d])^*$$

where $U$ ranges over the family of Stein open neighborhoods of $x$ and $^*$ denotes the duality functor in the category of $k$-vector spaces.

We choose a finite free resolution of $L_1$ on a neighborhood of $x$:

$$0 \to \mathcal{W}_{T^*X}^{N_r} \xrightarrow{P_{r-1}} \cdots \xrightarrow{P_0} \mathcal{W}_{T^*X}^{N_0} \to L_1 \to 0 \text{ for some } r \geq 0.$$

For a sufficiently small holomorphically convex compact neighborhood $K$ of $x$, the object $R\Gamma(K; R\text{Hom}_{W_{T^*X}}(L_1, \mathcal{O}_X^r))$ is represented by the complex

$$E^*(K) := 0 \to (\Gamma(K; \mathcal{O}_X^r))^N_0 \xrightarrow{P_0} \cdots \xrightarrow{P_{r-1}} (\Gamma(K; \mathcal{O}_X^r))^N_r \to 0,$$
where \((\Gamma(K; \mathcal{O}_X^r))^{N_0}\) stands in degree 0. Since
\[
R\text{Hom}_{W_{T^*X}}(\mathcal{O}_X^r, \mathcal{W}_{T^*X} \cdot d) \simeq \Omega_X^r,
\]
the object \(R\text{Hom}_{W}(\mathcal{O}_X^r, L_1 [d])\) is represented by the complex
\[
0 \to (\Omega_X^r)^{N_r} \xrightarrow{-P_{r-1}} \cdots \xrightarrow{-P_0} (\Omega_X^r)^{N_0} \to 0,
\]
where \((\Omega_X^r)^{N_0}\) stands in degree 0. Hence, for a sufficiently small Stein open neighborhood \(U\) of \(x\), \(R\Gamma_c(U; R\text{Hom}_{W}(\mathcal{O}_X^r, L_1 [2d]))\) is represented by the complex
\[
F_c^\bullet(U) := 0 \to (H_c^d(U; \Omega_X^r))^{N_r} \xrightarrow{-P_{r-1}} \cdots \xrightarrow{-P_0} (H_c^d(U; \Omega_X^r))^{N_0} \to 0
\]
where \((H_c^d(U; \Omega_X^r))^{N_0}\) stands in degree 0.

Let \(U_n\) be the open ball of \(X\) centered at \(x\) with radius \(1/n\). By the hypothesis, all morphisms
\[
F_c^\bullet(U_p) \to F_c^\bullet(U_n)
\]
are quasi-isomorphisms for \(p \geq n \gg 0\), and the cohomologies are finite-dimensional over \(k\). Therefore, the hypotheses of Theorem 4.7 are satisfied by Lemma 5.6 and we get
\[
\text{“lim” Hom}_k(F_c^\bullet(U_n), k) \simeq \text{“lim” } (F_c^\bullet(U_n))^\vee \text{ in Ind}(K^b(\text{Mod}^\text{dfn}(k))).
\]
Applying Proposition 5.8, we obtain
\[
H^j(R\text{Hom}_{W}(L_1, \mathcal{O}_X^r)) \simeq \lim_n H^j(E_c^\bullet(U_n))
\]
\[
\simeq \lim_n H^j\text{Hom}_k(F_c^\bullet(U_n), k)
\]
\[
\simeq \lim_n (H^jR\Gamma_c(U; R\text{Hom}_{W}(\mathcal{O}_X^r, L_1)) [2d])^*.
\]
q.e.d.

**Corollary 6.2.** In the situation of Theorem 6.1 (ii), assume moreover that \(X\) is compact of complex dimension \(2n\). Then the \(k\)-vector spaces \(\text{Ext}^j_{W_X}(L_1, L_0)\) and \(\text{Ext}^{2n-j}_{W_X}(L_0, L_1)\) are finite-dimensional and dual to each other.
7 Statement of the main theorem

For two subsets \( V \) and \( S \) of the real manifold \( X \), the normal cone \( C(S,V) \) is well defined in \( TX \). If \( V \) is a smooth and closed submanifold of \( X \), one denotes by \( C_V(S) \) the image of \( C(S,V) \) in the normal bundle \( T_VX \).

