Cohn localization of finite group rings.

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Abstract. The purpose of this paper is to give a complete description of the Cohn localization of the augmentation map $\mathbf{Z}[G] \to \mathbf{Z}$ when G is any finite group.

Keywords: Non-commutative localization, category of chain complexes.

Introduction. If A is a ring, denote by $M(A)$ the set of matrices with entries in A. Let $A \to B$ be a ring homomorphism. The Cohn localization $\Lambda = L(A \to B)$ is a ring obtained by formally inverting all matrices in $M(A)$ that become invertible in $M(B)$. The morphism $A \to B$ factors through Λ in a unique way. The ring Λ plays an important role in the theory of homological surgery but it is very difficult to determine in general.

If A is commutative, Λ is obtained by inverting all elements in the inverse image of B^*. In the general case Λ is much more complicated, in particular if $A \to B$ is the augmentation map of a group ring $\mathbf{Z}[G]$. In this case, the ring Λ is known if G is commutative or free [FV]. But in the other cases almost nothing is known.

The first result of this paper is a complete description of $L(\mathbf{Z}[G] \to \mathbf{Z})$ when G is a finite group.

Theorem A. *Let G be a finite group. For every prime p, denote by G_p the quotient of G by the normal closure of all q-Sylow subgroups of G with $q \neq p$ and by $f_p : \mathbf{Z}_{(p)}[G_p] \to \mathbf{Z}_{(p)}$ the corresponding augmentation map. Then the Cohn localization $\Lambda = L(\mathbf{Z}[G] \to \mathbf{Z})$ is given by the following pull-back diagram:*

$$\begin{array}{ccc} \Lambda & \longrightarrow & \prod_p \mathbf{Z}_{(p)}[G_p] \\ \Big\downarrow & & \Big\downarrow \prod_p f_p \\ \mathbf{Z} & \xrightarrow{\Delta} & \prod_p \mathbf{Z}_{(p)} \end{array}$$

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where Δ is the diagonal inclusion and the product is over all non-trivial G_p.

Consider a ring homomorphism $A \to B$. We'll say that B is a central localization (resp. a Ore localization) of A if B is the ring $S^{-1}A$ where S is a multiplicative set in the center of A (resp. a multiplicative set in A satisfying the Ore condition).

We say that B is stably flat over A if the two conditions hold:

- the multiplication map: $B \underset{A}{\otimes} B \to B$ is an isomorphism
- $\mathrm{Tor}_i^A(B, B) = 0$ for all $i > 0$.

The second result is the following:

Theorem B. *Let G be a finite group and Λ be the Cohn localization $L(\mathbf{Z}[G] \to \mathbf{Z})$. Then the following conditions are equivalent:*

1) Λ is a central localization of $\mathbf{Z}[G]$
2) Λ is a Ore localization of $\mathbf{Z}[G]$
3) Λ is a flat left $\mathbf{Z}[G]$-module
4) Λ is stably flat over $\mathbf{Z}[G]$
5) G is nilpotent.

1. Localization of complexes.

Throughout this section the ring homomorphism $f : A \to B$ is supposed to be surjective. The set of matrices in $M(A)$ sent by f to an invertible matrix in $M(B)$ will be denoted by W. The Cohn localization $L(A \to B)$ will be denoted by Λ.

Let R be a ring. The graded differential right R-modules which are projective and bounded from below define a category denoted by $\mathscr{C}_*(R)$. The objects in $\mathscr{C}_*(R)$ are called complexes (or R-complexes if needed). A morphism in $\mathscr{C}_*(R)$ is a linear map of some degree which commutes with the differentials (in the graded sense). A morphism in $\mathscr{C}_*(R)$ is a cofibration if it is injective with cokernel in $\mathscr{C}_*(R)$. A complex C is said to be finite if it is free and finitely generated.

Let $\mathscr{W}_0$ be the class of finite complexes $C \in \mathscr{C}_*(A)$ such that $C \otimes B$ is acyclic. Notice that a finite complex C of length 2 belongs to $\mathscr{W}_0$ if and only if the matrix of the only non zero differential of C lies in W. The class of complexes of lenght 2 in $\mathscr{W}_0$ will be denoted by W_1.

A complex $C \in \mathscr{C}_*(A)$ is said to be local if every morphism from a complex in $\mathscr{W}_0$ to C is null-homotopic. The class of local complexes in $\mathscr{C}_*(A)$ will be denoted by $\mathscr{L}$.

The class $\mathscr{W}_0$ is contained in a bigger class $\mathscr{W}$ defined as follows:

A complex $C \in \mathscr{C}_*(A)$ lies in $\mathscr{W}$ if and only if every morphism from C to a local complex is null-homotopic.

A morphism between two complexes is called a $\mathscr{W}-$equivalence if its mapping cone lies in $\mathscr{W}$.

We have also the notion of local right A-module: a right A-module M is said to be local if, for every morphism $f : L \to L'$ between finitely generated free A-modules with matrix in W, the map f^* from $\mathrm{Hom}(L', M)$ to $\mathrm{Hom}(L, M)$ is bijective. Using technics of [V], section 5 (see also [NR]), we get the following results:

1.1 Lemma: *A complex $C \in \mathscr{C}_*(A)$ lies in $\mathscr{W}$ if and only if every morphism from a finite complex to C factors through a complex in $\mathscr{W}_0$.*

1.2 Lemma: *$\mathscr{W}_0$ is the class of finite complexes in $\mathscr{W}$.*

1.3 Lemma: *Let C be a complex in $\mathscr{C}_*(A)$. Then we have:*

C lies in $\mathscr{L}$ if and only if every morphism from a complex in $\mathscr{W}$ to C is null-homotopic.

C lies in $\mathscr{W}$ if and only if every morphism from C to a complex in $\mathscr{L}$ is null-homotopic.

1.4 Lemma: *Let C be a finite complex. Then C belongs to $\mathscr{W}_0$ is and only if there exist a sequence of finite complexes:*

$$0 = C_0 \subset C_1 \subset \cdots \subset C_p$$

such that each C_{i+1}/C_i lies in W_1, and a homotopy equivalence between C_p and C.

1.5 Lemma: *A right A-module M is local if and only if the morphism $M \to M \otimes_A \Lambda$ is an isomorphism.*

1.6 Lemma: *Let M be a local right A-module. Then $\mathrm{Tor}_1^A(M, \Lambda)$ is trivial.*

1.7 Lemma: *A complex C is local if and only if its homology is local.*

1.8 Theorem: *There exist a functor Φ from $\mathscr{C}_*(A)$ to itself and a morphism λ from the identity of $\mathscr{C}_*(A)$ to Φ with the following properties:*

• The functor Φ sends cofibration to cofibration.

• For every complex C, $\Phi(C)$ is local and the morphism $\lambda_C : C \to \Phi(C)$ is a cofibration of degree 0 and a $\mathscr{W}-$equivalence.

• The functor Φ is homotopy exact. That is, if $\oplus_i C_i$ is a complex in $\mathscr{C}_*(A)$, the morphism $\oplus_i \Phi(C_i) \to \Phi(\oplus_i C_i)$ is a homotopy equivalence and if

$$0 \longrightarrow C \xrightarrow{i} C' \xrightarrow{j} C'' \longrightarrow 0$$

is a short exact sequence in $\mathscr{C}_*(A)$, then the map j induces a homotopy equivalence from $\Phi(C')/i_*(\Phi(C))$ to $\Phi(C'')$.

• For each complex C, the map $\lambda_C : C \to \Phi(C)$ is a homotopy equivalence if and only if C is local.