Consider now a complex symplectic manifold \((\mathfrak{X},\omega)\). The 2-form \( \omega \) gives the Hamiltonian isomorphism \( H \) from the cotangent to the tangent bundle to \( \mathfrak{X} \):

\[
H : T^*\mathfrak{X} \xrightarrow{\sim} TX, \quad \langle \theta, v \rangle = \omega(v, H(\theta)), \quad v \in TX, \theta \in T^*\mathfrak{X}.
\]

(7.1)

For a smooth Lagrangian submanifold \( \Lambda \) of \( \mathfrak{X} \), the isomorphism (7.1) induces an isomorphism between the normal bundle to \( \Lambda \) in \( X \) and its cotangent bundle:

\[
T_\Lambda X \simeq T^*\Lambda.
\]

(7.2)

Let us recall a few notations and conventions (see [12]). For a complex manifold \( X \) and a complex analytic subvariety \( Z \) of \( X \), one denotes by \( Z_{\text{reg}} \) the smooth locus of \( Z \), a complex submanifold of \( X \). For a holomorphic \( p \)-form \( \theta \) on \( X \), one says that \( \theta \) vanishes on \( Z \) and one writes \( \theta|_Z = 0 \) if \( \theta|_{Z_{\text{reg}}} = 0 \).

Proposition 7.1. Let \( \mathfrak{X} \) be a complex symplectic manifold and let \( \Lambda_0 \) and \( \Lambda_1 \) be two closed complex analytic isotropic subvarieties of \( \mathfrak{X} \). Then, after identifying \( T\mathfrak{X} \) and \( T^*\mathfrak{X} \) by (7.1), the normal cone \( C(\Lambda_0,\Lambda_1) \) is a complex analytic \( \mathbb{C}^\times \)-conic isotropic subvariety of \( T^*\mathfrak{X} \).

Note that the same result holds for real analytic symplectic manifolds, replacing “complex analytic variety” with “subanalytic subset” and “\( \mathbb{C}^\times \)-conic” with “\( \mathbb{R}^+ \)-conic”.

First we need two lemmas.

Lemma 7.2. Let \( X \) be a complex manifold and \( \theta \) a \( p \)-form on \( X \). Let \( Z \subset Y \) be closed subvarieties of \( X \). If \( \theta|_Y = 0 \), then \( \theta|_Z = 0 \).

Proof. By Whitney’s theorem, we can find an open dense subset \( Z' \) of \( Z_{\text{reg}} \) such that

- For any sequence \( \{y_n\}_n \) in \( Y_{\text{reg}} \) such that it converges to a point \( z \in Z' \)
- \( \{T_{y_n}Y\}_n \) converges to a linear subspace \( \tau \subset T_zX \), \( \tau \) contains \( T_zZ' \).

Since \( \theta \) vanishes on \( T_{y_n}Y \), it vanishes also on \( \tau \) and hence on \( T_zZ' \). q.e.d.

25
Lemma 7.3. Let $X$ be a complex manifold, $Y$ a closed complex subvariety of $X$ and $f: X \to \mathbb{C}$ a holomorphic function. Set $Z := f^{-1}(0)$, $Y' := (Y \setminus Z) \cap Z$. Consider a $p$-form $\eta$, a $(p-1)$-form $\theta$ on $X$ and set

$$\omega = df \wedge \theta + f \eta.$$ 

Assume that $\omega|_{Y} = 0$. Then $\theta|_{Y'} = 0$ and $\eta|_{Y'} = 0$.

Proof. We may assume that $Y = (Y \setminus Z)$.

Using Hironaka’s desingularization theorem, we may find a smooth manifold $\tilde{Y}$ and a proper morphism $p: \tilde{Y} \to Y$ such that, in a neighborhood of each point of $Y$, $p^* f$ may be written in a local coordinate system $(y_1, \ldots, y_n)$ as a product $\prod_{i=1}^{n} y_i^{a_i}$, where the $a_i$’s are non-negative integers. Let $\tilde{Z} = p^{-1}(Z)$. Then $(\tilde{Y} \setminus \tilde{Z}) \cap \tilde{Z} \to Y'$ is proper and surjective.

Hence, we may assume from the beginning that $Y$ is smooth and then $Y = X$. Moreover, it is enough to prove the result at generic points of $Y'$. Hence, we may assume, setting $(y_1, \ldots, y_n) = (t, x)$ ($x = (y_2, \ldots, y_n)$), that $f(t, x) = t^a$ for some $a > 0$. Write

$$\theta = t \theta_0 + dt \wedge \theta_1 + \theta_2,$$

where $\theta_1$ and $\theta_2$ depend only on $x$ and $dx$.

By the hypothesis,

$$0 = df \wedge \theta + f \eta = at^a dt \wedge \theta_0 + at^{a-1} dt \wedge \theta_2 + t^a \eta.$$

It follows that $\theta_2$ is identically zero. Hence:

$$\theta = t \theta_0 + dt \wedge \theta_1, \quad \eta = -adt \wedge \theta_0.$$ 

Therefore, $\theta|_{t=0} = \eta|_{t=0} = 0$. q.e.d.

Proof of Proposition 7.1. Recall that $X^a$ denotes the complex manifold $X$ endowed with the symplectic form $-\omega$ and that $\Delta$ denotes the diagonal of $X \times X^a$. Using the isomorphisms

$$TX \simeq T_\Delta(X \times X^a),$$

$$C(\Lambda_0, \Lambda_1) \simeq C(\Delta, \Lambda_0 \times \Lambda_1^f),$$

(the first isomorphism is associated with the first projection on $X \times X^a$) we are reduced to prove the result when $\Lambda_0$ is smooth.
Let \((x, u)\) be a local symplectic coordinate system on \(X\) such that
\[
\Lambda_0 = \{(x; u) : u = 0\}, \quad \omega = \sum_{i=1}^n du_i \wedge dx_i.
\]

Consider the normal deformation \(\tilde{X}_{\Lambda_0}\) of \(X\) along \(\Lambda_0\). Recall that we have a diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{p} & \tilde{X}_{\Lambda_0} & \xrightarrow{t} & \mathbb{C} \\
\downarrow & & \downarrow & & \downarrow \\
\Lambda_1 & \xrightarrow{p} & p^{-1}\Lambda_1 & \xrightarrow{t} & \mathbb{C}
\end{array}
\]

such that, denoting by \((x; \xi, t)\) the coordinates on \(\tilde{X}_{\Lambda_0}\),
\[
p(x; \xi, t) = (x; t\xi),
\]
\[
T_{\Lambda_0}\mathcal{X} \simeq \left\{(x; \xi, t) \in \tilde{X}_{\Lambda_0} : t = 0\right\},
\]
\[
C_{\Lambda_0}(\Lambda_1) \simeq p^{-1}(\Lambda_1) \setminus t^{-1}(0) \cap t^{-1}(0).
\]

Clearly, \(C_{\Lambda_0}(\Lambda_1)\) is a complex analytic variety. Moreover,
\[
p^*\omega = \sum_{i=1}^n d(t\xi_i) \wedge dx_i = dt \wedge \left(\sum_{i=1}^n \xi_i \wedge dx_i\right) + t \sum_{i=1}^n d\xi_i \wedge dx_i.
\]

Since \(p^*\omega\) vanishes on \((p^{-1}\Lambda_1)_{\text{reg}}\), \(\sum_{i=1}^n d\xi_i \wedge dx_i\) vanishes on \(C_{\Lambda_0}(\Lambda_1)\) by Lemma 7.3. q.e.d.

**Theorem 7.4.** Let \(\mathcal{X}\) be a complex symplectic manifold and let \(\mathcal{L}_i\) \((i = 0, 1)\) be two objects of \(D_{\text{rh}}^b\left(W_{\mathcal{X}}\right)\) supported by smooth Lagrangian submanifolds \(\Lambda_i\). Then

(i) the object \(R\text{Hom}_W(\mathcal{L}_1, \mathcal{L}_0)\) belongs to \(D_{\mathbb{C}, \text{c}}^b(k_{\mathcal{X}})\) and its microsupport is contained in the normal cone \(C(\Lambda_0, \Lambda_1)\),

(ii) the natural morphism
\[
R\text{Hom}_W(\mathcal{L}_1, \mathcal{L}_0) \to D_{\mathcal{X}}(R\text{Hom}_W(\mathcal{L}_0, \mathcal{L}_1[\dim_{\mathbb{C}} \mathcal{X}]))
\]

is an isomorphism.

The proof of (i) will be given in § 8 and (ii) is a particular case of Theorem 6.1.
**Conjecture 7.5.** Theorem 7.4 remains true without assuming that the $\Lambda_i$’s are smooth.