• If C is a $(p-1)$-connected complex, $\Phi(C)$ is $(p-1)$-connected and the map λ_C induces an isomorphism from $\mathrm{H}_p(C) \otimes_A \Lambda$ to $\mathrm{H}_p(\Phi(C))$.

Sketch of proof: Let's take a numbering of W_1: $W_1 = \{K_i\}$.

Let C be a complex. Denote by $E(C)$ the set of pairs (i, α) where α is a non zero morphism of degree 0 from K_i to C. The mapping cone of the map:

$$\oplus K_i \xrightarrow{\oplus\alpha} C$$

where the direct sum is over all $(i,\alpha) \in E(C)$, will be denoted by $\Phi_1(C)$. It is easy to see that, if C_i vanishes for $i < p$, the module $\Phi_1(C)_i$ vanishes also for $i < p$. Then $\Phi_1(C)$ is a well-defined complex in $\mathscr{C}_*(A)$. Moreover Φ_1 is a functor respecting the cofibrations.

The inclusion $C \subset \Phi_1(C)$ is a cofibration of degree 0 and every morphism from a complex in W_1 to C is null-homotopic in $\Phi_1(C)$. By iterating this construction, we get an infinite sequence:

$$C \to \Phi_1(C) \to \Phi_2(C) \to \Phi_3(C) \to \cdots$$

where all maps are cofibrations of degree 0. So $\Phi(C)$ is defined as the colimit of this sequence and $\Phi(C)$ is a well-defined complex in $\mathscr{C}_*(A)$.

It is clear that $\Phi_1(C)/C$ is a direct sum of complexes in W_1. Therefore $\Phi_1(C)/C$ and then $\Phi(C)/C$ lies in $\mathscr{W}$ and the morphism $\lambda_C : C \to \Phi(C)$ is a $\mathscr{W}-$equivalence of degree 0.

Denote by $\mathscr{W}'$ the class of complexes K such that every morphism from K to $\Phi(C)$ is null-homotopic. Let K be a complex in W_1 and f be a morphism from K to $\Phi(C)$. Since K is finitely generated $f(C')$ is contained in some $\Phi_n(C)$ and the map $K \to \Phi_n(C)$ is null homotopic in $\Phi_{n+1}(C)$. So every morphism from K to $\Phi(C)$ is null-homotopic and $\mathscr{W}'$ contains W_1. Thus, because of lemma 1.4, $\mathscr{W}'$ contains $\mathscr{W}_0$ and $\Phi(C)$ is local.

The other properties are easy to check. □

The functor Φ is universal in some sense. More precisely we have the following proposition which is easy to see:

1.9 Proposition: *Let $f : C \to C'$ be a cofibration of complexes. Then we have the following properties:*

Suppose f is a $\mathscr{W}-$equivalence. Then there is a morphism $g : C' \to \Phi(C)$, unique up to homotopy, such that: $\lambda_C = g \circ f$.

Suppose C' lies in $\mathscr{L}$. Then there is a morphism $g : \Phi(C) \to C$, unique up to homotopy, such thet: $f = g \circ \lambda_C$.

Consider the complex $E = \Phi(A)$, where A is considered as a complex concentrated in degree 0. The module $H_i(E)$ will be denoted by Λ_i.

1.10 Theorem: *The graded module Λ_* is a graded ring. The module Λ_i vanishes for $i < 0$ and the subring Λ_0 is the ring Λ.*

For each local complex C, $\mathrm{H}_*(C)$ is a graded right Λ_*-module.

For each complex C, there is a spectral sequence converging to $H_*(\Phi(C))$, where the E^2-term is:

$$E^2_{pq} = \mathrm{H}_p(C \underset{A}{\otimes} \Lambda_q)$$

and this spectral sequence is compatible with the action of Λ_*.

Proof: Since $E = \Phi(A)$ is local each Λ_i ia a right Λ-module. The last property of theorem 1.8 implies that Λ_i vanishes for $i < 0$ and that $\Lambda_0 = \Lambda$ as a right Λ-module.

Let a be a cycle in E. We have a morphism f_a from A to E sending u to au. This morphism induces a morphism $\widehat{f}_a$ from $\Phi(A)$ to $\Phi(E)$. But the map $\lambda_E : E \to \Phi(E)$ is a homotopy equivalence. So, by taking a homotopy inverse of this map, we get a morphism (still denoted by $\widehat{f}_a$) from E to E. This morphism is well defined up to homotopy and its homotopy class depends only on the homology class of a. Let b be a cycle in E. We set: $a.b = \widehat{f}_a(b)$. The homology class of $a.b$ depends only on the homology classes of a and b. So we have a product in Λ_* defined by: $[a].[b] = [a.b]$ where $[u]$ is the homology class of a cycle u. It is easy to see that this product is compatible with the degree on Λ_* and with the ring structure on $\Lambda_0 = \Lambda$.

The correspondance $a \mapsto \widehat{f}_a$ is $\mathbf{Z}$-linear up to homotopy and the product in Λ_* is bilinear over $\mathbf{Z}$. Consider three cycles a, b, c in E. We have:

$$a.(b.c) \sim \widehat{f}_a(b.c) \sim \widehat{f}_a\Big(\widehat{f}_b(c)\Big) \sim \widehat{f}_a \circ \widehat{f}_b(c) \sim \Big(\widehat{f}_a \circ \widehat{f}_b(1)\Big).c$$

$$\sim \Big(\widehat{f}_a(b)\Big).c \sim (a.b).c$$

and the product in Λ_* is associative.

Let C be a local complex. If c is a cycle in C, we define the morphism f_c from A to C and the morphism $\widehat{f}_c$ from $\Phi(A)$ to C as above. So the action of Λ_* on $\mathrm{H}_*(C)$ is defined by: $[c].[a] = [\widehat{f}_c(a)]$. It is easy to see that this action induces a structure of graded right Λ_*-module on $\mathrm{H}_*(C)$.

Consider a complex $C = C_*$. For each integer i, denote by K_i the i-skeletton of C. We have a sequence of cofibrations:

$$\ldots \subset \Phi(K_{i-1}) \subset \Phi(K_i) \subset \Phi(K_{i+1}) \subset \ldots$$

inducing a filtration of some subcomplex $U \subset \Phi(C)$. Since $\Phi(C)/\Phi(K_p) \sim \Phi(C/K_p)$ is p-connected, the inclusion $U \subset \Phi(C)$ is a homotopy equivalence. So we get a spectral sequence converging to the homology of $\Phi(C)$ with E^1-term:

$$E^1_{pq} = \mathrm{H}_{p+q}(\Phi(K_p)/\Phi(K_{p-1})) \simeq \mathrm{H}_{p+q}(\Phi(K_p/K_{p-1})) = \mathrm{H}_{p+q}(\Phi(C_p)) \simeq C_p \underset{A}{\otimes} \Lambda_q$$

Moreover the term E^1_{p*} is a right Λ_*-module and the result follows. □

1.11 Corollary: *Let $p > 0$ be an integer. Suppose Λ_i vanishes for $0 < i < p$. Then $Tor_i^A(\Lambda, \Lambda)$ vanishes for $0 < i \leq p$ and Λ_p is isomorphic to $Tor_{p+1}^A(\Lambda, \Lambda)$ as a Λ-bimodule.*

Proof: Consider a projective resolution C of the right A-module Λ. Since C is local $\Lambda_C : C \to \Phi(C)$ is a homotopy equivalence and the spectral sequence of theorem 1.10 implies the desired result. □

1.12 Remark: Because of this corollary, Λ_i vanish for all $i > 0$ if and only if Λ is stably flat over A. An example of non stably flat localization rings was found by Schofield [S] but theorem B implies a lot of explicit other examples.