Remark that the analogous of Conjecture 7.5 for complex contact manifolds is true over the field $\mathbb{C}$, as we shall see in § 9.

**Corollary 7.6.** Let $L_0$ and $L_1$ be two regular holonomic $\mathcal{W}_X$-modules supported by smooth Lagrangian submanifolds. Then the object $R\mathcal{H}om_{\mathcal{W}_X}(L_1, L_0)$ of $\mathbb{D}^b_{\mathbb{C}-c}(k_X)$ is perverse.

**Proof.** Since the problem is local, we may assume that $X = T^*X$, $\Lambda_0 = T^*_X X$ and $L_0 = \mathcal{O}_X$.

By Theorem 7.4 (ii), it is enough to check that if $Y$ is a locally closed smooth submanifold of $X$ of codimension $l$, then $H^j(\Gamma_Y(\mathcal{R}Hom_{\mathcal{W}_{T^*X}}(L_1, L_0)))|_Y$ vanishes for $j < l$. This follows from Lemma 5.9. q.e.d.

**Remark 7.7.** It would be interesting to compare $R\mathcal{H}om_{\mathcal{W}_X}(L_1, L_0)$ with the complexes obtained in [1].

**8 Proof of Theorem 7.4**

In this section $X$ denotes a complex manifold. As usual, $\mathcal{O}_X$ is the structure sheaf and $\mathcal{D}_X$ is the sheaf of rings of (finite–order) differential operators. For a coherent $\mathcal{D}_X$-module $M$, we denote by $\text{char}(M)$ its characteristic variety. This notion extends to the case where $M$ is a countable union of coherent sub-$\mathcal{D}_X$-modules. In this case, one sets

$$\text{char}(M) = \bigcup_{N \subset M} \text{char}(N)$$

where $N$ ranges over the family of coherent $\mathcal{D}_X$-submodules of $M$, and, for a subset $S$ of $T^*X$, $\overline{S}$ means the closure of $S$.

**Lemma 8.1.** Let $M_0$ be a coherent $\mathcal{D}_X$-module. Then

$$SS(R\mathcal{H}om_{\mathcal{D}_X}(M_0, \mathcal{O}_X^\tau(0))) \subset \text{char}(M_0).$$

**Proof.** Apply [19, Ch.3 Th. 3.2.1]. q.e.d.

**Lemma 8.2.** Let $M$ be a coherent $\mathcal{D}_X[\tau^{-1}]$-module. Then

$$SS(R\mathcal{H}om_{\mathcal{D}_X[\tau^{-1}]}(M, \mathcal{O}_X^\tau(0))) \subset \text{char}(M).$$
Proof. Let $M_0 \subset M$ be a coherent $\mathcal{D}_X$-submodule which generates $M$. Then

$$\text{char}(\mathcal{D}_X[\tau^{-1}] \otimes_{\mathcal{D}_X} M_0) = \text{char}(M_0) \subset \text{char}(M).$$

Consider the exact sequence of coherent $\mathcal{D}_X[\tau^{-1}]$-modules

$$(8.3) \quad 0 \to N \to \mathcal{D}_X[\tau^{-1}] \otimes_{\mathcal{D}_X} M_0 \to M \to 0.$$ 

Applying the functor $R\mathcal{H}om_{\mathcal{D}_X[\tau^{-1}]}(\text{•}, \mathcal{O}_X^\tau)$ to the exact sequence (8.3), we get a distinguished triangle $G' \to G \to G'' \to 1$. Note that $G \simeq R\mathcal{H}om_{\mathcal{D}_X}(M_0, \mathcal{O}_X^\tau)$, since $\mathcal{D}_X[\tau^{-1}]$ is flat over $\mathcal{D}_X$.

Let $\theta = (x_0; p_0) \in T^*X$ with $\theta \notin \text{char}(M)$ and let $\psi$ be a real function on $X$ such that $\psi(x_0) = 0$ and $d\psi(x_0) = p_0$. Consider the distinguished triangle

$$(8.4) \quad (R\Gamma_{\psi \geq 0}(G'))_{x_0} \to (R\Gamma_{\psi \geq 0}(G))_{x_0} \to (R\Gamma_{\psi \geq 0}(G''))_{x_0} \to 1.$$ 

By Lemma 8.1, we have:

$$H^j((R\Gamma_{\psi \geq 0}(G))_{x_0}) \simeq 0$$

for all $j \in \mathbb{Z}$. The objects of the distinguished triangle (8.4) are concentrated in degree $\geq 0$. Therefore, $H^j((R\Gamma_{\psi \geq 0}(G'))_{x_0}) \simeq 0$ for $j \leq 0$.

Since $\text{char}(N) \subset \text{char}(\mathcal{D}_X[\tau^{-1}] \otimes_{\mathcal{D}_X} M_0)$, we get $H^j((R\Gamma_{\psi \geq 0}(G''))_{x_0}) \simeq 0$ for $j \leq 0$. By repeating this argument, we deduce that $H^j((R\Gamma_{\psi \geq 0}(G'))_{x_0}) \simeq H^{j-1}((R\Gamma_{\psi \geq 0}(G''))_{x_0}) \simeq 0$ for all $j \in \mathbb{Z}$.

Note that the statement of Theorem 7.4 is local and invariant by quantized symplectic transformation. From now on, we denote by $(x; u)$ a local symplectic coordinate system on $X$ such that

$$X := \Lambda_0 = \{(x; u) \in X ; u = 0\}.$$ 

We denote by $(x; \xi)$ the associated homogeneous symplectic coordinates on $T^*X$. The differential operator $\partial_{x_i}$ on $X$ has order 1 and principal symbol $\xi_i$. The monomorphism (1.7) extends as a monomorphism of rings

$$\mathcal{D}_X[\tau^{-1}, \tau] \hookrightarrow \mathcal{W}_{T^*X}.$$ 

Note that the total symbol of the operator $\partial_{x_i}$ of $\mathcal{W}_{T^*X}$ is $u_i \cdot \tau$. 

29
We may assume that there exists a holomorphic function \( \varphi : X \to \mathbb{C} \) such that
\[
\Lambda_1 = \{(x; u) \in X : u = \operatorname{grad} \varphi(x)\}.
\]
Here, \( \operatorname{grad} \varphi = (\varphi'_1, \ldots, \varphi'_n) \) and \( \varphi'_i = \frac{\partial \varphi}{\partial x_i} \).

If \( \Lambda_0 = \Lambda_1 \), Theorem 7.4 is immediate. We shall assume that \( \Lambda_0 \neq \Lambda_1 \) and thus that \( \varphi \) is not a constant function. We may assume that
\[
\begin{cases}
\mathcal{L}_0 = \mathcal{O}_X, \\
\mathcal{L}_1 = \mathcal{W}_{T^*X}/\mathcal{I}_1, \mathcal{I}_1 \text{ being the ideal generated by } \{\partial x_i - \varphi'_i \tau\}_{i=1,\ldots,n}.
\end{cases}
\]
(8.6)

To \( \varphi : X \to \mathbb{C} \) are associated the maps
\[
T^*X \xrightarrow{\varphi_*} X \times \mathbb{C} \xrightarrow{T^*\varphi} T^*\mathbb{C},
\]
and the \((\mathcal{D}_X, \mathcal{D}_\mathbb{C})\)-bimodule \( \mathcal{D}_X \varphi_* \mathbb{C} \). Let \( t \) be the coordinate on \( \mathbb{C} \). By identifying \( \partial_t \) and \( \tau \), we regard \( \mathcal{D}_X \varphi_* \mathbb{C} \) as a \( \mathcal{D}_X[\tau] \)-module. We set
\[
V = \overline{\text{Im} \varphi_*}, \text{ the closure of } \varphi_* (T^* \mathbb{C} \times \mathbb{C} X).
\]

Lemma 8.3. Regarding \( \mathcal{D}_X \varphi_* \mathbb{C} \) as a \( \mathcal{D}_X \)-module, one has \( \text{char}(\mathcal{D}_X \varphi_* \mathbb{C}) \subset V \).

Note that \( \mathcal{D}_X \varphi_* \mathbb{C} \) is not a coherent \( \mathcal{D}_X \)-module in general.