1.13 Proposition: *Let C be a complex in $\mathscr{C}_*(A)$ and p be an integer. Suppose C is $(p-1)$-connected, $\mathrm{H}_p(C)$ is finitely generated and $\mathrm{H}_p(C \otimes_A B) = 0$. Then $\mathrm{H}_i(C \otimes_A \Lambda)$ vanishes for all $i \leq p$.*

Proof: Because of these properties, C is homotopy equivalent to a complex C' such that: $C'_i = 0$ for $i < p$ and C'_p is a finitely generated free A-module. Moreover the map $d \otimes B : C'_{p+1} \otimes B \to C'_p \otimes B$ is surjective. Since the morphism $A \to B$ is surjective there is a morphism α from C'_p to C'_{p+1} such that $(d \circ \alpha) \otimes B$ is the identity. Consider the following commutative diagram:

$$\begin{array}{ccc} C'_p & \xrightarrow{\alpha} & C'_{p+1} \\ {\scriptstyle d\circ\alpha}\downarrow & & \downarrow{\scriptstyle d} \\ C'_p & \xrightarrow{1} & C'_p \end{array}$$

The vertical map on the left defines a finite complex C'' concentrated in degrees p and $p+1$ and the horizontal maps induce a morphism from C'' to C'. Since $d \circ \alpha$ is the identity after tensorization by B, C'' is a complex in W_1 and $C'' \otimes \Lambda$ is acyclic. On the other hand $C'' \to C'$ is surjective in degree p. So the map from $\mathrm{H}_p(C'' \otimes \Lambda)$ to $\mathrm{H}_p(C' \otimes \Lambda)$ is trivial and surjective. The result follows. □

1.14 Corollary: *Let M be a finitely generated right A-module. Suppose $M \otimes_A B$ is the trivial module. Then $M \otimes_A \Lambda$ is trivial too.*

1.15 Proposition: *Let G be a group and H be the subgroup generated by the finitely generated perfect subgroups of G. Then the canonical map from $L(\mathbf{Z}[G] \to \mathbf{Z})$ to $L(\mathbf{Z}[G/H] \to \mathbf{Z})$ is an isomorphism.*

Proof: Let Γ be a finitely generated perfect group. Denote by Λ' the localization ring $L(\mathbf{Z}[\Gamma] \to \mathbf{Z})$ and by M the kernel of the augmentation map $\mathbf{Z}[\Gamma] \to \mathbf{Z}$. It is easy to see that M is a finitely generated right $\mathbf{Z}[\Gamma]$-module and $M \otimes \mathbf{Z}$ is trivial. Because of corollary 1.14, the module $M \otimes \Lambda'$ is trivial too and the exact sequence:

$$M \longrightarrow \mathbf{Z}[\Gamma] \longrightarrow \mathbf{Z} \longrightarrow 0$$

induces an isomorphism:

$$\mathbf{Z}[\Gamma] \otimes \Lambda' \xrightarrow{\sim} \mathbf{Z} \otimes \Lambda'$$

Then, for every $\gamma \in \Gamma$, $\gamma - 1$ vanishes in Λ' and we have:

$$\Lambda' \simeq L(\mathbf{Z} \to \mathbf{Z}) = \mathbf{Z}$$

Let γ be an element in H. This element is contained in a finitely generated perfect subgroup $\Gamma \subset H$. We have a commutative diagram:

$$\begin{array}{ccc} L(\mathbf{Z}[\Gamma] \to \mathbf{Z}) & \stackrel{\sim}{\longrightarrow} & L(\mathbf{Z} \to \mathbf{Z}) \\ \downarrow & & \downarrow \\ L(\mathbf{Z}[G] \to \mathbf{Z}) & \longrightarrow & L(\mathbf{Z}[G/H] \to \mathbf{Z}) \end{array}$$

The element $\gamma - 1 \in \mathbf{Z}[G]$ induces an element $u \in L(\mathbf{Z}[G] \to \mathbf{Z})$. But this element is coming from an element v in $L(\mathbf{Z}[\Gamma] \to \mathbf{Z})$ which is killed in $L(\mathbf{Z} \to \mathbf{Z})$. Since the map $L(\mathbf{Z}[\Gamma] \to \mathbf{Z}) \to L(\mathbf{Z} \to \mathbf{Z})$ is an isomorphism v is trivial and u is trivial too. Thus every element in H is sent to 1 in $L(\mathbf{Z}[G] \to \mathbf{Z})$. The result follows. $\square$

1.16 Lemma: *Let $f : A \to B$ be a surjective ring homomorphism. Then the map $A \to L(A \to B)$ is an isomorphism if and only if every element in $f^{-1}(1)$ is invertible.*

Proof: Denote by Λ the localization ring $L(A \to B)$. Suppose $A \to \Lambda$ is an isomorphism. Let a be an element in $f^{-1}(1)$. This element represents a 1×1-matrix which is invertible in $M(B)$ and then in $M(\Lambda)$. Because $A \to \Lambda$ is bijective, the matrix is invertible. Thus a is invertible and $f^{-1}(1)$ is contained in A^*.

Suppose now $f^{-1}(1)$ is contained in A^*. Consider a matrix M in $M(A)$ such than $f(M)$ is invertible. Let N be a matrix in $M(A)$ with $f(N) = f(M)^{-1}$. Since the diagonal entries of MN are invertible it is possible to multiply MN on the left and the right by elementary matrices in order to obtain a diagonal matrix and the same holds for NM. Therefore MN and NM are invertible and M is invertible too. Then every matrix sent to an invertible matrix in $M(B)$ is invertible and the Cohn localization of $A \to B$ is the ring A. $\square$

1.17 Proposition: *Let p be a prime and G be a finite p-group. Denote by Λ the localization $L(\mathbf{Z}_{(p)}[G] \to \mathbf{Z}_{(p)})$. Then the map $\mathbf{Z}_{(p)}[G] \to \Lambda$ is an isomorphism.*

Proof: Let $f : A \to B$ be a ring homomorphism. We say that f is local if f is surjective and every element in $f^{-1}(1)$ is invertible. It is easy to see that a composite of local maps is local.

The proposition is obvious if the order of G is 1. We'll prove the result by induction. Suppose the order of G is $p^n > 1$ and the proposition is true for every group of order p^i, $i < n$. Let z be a central element in G of order p and G' be the quotient $G/ < z >$. Consider the maps:

$$\mathbf{Z}_{(p)}[G] \xrightarrow{f} \mathbf{Z}_{(p)}[G'] \xrightarrow{g} \mathbf{Z}_{(p)}$$

Because of lemma 1.16 g is local and we have to prove that $g \circ f$ is also local.

Let f' be the reduction of f mod p:

$$f' : \mathbf{F}_p[G] \longrightarrow \mathbf{F}_p[G']$$

Set: $z = 1+u$. The relation: $z^p = 1$ becomes mod p: $u^p = 0$. Let U be an element in $f'^{-1}(1)$. This element has the form: $U = 1 - uV$ for some element $V \in \mathbf{F}_p[G]$. Then U is invertible with inverse:

$$U^{-1} = 1 + uV + u^2V^2 + \ldots + u^{p-1}V^{p-1}$$

and f' is local. Let U be an element of $f^{-1}(1) \subset \mathbf{Z}_{(p)}[G]$. The multiplication by U is an endomorphism φ on $\mathbf{Z}_{(p)}[G]$ considered as a free $\mathbf{Z}_{(p)}$-module. Because U is invertible mod p, the determinant of φ is non-zero mod p and then invertible in $\mathbf{Z}_{(p)}$. Consequently, U is invertible and f and $g \circ f$ are local. The result follows. $\square$

2. Proof of the main theorems.

Let G be a finite group. The localisation $L(\mathbf{Z}[G] \to \mathbf{Z})$ will be denoted by Λ.

2.1 Lemma: *Let x and y be two commuting elements in G with coprime orders. Then $(x-1)(y-1)$ vanishes in Λ.*

Proof: Suppose we have: $x^p = 1$ and $y^q = 1$ for coprime integers p, q. Let's take two integers a and b with: $ap + bq = 1$ and set: $z = xy$.

We have in $Z[G]$:

$$0 = (z^{pq} - 1)(z - 1) = (z^p - 1)(z^q - 1)\varphi_{p,q}(z)$$

where $\varphi_{p,q}$ is a product of cyclotomic polynomials. It is easy to see that $\varphi_{p,q}(1)$ is equal to 1 and $\varphi_{p,q}(z)$ is invertible in Λ. So we have in Λ:

$$(z^p - 1)(z^q - 1) = 0 \implies (y^p - 1)(x^q - 1) = 0 \implies (y^{ap} - 1)(x^{bq} - 1) = 0$$

$$\implies (y^{ap+bq} - 1)(x^{ap+bq} - 1) = 0 \implies (y - 1)(x - 1) = 0$$

$\square$

2.2 Lemma: *Let H be an elementary abelian p-subgroup of G and $N(H)$ be its normalizer. Let z be an element in $N(H)$ of order coprime to p. Then, for every $x \in H$, $(x-1)(z-1)$ and $(z-1)(x-1)$ vanish in Λ.*

Proof: Let q be the order of z. The conjugation by z is an action of $\mathbf{Z}/q$ on H and H is a representation of the group $\mathbf{Z}/q$ in characteristic p. Since q and p are coprime, H is a direct sum of irreducible representations H_i. Each H_i is the underlying group of a field $\mathbf{F}_i$ of characteristic p and the action of z on $\mathbf{F}_i$ is the multiplication by some element $\alpha_i \in \mathbf{F}_i^*$ of order q_i dividing q. Since $\mathbf{F}_i$ contains an element of order q_i, the cardinal n_i of H_i is congruent to 1 mod q_i: $n_i = 1 + h_i q_i$.

Consider the element $u = -h_i(1+z+\ldots+z^{q_i-1})+N_i$ in $\mathbf{Z}[G]$ where N_i is the sum of all elements of H_i. It is easy to see that the augmentation of u is $-q_ih_i + n_i = 1$ and u is invertible in Λ. For any $x \in H_i$ we have:

$$u(1-z)(1-x) = (-h_i\sum_{j<q_i} z^j+N_i)(1-z)(1-x) = -h_i(1-z^{q_i})(1-x)+N_i(1-z)(1-x)$$

But z^{q_i} and x are two commuting elements with coprime orders and z commutes with N_i. Then we have in Λ:

$$u(1-z)(1-x) = N_i(1-z)(1-x) = (1-z)N_i(1-x) = 0$$

and $(1-z)(1-x)$ vanishes in Λ.

We have also in Λ:

$$(1-x)(1-z)u = (1-x)(1-z)(-h_i\sum_{j<q_i} z^j+N_i) = -h_i(1-x)(1-z^{q_i})+(1-x)(1-z)N_i$$

$$= (1-x)N_i(1-z) = 0$$

and $(1-x)(1-z)$ vanishes also in Λ. The desired result follows easily. □

Consider two distinct primes p and q, a finite field $\mathbf{F}$ of characteristic p and an element $\alpha \in \mathbf{F}^*$ of order q. Using these data we'll construct a p-group $\Gamma(\mathbf{F},\alpha)$ equipped with an automorphism σ of order q.

Denote by $E = E(\mathbf{F},\alpha)$ the quotient of the second exterior power of $\mathbf{F}$ over $\mathbf{F}_p$ by the following relations:

$$\forall x, y \in \mathbf{F}, \quad (\alpha x){\wedge}(\alpha y) \equiv x{\wedge}y$$

So we define the group $\Gamma(\mathbf{F},\alpha)$ by generators and relations. The generators are the $s(x)$ and the $\lambda(u)$ for $x \in \mathbf{F}$ and $u \in E$. The relations are the following:

$$\forall u, v \in E, \quad \lambda(u+v) = \lambda(u)\lambda(v)$$

$$\forall x, y \in \mathbf{F}, \quad s(x+y) = s(x)s(y)\lambda(x{\wedge}((\alpha-1)y))$$
$$\forall (x,u) \in \mathbf{F}\times E, \quad \lambda(u)s(x) = s(x)\lambda(u)$$

2.3 Lemma: *The group $\Gamma(\mathbf{F},\alpha)$ is a central extension by E of the additive group of $\mathbf{F}$. More precisely we have an exact sequence:*

$$0 \longrightarrow E \xrightarrow{\lambda} \Gamma(\mathbf{F},\alpha) \xrightarrow{\pi} \mathbf{F} \longrightarrow 0$$

where π sends $\lambda(u)$ to 0 and $s(x)$ to x. Moreover this exact sequence is compatible with an action σ satisfying the following:

$$\forall u \in E, \quad \sigma(u) = u \qquad\qquad \forall x \in \mathbf{F}, \quad \sigma(s(x)) = s(\alpha x), \quad \sigma(x) = \alpha x$$

Proof: It is just an easy computation.

2.4 Lemma: *Let* $\mathbf{F}$ *be a finite field of characteristic* p *and* α *be an element in* $\mathbf{F}^*$ *of prime order* q*. Let* C *be a non-trivial cyclic* q*-group and* z *be a generator of* C*. The group* C *acts on* $\Gamma = \Gamma(\mathbf{F}, \alpha)$ *by:* $z.u = \sigma(u)$*, for every* $u \in \Gamma$*. Let* G_1 *be the semi-direct product* $\Gamma \rtimes C$ *of* Γ *by* C*. Then the morphism* $L(\mathbf{Z}[G_1] \to \mathbf{Z}) \to L(\mathbf{Z}[C] \to \mathbf{Z})$ *is an isomorphism.*

Proof: Denote by Λ_1 the ring $L(\mathbf{Z}[G_1] \to \mathbf{Z})$ and by $\equiv$ the equality in Λ_1. Because of lemma 2.1, we have for each $u \in E$ and $x \in \mathbf{F}$:

$$(\lambda(u) - 1)(z - 1) \equiv 0$$

$$\Longrightarrow \quad (\lambda(u) - 1)(z - 1)s(x) \equiv 0 \quad \Longrightarrow \quad (\lambda(u) - 1)(s(\alpha x)z - s(x)) \equiv 0$$
$$\Longrightarrow \quad (\lambda(u) - 1)(s(\alpha x) - s(x)) \equiv 0 \quad \Longrightarrow \quad (\lambda(u) - 1)(s(\alpha x)s(x)^{-1} - 1) \equiv 0$$

But we have the following:

$$s(\alpha x)s(x)^{-1} = s((\alpha - 1)x)s(x)\lambda((\alpha - 1)x, (\alpha - 1)x)s(x)^{-1} = s((\alpha - 1)x)$$

Then we have:

$$(\lambda(u) - 1)(s((\alpha - 1)x) - 1) \equiv 0$$

and that implies:

$$\forall (u, x) \in E \times \mathbf{F}, \quad (\lambda(u) - 1)(s(x) - 1) \equiv 0$$