Proof. Let \( \Gamma_\varphi \) be the graph of \( \varphi \) in \( X \times \mathbb{C} \) and let \( \mathcal{B}_\varphi \) be the coherent left \( \mathcal{D}_X \times \mathbb{C} \)-module associated to this submanifold. (Note that \( \text{char}(\mathcal{B}_\varphi) = \Lambda_\varphi \), the conormal bundle to \( \Gamma_\varphi \) in \( T^*(X \times \mathbb{C}) \).) We shall identify \( \Gamma_\varphi \) with \( X \) by the first projection on \( X \times \mathbb{C} \) and the left \( \mathcal{D}_X \)-modules \( \mathcal{B}_\varphi \) with \( \mathcal{D}_X \varphi_* \mathbb{C} \).

Denote by \( \delta (t - \varphi) \) the canonical generator of \( \mathcal{B}_\varphi \). Then
\[
\mathcal{D}_X \varphi_* \mathbb{C} = \mathcal{D}_X \times \mathbb{C} \cdot \delta (t - \varphi) = \sum_{n \in \mathbb{N}} \mathcal{D}_X \partial_t^n \cdot \delta (t - \varphi).
\]

Define
\[
\mathcal{N}_\varphi := \sum_{n \in \mathbb{N}} \mathcal{D}_X (t \partial_t)^n \cdot \delta (t - \varphi).
\]

By [9, Th. 6.8], the \( \mathcal{D}_X \)-module \( \mathcal{N}_\varphi \) is coherent and its characteristic variety is contained in \( V \). All \( \mathcal{D}_X \)-submodules \( \partial_t^n \mathcal{N}_\varphi \) of \( \mathcal{D}_X \varphi_* \mathbb{C} \) are isomorphic since \( \partial_t \) is injective on \( \mathcal{B}_\varphi \). Then the result follows from
\[
\mathcal{D}_X \varphi_* \mathbb{C} = \sum_{n \in \mathbb{N}} \partial_t^n \mathcal{N}_\varphi.
\]
q.e.d.
We consider the left ideal and the modules:

\[
\begin{align*}
\mathcal{I} &:= \text{the left ideal of } \mathcal{D}_X[\tau^{-1}, \tau] \text{ generated by } \{\partial_{x_i} - \varphi_i' \tau\}_{i=1,\ldots,n}, \\
\mathcal{M} & = \mathcal{D}_X[\tau^{-1}]/(\mathcal{I} \cap \mathcal{D}_X[\tau^{-1}]), \\
\mathcal{M}_0 & = \mathcal{D}_X/(\mathcal{I} \cap \mathcal{D}_X).
\end{align*}
\]

(8.7)

**Lemma 8.4.**

(i) \(\mathcal{M}\) is \(\mathcal{D}_X[\tau^{-1}]-coherent\),

(ii) \(\mathcal{M}_0\) is \(\mathcal{D}_X\)-coherent,

(iii) we have an isomorphism of \(\mathcal{D}_X\)-modules \(\mathcal{D}_X \cdot \delta(t - \varphi) \simeq \mathcal{M}_0\), where \(\mathcal{D}_X \cdot \delta(t - \varphi)\) is the \(\mathcal{D}_X\)-submodule of \(\mathcal{D}_X[\tau^{-1}]\), generated by \(\delta(t - \varphi)\),

(iv) \(\text{char}(\mathcal{M}) = \text{char}(\mathcal{M}_0) \subset V\).

**Proof.**

(i) follows from Lemma 1.6 (ii).

(ii) follows from Lemma 1.5 (iii).

(iii) Clearly, \(\mathcal{D}_X \cdot \delta(t - \varphi) \simeq \mathcal{D}_X/\mathcal{D}_X \cap \mathcal{I}\).

(iv) (a) By (iii) and Lemma 8.3 we get the inclusion \(\text{char}(\mathcal{M}_0) \subset V\).

(iv) (b) Since \(\mathcal{M}_0 \subset \mathcal{M}\), the inclusion \(\text{char}(\mathcal{M}_0) \subset \text{char}(\mathcal{M})\) is obvious.

(iv) (c) Denote by \(u\) the canonical generator of \(\mathcal{M}\). Then \(\mathcal{M} = \bigcup_{n \leq 0} \mathcal{D}_X \tau^n u\).

Since there are epimorphisms \(\mathcal{M}_0 \rightarrow \mathcal{D}_X \tau^n u\), the inclusion \(\text{char}(\mathcal{M}) \subset \text{char}(\mathcal{M}_0)\) follows.

\[\text{q.e.d.}\]

Set

\[
F_0 = R\text{Hom}_{\mathcal{D}_X[\tau^{-1}]}(\mathcal{M}, \mathcal{O}_X^\tau(0)),
\]

where the module \(\mathcal{M}\) is defined in (8.7).

**Lemma 8.5.** \(SS(F_0)\) is a closed \(\mathbb{C}^\times\)-conic complex analytic Lagrangian subset of \(T^*X\) contained in \(C(\Lambda_0, \Lambda_1)\).

**Proof.** By Lemmas 8.2 and 8.4, \(SS(F_0) \subset V \cap (\pi^{-1}(\Lambda_0 \cap \Lambda_1))\), and one immediately checks that

\[
V \cap \pi^{-1}(\Lambda_0 \cap \Lambda_1) = C(\Lambda_0, \Lambda_1).
\]

(8.9)

Since \(SS(F_0)\) is involutive by [12] and is contained in a \(\mathbb{C}^\times\)-conic analytic isotropic subset by Proposition 7.1, it is a closed \(\mathbb{C}^\times\)-conic complex analytic Lagrangian subset of \(T^*\Lambda_0\) by [12, Prop. 8.3.13].

\[\text{q.e.d.}\]

**Lemma 8.6.** Let \(F_0\) be as in (8.8). Then for each \(x \in X\) and each \(j \in \mathbb{Z}\), the \(k_0\)-module \(H^j(F_0)_x\) is finitely generated.

31
Proof. Let $x_0 \in X$ and choose a local coordinate system around $x_0$. Denote by $B(x_0; \varepsilon)$ the closed ball of center $x_0$ and radius $\varepsilon$. By a result of [7] (see also [12, Prop. 8.3.12]), we deduce from Lemma 8.5 that the natural morphisms

$$R\Gamma(B(x_0; \varepsilon_1); F_0) \to R\Gamma(B(x_0; \varepsilon_0); F_0)$$

are isomorphisms for $0 < \varepsilon_0 \leq \varepsilon_1 \ll 1$.

We represent $F_0$ by a complex:

$$0 \to (\mathcal{O}^\tau(0))^{N_0} \xrightarrow{d_0} \cdots \to (\mathcal{O}^\tau(0))^{N_n} \to 0,$$

(8.10)

where the differentials are $k_0$-linear. It follows from Lemma 5.5 and Theorem 4.3 that the cohomology objects $H^j(F_0)_{x_0}$ are finitely generated.

q.e.d.

End of the proof of Theorem 7.4. As already mentioned, part (ii) is a particular case of Theorem 6.1.

Let us prove part (i). By “dévissage” we may assume that $\mathcal{L}_0$ and $\mathcal{L}_1$ are concentrated in degree 0. We may also assume that $\mathcal{L}_0$ and $\mathcal{L}_1$ are as in (8.6).

Let $F_0$ be as in (8.8), and set $F = R\text{Hom}_{\mathcal{W}_{T^*X}}(\mathcal{L}_1, \mathcal{O}^r_X)$. Since $\mathcal{L}_1 \simeq \mathcal{W}_{T^*X} \otimes_{\mathcal{D}_X[\tau-1]} \mathcal{M}$, we obtain

$$F \simeq F_0 \otimes_{k_0} k$$

by Lemmas 1.6 and 1.7.

Hence we have $SS(F) \subset SS(F_0)$, and the weak constructibility as well as the estimate of $SS(F)$ follows by Lemma 8.2 and Lemma 8.5. The finiteness result follows from Lemma 8.6.

q.e.d.

9 Serre functors on contact and symplectic manifolds

In the definition below, $\mathbb{K}$ is a field and $^*$ denotes the duality functor for $\mathbb{K}$-vector spaces.

Definition 9.1. Consider a $\mathbb{K}$-triangulated category $T$.

(i) The category $T$ is Ext-finite if for any $L_0, L_1 \in T$, $\text{Ext}^j_T(L_1, L_0)$ is finite-dimensional over $\mathbb{K}$ for all $j \in \mathbb{Z}$ and is zero for $|j| \gg 0$.
(ii) Assume that $\mathcal{T}$ is Ext-finite. A Serre functor (see [2]) $S$ on $\mathcal{T}$ is an equivalence of $\mathbb{K}$-triangulated categories $S: \mathcal{T} \leftrightarrow \mathcal{T}$ satisfying
\[
(Hom_{\mathcal{T}}(L_1, L_0))^* \simeq Hom_{\mathcal{T}}(L_0, S(L_1))
\]
functorially in $L_0, L_1 \in \mathcal{T}$.