Denote by I the two-sided ideal of $\mathbf{Z}[G_1]$ generated by the $s(x) - 1$. We have:

$$\forall (u, v) \in E \times I, \quad (\lambda(u) - 1)v \equiv 0$$

Because of the relation: $s(x + y) = s(x)s(y)\lambda(x\wedge(\alpha - 1)y)$, we have in $\mathbf{Z}[G_1]/I$:

$$1 = s(x)s(y)\lambda(x\wedge(\alpha - 1)y) = s(y)\lambda(x\wedge(\alpha - 1)y) = \lambda(x\wedge(\alpha - 1)y)$$

and $\lambda(x\wedge y) - 1$ lies in I for every x and y in $\mathbf{F}$. Therefore $\lambda(u) - 1$ belongs to I for every $u \in E$ and we have:

$$\forall u, v \in E, \quad (\lambda(u) - 1)(\lambda(v) - 1) \equiv 0$$

Let A be the quotient of $\mathbf{Z}[G_1]$ by the two-sided ideal generated by the elements $(\lambda(u) - 1)(\lambda(v) - 1)$, $(\lambda(u) - 1)(s(x) - 1)$ and $(\lambda(u) - 1)(z - 1)$ for $u, v \in E$ and $x \in \mathbf{F}$. It is easy to see that we have an exact sequence of right $\mathbf{Z}[G_1]$-modules:

$$E \longrightarrow A \longrightarrow \mathbf{Z}[\mathbf{F} \rtimes C] \longrightarrow 0$$

where the map $E \to A$ is the map $u \mapsto \lambda(u) - 1$. But G_1 acts trivially on E and E is a local $\mathbf{Z}[G_1]$-module. So we have an exact sequence:

$$E \longrightarrow A \otimes \Lambda_1 \longrightarrow \mathbf{Z}[\mathbf{F} \rtimes C] \otimes \Lambda_1 \longrightarrow 0$$

On the other hand the elements in $\mathbf{Z}[G_1]$ killed in A are killed in Λ_1. Then we have: $A \otimes \Lambda_1 = \Lambda_1$ and the exact sequence is:

$$E \longrightarrow \Lambda_1 \longrightarrow \mathbf{Z}[\mathbf{F} \rtimes C] \otimes \Lambda_1 \longrightarrow 0$$

The module $\mathbf{Z}[\mathbf{F} \rtimes C] \otimes \Lambda_1$ is actually the Cohn localization $\Lambda'_1 = L(\mathbf{Z}[\mathbf{F} \rtimes C] \to \mathbf{Z})$.

Denote by $\equiv$ the equality in Λ'_1 and for each $x \in \mathbf{F}$ denote by $[x]$ the corresponding element in $\mathbf{Z}[\mathbf{F} \rtimes C]$. Because of lemme 2.2 we have, for $x \in \mathbf{F}$:

$$([x]-1)(z-1) \equiv 0 \equiv (z-1)([x]-1) \quad \Longrightarrow \quad [x]z \equiv z[x] = [\alpha x]z \quad \Longrightarrow \quad [x] \equiv [\alpha x]$$

$$\Longrightarrow \quad [(\alpha - 1)x] \equiv 1$$

So $[x]$ is sent to $1 \in \Lambda'_1$ for every $x \in \mathbf{F}$ and Λ'_1 is the Cohn localization of $\mathbf{Z}[C] \to \mathbf{Z}$. Thus we have the exact sequence:

$$E \longrightarrow \Lambda_1 \longrightarrow L(\mathbf{Z}[C] \to \mathbf{Z}) \longrightarrow 0$$

Consider the following commutative diagram:

$$\begin{array}{ccc} \mathbf{Z}[C] & \longrightarrow & \mathbf{Z} \\ \big\downarrow & & \big\downarrow \\ \mathbf{Z}_{(q)}[C] & \longrightarrow & \mathbf{Z}_{(q)} \end{array}$$

This diagram induces the following commutative diagram:

$$\begin{array}{ccc} \mathbf{Z}[C] & \longrightarrow & L(\mathbf{Z}[C] \to \mathbf{Z}) \\ \big\downarrow & & \big\downarrow \\ \mathbf{Z}_{(q)}[C] & \longrightarrow & L(\mathbf{Z}_{(q)}[C] \to \mathbf{Z}_{(q)}) \end{array}$$

Because of proposition 1.17, the bottom map is an isomorphism and $\mathbf{Z}[C]$ injects into $L(\mathbf{Z}[C] \to \mathbf{Z})$. So we have the following exact sequence:

$$E \longrightarrow B \longrightarrow \mathbf{Z}[C] \longrightarrow 0$$

where B is the image of $\mathbf{Z}[G_1]$ in Λ_1.

By sending z to z, we get a section of $B \to \mathbf{Z}[C]$ and B is linearly isomorphic to $\mathbf{Z}[C] \oplus E'$ where E' is a quotient of E. If $\mu : E \to E'$ is the quotient map, we have the following decompositions (for $x \in \mathbf{F}$ and $u \in E$):

$$z^k = z^k \oplus 0 \qquad\qquad s(x) = 1 \oplus \varphi(x) \qquad\qquad \lambda(u) = 0 \oplus \mu(u)$$

where φ is some map from $\mathbf{F}$ to E'. The relation $s(x+y) = s(x)s(y)\lambda(x\wedge(\alpha-1)y)$ becomes:

$$\varphi(x+y) = \varphi(x) + \varphi(y) + \mu(x\wedge(\alpha-1)y)$$

So we have:

$$\mu(x\wedge(\alpha-1)y) = \mu(y\wedge(\alpha-1)x) = \mu((1-\alpha)x\wedge y) = \mu(x\wedge y) - \mu(\alpha x\wedge y) = \mu(x\wedge y) - \mu(x\wedge\alpha^{-1}y)$$

$$\Longrightarrow \mu(x\wedge(\alpha-2+\alpha^{-1})y) = \mu(x\wedge(\alpha-1)^2\alpha^{-1}y) = 0$$

But $(\alpha-1)^2\alpha^{-1}$ is non-zero and $\mu(x\wedge y)$ vanishes for every x, y in $\mathbf{F}$. Therefore μ is the zero map and E' is the trivial module. Consequently the ring B is isomorphic to $\mathbf{Z}[C]$. The result follows. □

2.5 Lemma: *Let H be a p-subgroup of G and z be an element in the normalizer $N(H)$ of H. Suppose the order of z is coprime to p. Then, for every $x \in H$, $(x-1)(z-1)$ and $(z-1)(x-1)$ vanish in Λ.*

Proof: Let f be the canonical morphism from $\mathbf{Z}[G]$ to Λ. Set: $H' = f(H)$, $G' = f(G)$ and $z' = f(z)$. The group H' is a p-subgroup of G' and z' is an element in the normalizer $N(H')$ of order q' prime to p. Notice that Λ is the localization ring $L(\mathbf{Z}[G'] \to \mathbf{Z})$.

Suppose z' is contained in the centralizer of H'. Because of lemma 2.1, the elements $(x'-1)(z'-1)$ and $(z'-1)(x'-1)$ vanish in Λ for each $x' \in H'$. The result follows in this case.

Suppose now z' is not contained in the centralizer of H'. Define the subgroups $H_k \subset H'$ by: $H_0 = H'$ and, for every $k > 0$, H_k is the subgroup of H' generated by $[H', H_{k-1}]$ and the elements u^p for $u \in H_{k-1}$.