(iii) If moreover there exists an integer $d$ such that $S$ is isomorphic to the shift by $d$, then one says that $\mathcal{T}$ is a $\mathbb{K}$-triangulated Calabi-Yau category of dimension $d$.

Let $\mathcal{Y}$ be a complex contact manifold. The algebroid stack $\mathcal{E}_\mathcal{Y}$ of microdifferential operators on $\mathcal{Y}$ has been constructed in [8] and the triangulated categories $D_{\text{coh}}^b(\mathcal{E}_\mathcal{Y}), D_{\text{hol}}^b(\mathcal{E}_\mathcal{Y})$ and $D_{\text{rh}}^b(\mathcal{E}_\mathcal{Y})$ are naturally defined.

**Theorem 9.2.** For a complex contact manifold $\mathcal{Y}$, we have

(i) for $M$ and $N$ in $D_{\text{rh}}^b(\mathcal{E}_\mathcal{Y})$, the object $F = R\text{Hom}_{\mathcal{E}_\mathcal{Y}}(M, N)$ belongs to $D^b_{\mathcal{C}-c}(\mathcal{C}_\mathcal{Y})$,

(ii) if $\mathcal{Y}$ is compact, then $D_{\text{rh}}^b(\mathcal{E}_\mathcal{Y})$ is a Calabi-Yau $\mathcal{C}$-triangulated category of dimension $\dim_{\mathcal{C}} \mathcal{Y} - 1$.

Sketch of proof. (i) is well-known and follows from [10] (see [21] for further developments). The idea of the proof is as follows. The assertion being local and invariant by quantized contact transformations, we may assume that $\mathcal{Y}$ is an open subset of the projective cotangent bundle $P^*Y$ to a complex manifold. Then, using the diagonal procedure, we reduce to the case $F = R\text{Hom}_{\mathcal{D}_Y}(M, \mathcal{C}_Z|_Y)$, where $M$ is a regular holonomic $\mathcal{D}_Y$-module and $\mathcal{C}_Z|_Y$ is the $\mathcal{E}_{P^*Y}$-module associated to a complex hypersurface $Z$ of $Y$.

(ii) follows from (i) as in the proof of Theorem 6.1. q.e.d.

**Remark 9.3.** (i) If Conjecture 7.5 is true, that is, if Theorem 7.4 holds for any Lagrangian varieties, then, for any compact complex symplectic manifold $X$, the $\mathbb{K}$-triangulated category $D_{\text{rh}}^b(\mathcal{W}_X)$ is a Calabi-Yau $\mathbb{K}$-triangulated category of dimension $\dim_{\mathbb{K}} X$. Note that this result is true when replacing the notion of regular holonomic module by the notion of good modules, i.e., coherent modules admitting globally defined $\mathcal{W}_X(0)$-submodules which generate them. This follows from a theorem of Schapira-Schneider to appear.

(ii) Note that Proposition 1.4.8 of [10] is not true, but this proposition is not used in loc. cit. Indeed, for a compact complex manifold $X$, if $\mathcal{T} := D_{\text{rh}}^b(\mathcal{D}_X)$ denotes the full triangulated subcategory of $D^b(\mathcal{D}_X)$ consisting
of objects with regular holonomic cohomologies, it is well known that the duality functor is not a Serre functor on $T$.

(iii) It may be interesting to notice that the duality functor is not a Serre functor in the setting of regular holonomic $\mathcal{D}$-modules, but is a Serre functor in the microlocal setting, that is, for regular holonomic $\mathcal{E}$-modules. This may be compared to [12, Prop. 8.4.14].

References

http://www.math.nyu.edu/~tschinke/manin/manin-index.html


Masaki Kashiwara  
Research Institute for Mathematical Sciences  
Kyoto University  
Kyoto, 606-8502, Japan  
e-mail: masaki@kurims.kyoto-u.ac.jp

Pierre Schapira  
Institut de Mathématiques  
Université Pierre et Marie Curie  
175, rue du Chevaleret, 75013 Paris,  
France  
e-mail: schapira@math.jussieu.fr  
http://www.math.jussieu.fr/~schapira/