Since H' is a p-group, H_k is trivial for k big enough and H_k is stable under every automorphism of H'. Since z' doesn't commute with all elements in H', there is an integer k such that z' centralizes H_{k+1} but not H_k. The group C' generated by z' acts on H_k and on H_k/H_{k+1}.

Suppose C' acts trivially on H_k/H_{k+1}. Then for every $u \in H_k$, there is a $v \in H_{k+1}$ with: $z'uz'^{-1} = uv$. Since z' commutes with v we have: $z'^i u z'^{-i} = uv^i$ for all i and then: $v^{q'} = 1$. Since v lies in a p-group, v is equal to 1 and z' commutes with u. But z' doesn't commute with all elements in H_k and C' doesn't act trivially on H_k/H_{k+1}.

So there is a prime q and an element $z_1 \in C'$ such that z_1 generates a q-group $C \subset C'$ and the action of z_1 on H_k/H_{k+1} is non-trivial and of order q. Actually H_k/H_{k+1} is a $\mathbf{F}_p[\mathbf{Z}/q]$-module with a non-trivial $\mathbf{Z}/q$-action. Since $\mathbf{F}_p[\mathbf{Z}/q]$ is semi-simple, H_k/H_{k+1} contains a simple $\mathbf{F}_p[\mathbf{Z}/q]$-module M where the action of z_1 is non-trivial. Then there exist a finite field $\mathbf{F}$, an element $\alpha \in \mathbf{F}^*$ of order q and an isomorphism $\varphi : \mathbf{F} \overset{\sim}{\to} M$ such that: $\varphi(\alpha u) = z_1.\varphi(u)$ for every $u \in \mathbf{F}$.

Sub-lemma: *There is a commutative diagram:*

$$\begin{array}{ccc} \Gamma(\mathbf{F},\alpha) & \xrightarrow{\widetilde{\varphi}} & H_k \\ \Big\downarrow \pi & & \Big\downarrow \\ \mathbf{F} & \xrightarrow{\varphi} & H_k/H_{k+1} \end{array}$$

where $\widetilde{\varphi}$ is a group homomorphism satisfying the following:

$$\forall u \in \Gamma(\mathbf{F},\alpha), \quad \widetilde{\varphi}(\sigma(u)) = z_1 \widetilde{\varphi}(u) z_1^{-1}$$

Using this homomorphism $\widetilde{\varphi}$, we get a group homomormhism f from $\Gamma(\mathbf{F},\alpha) \rtimes C$ to G' and the map:

$$\Gamma(\mathbf{F},\alpha) \rtimes C \xrightarrow{f} G'' \longrightarrow \Lambda$$

factors through $L(\mathbf{Z}[\Gamma(\mathbf{F},\alpha) \rtimes C] \to \mathbf{Z}) = L(\mathbf{Z}[C] \to \mathbf{Z})$. Therefore the map $\Gamma(\mathbf{F},\alpha) \longrightarrow G'$ is trivial. But that's impossible because $\varphi : \mathbf{F} \to H_k/H_{k+1}$ is not trivial. Thus the case where z' is not contained in the centralizer of H' doesn't appear and the lemma is proven. So the last thing to do is to prove the sub-lemma.

Proof of the sub-lemma: Let x be an element in $\mathbf{F}$. Let u be an element in H_k which lifts $\varphi(x) \in H_k/H_{k+1}$. Since z_1 centralizes H_{k+1}, the commutator $v = [z_1, u] = z_1 u z_1^{-1} u^{-1}$ doesn't depend on u. Moreover the image of v in H_k/H_{k+1} is equal to $\varphi((\alpha-1)x)$. Therefore there is a unique map $\theta : \mathbf{F} \to H_k$ such that:

$$\forall x \in \mathbf{F}, \quad \theta(x) \equiv \varphi(x) \bmod \ H_{k+1}$$

$$\forall x \in \mathbf{F}, \quad [z_1, \theta(x)] = \theta((\alpha-1)x)$$

Let u be an element in H_{k+1}. The element $v = \theta(x)^{-1} u \theta(x)$ lies in H_{k+1} and we have:

$$u\theta((\alpha-1)x) = u z_1 \theta(x) z_1^{-1} \theta(x)^{-1} = z_1 u \theta(x) z_1^{-1} \theta(x)^{-1} = z_1 \theta(x) v z_1^{-1} \theta(x)^{-1}$$

$$= z_1 \theta(x) z_1^{-1} v \theta(x)^{-1} = z_1 \theta(x) z_1^{-1} \theta(x)^{-1} u = \theta((\alpha-1)x) u$$

Therefore $\theta(x)$ centralizes H_{k+1} for each $x \in \mathbf{F}$.

Let x and y be two elements in $\mathbf{F}$ and two elements u and v in H_k lifting $\varphi(x)$ and $\varphi(y)$. Then the element $uvu^{-1}v^{-1}$ depends neither on u nor on v and will be denoted by $\mu(x,y)$. So we have:

$$\mu(x,y) = uvu^{-1}v^{-1} = \theta(x)\theta(y)\theta(x)^{-1}\theta(y)^{-1}$$

and $\mu(x,y)$ lies in the center Z of H_k. It is easy to see the following:

$$\mu(x, y+y') = \mu(x,y)\mu(x,y')$$

$$\mu(x,y) = \mu(y,x)^{-1}$$

$$\mu(x,x) = 1$$

$$\mu(x,y) = z\mu(x,y)z^{-1} = \mu(\alpha x, \alpha y)$$

and there is a group homomorphism g from $E = E(\mathbf{F},\alpha)$ to Z such that:

$$\mu(x,y) = g((\alpha-1)x \wedge (\alpha-1)y)$$

Consider two elements x and y in $\mathbf{F}$ and two elements u and v in H_k lifting $\varphi(x)$. Set: $x' = (\alpha-1)x$ and $y' = (\alpha-1)y$. We have:

$$\theta(y')^{-1}\theta(x')^{-1}\theta(x'+y') = vz_1v^{-1}z_1^{-1}uzu^{-1}z_1^{-1}z_1uvz_1^{-1}(uv)^{-1} = vz_1v^{-1}z_1^{-1}uz_1vz_1^{-1}(uv)^{-1}$$

$$= [vz_1v^{-1}z_1^{-1}, u] = \mu((1-\alpha)y, x) = g((\alpha-1)x \wedge (\alpha-1)^2 y) = g(x' \wedge (\alpha-1)y')$$

and that implies:

$$\theta(x'+y') = \theta(x')\theta(y')g(x' \wedge (\alpha-1)y')$$

and then:

$$\theta(x+y) = \theta(x)\theta(y)g(x \wedge (\alpha-1)y)$$

for each x, y in $\mathbf{F}$. So we define the morphism $\widetilde{\varphi}$ by:

$$\forall x \in \mathbf{F}, \quad s(x) \mapsto \theta(x)$$

$$\forall u \in E, \quad \lambda(u) \mapsto g(u)$$

It is easy to see that $\widetilde{\varphi}$ is a well defined group homomorphism from $\Gamma(\mathbf{F},\alpha)$ to H_k which is equivariant with respect to the action of σ on $\Gamma(\mathbf{F},\alpha)$ and the conjugation by z_1 on H_k. The result follows. □

2.6 Lemma: *Let G be a finite group. Then the image of G in Λ is nilpotent.*

Proof: Let G' be the image of the map $G \to \Lambda^*$. Because of proposition 1.15 G' doesn't contain any perfect subgroup and G' is solvable. Thus there is a filtration:

$$1 = H_0 \subset H_1 \subset \ldots \subset H_n = G'$$

of G' by normal subgroups such that H_k/H_{k-1} is a group of prime order.

Let $k > 0$ be an integer. Suppose H_{k-1} is nilpotent. Denote by p the order of H_k/H_{k-1}. There is an element $z \in H_k$ of order p^n for some $n > 0$ which generates H_k/H_{k-1}. Let S be a q-Sylow subgroup of H_{k-1} with $q \neq p$. Because of lemma 2.5, z commutes with the elements of S and S is normal in H_k. The p-Sylow subgroup of H_k is generated by the p-Sylow subgroup of H_{k-1} and z. Therefore every Sylow subgroup of H_k is normal and H_k is nilpotent. So, by induction, every H_k is nilpotent and G' is nilpotent too. □

Let G' be the product of the G_p's. The morphism $G \to G'$ is surjective and G' is the universel nilpotent quotient of G. Because of lemma 2.6, the map $L(\mathbf{Z}[G] \to \mathbf{Z}) \to L(\mathbf{Z}[G'] \to \mathbf{Z})$ is an isomorphism.

For each prime p, set:

$$n_p = \mathrm{card}(G_p) \qquad N_p = \sum_{x\in G_p} x \qquad \omega_p = n_p - N_p = \sum_{x\in G_p} (1-x)$$

ω_p lies in the center of $\mathbf{Z}[G']$ and we have: $\omega_p^2 = n_p\omega_p$. Consider the multiplicative sets $S_p = 1+\mathbf{Z}\omega_p$ and $\Sigma_p = 1+\mathbf{Z}n_p$. Let R be the subring of $\mathbf{Z}[G']$ generated by the ω_p's and I be the ideal of R generated by the ω_p's. We have another multiplicative set: $S = 1+I$. Sets S and S_p are contained in the center of $\mathbf{Z}[G']$ and are sent to $\{1\}$ by the augmentation map. The set Σ_p has the following property:

$$\Sigma_p^{-1}\mathbf{Z} = \mathbf{Z}_{(p)}$$

If D is a commutative square in a category of modules, we say that D is exact if D is cartesian and cocartesian.

2.7 Lemma: *Suppose G is nilpotent. Let f_p be the augmentation map $\mathbf{Z}_{(p)}[G_p] \to \mathbf{Z}_{(p)}$ and $\Delta : \mathbf{Z} \to \prod \mathbf{Z}_{(p)}$ be the diagonal inclusion. Then the following square is exact:*

$$\begin{array}{ccc} S^{-1}\mathbf{Z}[G] & \longrightarrow & \prod_p \mathbf{Z}_{(p)}[G_p] \\ \big\downarrow & & \big\downarrow \prod_p f_p \\ \mathbf{Z} & \xrightarrow{\Delta} & \prod_p \mathbf{Z}_{(p)} \end{array}$$

Proof: For a prime p, consider the following exact square:

$$(D_p) \qquad \begin{array}{ccc} \mathbf{Z}[G_p] & \longrightarrow & \mathbf{Z}[G_p]/(N_p) \\ \big\downarrow & & \big\downarrow \\ \mathbf{Z} & \longrightarrow & \mathbf{Z}/(n_p) \end{array}$$

Let p and q be two distinct primes. Let G' be the group such that G is isomorphic to $G_p \times G_q \times G'$. By tensoring (D_q) by $\mathbf{Z}[G_p]/(N_p) \otimes \mathbf{Z}[G']$, we get an exact square:

$$\begin{array}{ccc} \mathbf{Z}[G]/(N_p) & \longrightarrow & \mathbf{Z}[G]/(N_p, N_q) \\ \big\downarrow & & \big\downarrow \\ \mathbf{Z}[G/G_q]/(N_p) & \longrightarrow & \mathbf{Z}/(n_q)[G/G_q]/(N_p) \end{array}$$

It is easy to see that S modulo N_p and N_q contains $1+\mathbf{Z}n_p+\mathbf{Z}n_q$ and then 0. So we have:

$$S^{-1}\mathbf{Z}[G]/(N_p, N_q) \simeq S^{-1}\mathbf{Z}/(n_q)[G/G_q]/(N_p) = 0$$

and that implies:

$$S^{-1}\mathbf{Z}[G]/(N_p) \stackrel{\sim}{\to} S^{-1}\mathbf{Z}[G/G_q]/(N_p)$$

Applying this formula for all $q \neq p$, we get:

$$S^{-1}\mathbf{Z}[G]/(N_p) \simeq S^{-1}\mathbf{Z}[G_p]/(N_p) \simeq S_p^{-1}\mathbf{Z}[G_p]/(N_p) \simeq \Sigma_p^{-1}\mathbf{Z}[G_p]/(N_p) \simeq \mathbf{Z}_{(p)}[G_p]/(N_p)$$

If we kill G_p in this isomorphism, we get:

$$S^{-1}\mathbf{Z}/(n_p)[G/G_p] \simeq \mathbf{Z}/(n_p)$$

On the other hand, by tensoring (D_p) by $\mathbf{Z}[G/G_p]$, we get an exact square:

$$\begin{array}{ccc}
\mathbf{Z}[G] & \longrightarrow & \mathbf{Z}[G]/(N_p) \\
\big\downarrow & & \big\downarrow \\
\mathbf{Z}[G/G_p] & \longrightarrow & \mathbf{Z}/(n_p)[G/G_p]
\end{array}$$

So, by localizing this square with S, we get the following exact square:

$$\begin{array}{ccc}
S^{-1}\mathbf{Z}[G] & \longrightarrow & \mathbf{Z}_{(p)}[G_p]/(N_p) \\
\big\downarrow & & \big\downarrow \\
\mathcal{S}^{-1}Z[G/G_p] & \longrightarrow & \mathbf{Z}/(n_p)
\end{array}$$

Consider the following commutative diagram:

$$\begin{array}{ccccc}
S^{-1}\mathbf{Z}[G] & \longrightarrow & \mathbf{Z}_{(p)}[G_p] & \longrightarrow & \mathbf{Z}_{(p)}[G_p]/(N_p) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{S}^{-1}Z[G/G_p] & \longrightarrow & \mathbf{Z}_{(p)} & \longrightarrow & \mathbf{Z}/(n_p)
\end{array}$$

The square on the right is exact and the composite square is also exact. Therefore the square on the left is exact. Then it is easy to check the desired formula by induction on the number of Sylow subgroups of G. □

Proof of theorem A: Since S is sent to $\{1\}$ by the augmentation map, the morphism $\mathbf{Z}[G'] \to \Lambda$ factors through the ring $S^{-1}\mathbf{Z}[G']$ and we have:

$$L(\mathbf{Z}[G] \to \mathbf{Z}) \simeq L(\mathbf{Z}[G'] \to \mathbf{Z}) \simeq L(S^{-1}\mathbf{Z}[G'] \to \mathbf{Z})$$

On the other hand the $\mathbf{Z}[G]$-modules $\mathbf{Z}$, $\mathbf{Z}_{(p)}[G_p]$ and $\mathbf{Z}_{(p)}$ are local. Because of lemma 2.7, the module $S^{-1}\mathbf{Z}[G']$ is also local and the morphism $S^{-1}\mathbf{Z}[G'] \to \Lambda$ is an isomorphism. □

Proof of theorem B: It is easy to check the implications:

$$1) \implies 2) \implies 3) \implies 4)$$

Actually 2) and 3) are equivalent in any case by a result of Teichner [T].

If G is nilpotent, lemma 2.7 shows that Λ is the ring $S^{-1}\mathbf{Z}[G]$ and Λ is a central localization of $\mathbf{Z}[G]$. So condition 5) implies the condition 1). The last thing to do is to prove the implication: 4) $\implies$ 5).

Suppose Λ is stably flat over $\mathbf{Z}[G]$. Consider the left $\mathbf{Z}[G]$-module $U = \Lambda \otimes_{\mathbf{Z}} \Lambda$ where the action of G on U is defined by:

$$g.(u \otimes v) = ug^{-1} \otimes gv$$

for every $(g, u, v) \in G \times \Lambda \times \Lambda$. It is easy to see that $\mathrm{Tor}_i^{\mathbf{Z}[G]}(\Lambda, \Lambda)$ is isomorphic to $\mathrm{H}_i(G, U)$ and $\mathrm{H}_i(G, U)$ vanishes for all $i > 0$.

For each prime p, denote by H_p the kernel of the quotient map $G \to G_p$. The intersection H of the H_p's is the kernel of the map $G \to \prod_p G_p$.

Let p be a prime. We have: $\mathrm{H}_i(G, U)_{(p)} = \mathrm{H}_i(G, U_{(p)})$ and $\mathrm{H}_i(G, U_{(p)})$ vanishes for $i > 0$. Because of theorem A, we have an exact sequence of $\mathbf{Z}_{(p)}[G]$-modules:

$$0 \longrightarrow E_p \longrightarrow \Lambda_{(p)} \longrightarrow \mathbf{Z}_{(p)}[G_p] \longrightarrow 0$$

where E_p is a $\mathbf{Q}[G]$-module. Therefore we get another exact sequence:

$$0 \longrightarrow E'_p \longrightarrow U_{(p)} \longrightarrow \mathbf{Z}_{(p)}[G_p] \otimes \mathbf{Z}_{(p)}[G_p] \longrightarrow 0$$

for some $\mathbf{Q}[G]$-module E'_p. Since $\mathbf{Q}[G]$ is semi-simple this exact sequence splits and $\mathrm{H}_i(G, \mathbf{Z}_{(p)}[G_p \times G_p])$ vanishes for all $i > 0$. In this homology module, G acts on $G_p \times G_p$ by: $g(u, v) = (ug^{-1}, gv)$. Consider the map $f : G_p \times G_p \to G_p \times G_p$ sending (u, v) to (v, uv). This map induces an isomorphism from $\mathbf{Z}_{(p)}[G_p \times G_p]$ to $V = \mathbf{Z}_{(p)}[G_p] \otimes \mathbf{Z}_{(p)}[G_p]$, where the action of G on V is given by:

$$g(u \otimes v) = gu \otimes v$$

Therefore $\mathrm{H}_i(G, V)$ vanishes and we have:

$$\mathrm{H}_i(G, \mathbf{Z}_{(p)}[G_p]) \otimes \mathbf{Z}_{(p)}[G_p] \simeq 0 \implies \mathrm{H}_i(G, \mathbf{Z}_{(p)}[G_p]) \simeq \mathrm{H}_i(H_p, \mathbf{Z}_{(p)}) \simeq 0$$

Because of the next lemma the order of H_p is coprime to p. Therefore the order of H is coprime to every prime and $G \simeq \prod_p G_p$ is nilpotent. □

Lemma: *Let G be a finite group and p be a prime. Suppose $\mathrm{H}_i(G, \mathbf{Z}_{(p)}) = 0$ for all $i > 0$. Then the order of G is coprime to p.*

Proof: Let G be a finite group. The Krull dimension of the $\mathbf{F}_p$-algebra $\mathrm{H}^*(G, \mathbf{F}_p)$ (or equivalently of its center) is zero if the homology $\mathrm{H}_i(G, \mathbf{Z}_{(p)})$ vanishes for $i > 0$. But this dimension is known to be the maximal integer n such that G contains an elementary abelian p-group $(\mathbf{Z}/p)^n$ (see Quillen [Q] or Adem [A]).

Since this dimension is zero, G doesn't contain any non-trivial elementary abelian p-group and the order of G is coprime to p. □

An example: Let p be an odd number and G be a dihedral group of order $2p$. Let t be the generator of $G_2 = G/[G,G] \simeq \mathbf{Z}/2$. Denote by Λ the localization ring $L(\mathbf{Z}[G] \to \mathbf{Z})$ and by Λ_* the graded ring $\mathrm{H}_*(\Phi(\mathbf{Z}[G]))$. Then we have:

$$\Lambda = \mathbf{Z} \oplus \mathbf{Z}_{(2)}(t-1) \subset \mathbf{Z}_{(2)}[G_2] = \mathbf{Z}_{(2)}[\mathbf{Z}/2]$$

Moreover there is an element α of degree 2 such that Λ_* is the graded ring:

$$\Lambda[\alpha]/(p\alpha) = \Lambda \oplus (\mathbf{Z}/p)\alpha \oplus (\mathbf{Z}/p)\alpha^2 \oplus (\mathbf{Z}/p)\alpha^3 \oplus \dots$$

A counter-example: Let H be a sub-group of a group G such that $H = [G, H]$. If G is finite the localization of $\mathbf{Z}[G] \to \mathbf{Z}$ is isomorphic to the localization of $\mathbf{Z}[G/H] \to \mathbf{Z}$. But this property is not true in general even if H is finite as will be shown in the following example:

Let p be an odd prime. Consider the group G generated by two elements x and t with the only relations: $x^p = 1$ and $tx = x^{-1}t$. The center Z of G is a free group generated by $y = t^2$ and G/Z is a dihedral group of order $2p$. Let $H \subset G$ be the subgroup generated by x. The group H is isomorphic to $\mathbf{Z}/p$ and we have: $H = [G, H]$. The Cohn localization of the augmentation map $\mathbf{Z}[G] \to \mathbf{Z}$ will be denoted by Λ. We will prove that the morphism $\mathbf{Z}[G] \to \Lambda$ is injective and doesn't factors through $\mathbf{Z}[G/H]$.

Consider the field $F_1 = \mathbf{Q}(t)$ of rational fractions in t. The correspondance $x \mapsto 1$ and $t \mapsto t$ induces a morphism f_1 from $\mathbf{Z}[G]$ to F_1. It is clear that F_1 is a local $\mathbf{Z}[G]$-module.

Consider the field $F_2 = \mathbf{Q}[\zeta_p](y)$ of rational fractions in y with coefficients in the cyclotomic field $\mathbf{Q}[\zeta_p]$. The complex conjugation $\zeta_p \mapsto \zeta_p^{-1}$ induces an involution $u \mapsto \overline{u}$ in F_2. Set: $A = F_2 \oplus F_2 t$. This module is a left F_2-vector space. We define a product on A by:

$$(a + bt)(c + dt) = ac + b\overline{d}y + (ad + b\overline{c})t$$

and A is a ring. The involution on F_2 extends to an anti-involution on A by setting: $\overline{t} = -t$. It is easy to see that $u + \overline{u}$ and $u\overline{u}$ are in the center of A for each $u \in A$. Thus each non-zero element $u \in A$ is invertible with inverse:

$$u^{-1} = (u\overline{u})^{-1}\overline{u}$$

and A is a division ring. Actually A can be embedded in the skew field of quaternions.

The correspondance $x \mapsto \zeta_p$ and $t \mapsto t$ induces a morphism f_2 from $\mathbf{Z}[G]$ to A. As above A is a local $\mathbf{Z}[G]$-module.

Using f_1 and f_2 we have a morphism $f : \mathbf{Z}[G] \to F_1 \times A$ which factors through Λ. But it is easy to see that f is injective and the morphism $\mathbf{Z}[G] \to \Lambda$ is injective too.

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