# Algebraic structures on modules of diagrams

by P. Vogel[1]

**Abstract.**

There exists a graded algebra $\Lambda$ acting in a natural way on many modules of 3-valent diagrams. Every simple Lie superalgebra with a nonsingular invariant bilinear form induces a character on $\Lambda$. The classical and exceptional Lie algebras and the Lie superalgebra $\mathrm{D}(2,1,\alpha)$ produce eight distinct characters on $\Lambda$ and eight distinct families of weight functions on chord diagrams. As a consequence we prove that weight functions coming from semisimple Lie superalgebras do not detect every element in the module $\mathcal{A}$ of chord diagrams. A precise description of $\Lambda$ is conjectured.

## Introduction.

V. Vassiliev [Va] has recently defined a new family of knot invariants. Actually every knot invariant with values in an abelian group may be seen as a linear map from the free $\mathbf{Z}$-module $\mathbf{Z}[\mathcal{K}]$ generated by isomorphism classes of knots. This module is a Hopf algebra and has a natural filtration $\mathbf{Z}[\mathcal{K}] = I_0 \supset I_1 \supset \dots$ defined in terms of singular knots, and a Vassiliev invariant of order $n$ is an invariant which is trivial on $I_{n+1}$. The coefficients of Jones [J], HOMFLY [H], Kauffman [Ka] polynomials are Vassiliev invariants.

The associated graded Hopf algebra $\mathrm{Gr}\mathbf{Z}[\mathcal{K}] = \bigoplus_n I_n/I_{n+1}$ is finitely generated over $\mathbf{Z}$ in each degree but its rank is completely unknown. Actually $\mathrm{Gr}\mathbf{Z}[\mathcal{K}]$ is a certain quotient of the graded Hopf algebra $\mathcal{A}$ of chord diagrams [BN]. Every Vassiliev invariants of order $n$ induces a weight function of degree $n$, (i.e. a linear form of degree $n$ on $\mathcal{A}$). Conversely every weight function can be integrated (via the Kontsevich integral) to a knot invariant. Very few things are known about the algebra $\mathcal{A}$. Rationally, $\mathcal{A}$ is the symmetric algebra on a graded module $\mathcal{P}$, and the so-called Adams operations split $\mathcal{A}$ and $\mathcal{P}$ in a direct sum of modules defined in terms of unitrivalent diagrams. The rank of $\mathcal{P}$ is known in degree $< 10$.

Every Lie algebra equipped with a nonsingular invariant bilinear form and a finite dimensional representation induces a weight function on $\mathcal{A}$. It was conjectured in [BN]

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that the weight functions corresponding to the classical simple Lie algebras detect every nontrivial element in $\mathcal{A}$.

In this paper[2], we define a graded algebra $\Lambda$ acting on many modules of diagrams like $\mathcal{P}$ and every Lie algebra equipped with a nonsingular invariant bilinear form induces a character on $\Lambda$. With this procedure, we construct eight characters from $\Lambda$ to a polynomial algebra of two variables for three of them, and to $\mathbf{Q}[t]$ for the others. These eight characters are algebraically independent. As a consequence, we construct a primitive element in $\mathcal{A}$ which is rationally nontrivial and killed by every semisimple Lie algebra and Lie superalgebra equipped with a nonsingular invariant bilinear form and a finite-dimensional representation.

In the first section many modules of diagrams are defined.

In Section 2, we construct a transformation $t$ of degree 1 acting on some of these modules.

In Section 3, we construct the algebra $\Lambda$. This algebra contains the element $t$.

In Section 4, some modules of diagrams are completely described in term of $\Lambda$.

In Section 5, we define many elements in $\Lambda$ and construct a graded algebra homomorphism from $R_0$ to $\Lambda$, where $R_0$ is a subalgebra of a polynomial algebra $R$ with three variables of degree 1, 2 and 3.

In Section 6, we construct many weight functions and show that every simple quadratic Lie superalgebra induces a well-defined character on $\Lambda$

In Section 7, we construct the eight characters.

Using these characters, many results on $\Lambda$ are proven in the last section. In particular the morphism $R_0 \longrightarrow \Lambda$ factors through a quotient $R_0/I$ where $I$ is an ideal in $R$ generated by a polynomial $P \in R_0$ of degree 16 and the induced morphism $R_0/I \longrightarrow \Lambda$ is conjectured to be an isomorphism.

## 1. Modules of diagrams.

Denote by 3-valent graph a graph where every vertex is 1-valent or 3-valent. A 3-valent graph is defined by local conditions. So in such a graph an edge may be a loop and two distinct egdes may have common boundary points. The set of 1-valent vertices of a 3-valent graph $K$ will be called its boundary and denoted by $\partial K$.

Let $\Gamma$ be a curve, i.e. a compact 1-dimensional manifold and $X$ be a finite set. A $(\Gamma, X)$-*diagram* is a finite 3-valent graph $D$ equipped with the following data:

— an isomorphism from the disjoint union of $\Gamma$ and $X$ to a subgraph of $D$ sending $\partial\Gamma \cup X$ bijectively to $\partial D$

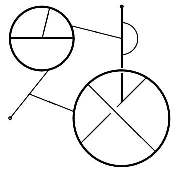
— for every 3-valent vertex $x$ of $D$, a cyclic ordering of the set of oriented edges ending at $x$.

The class of $(\Gamma, X)$-diagrams will be denoted by $\mathcal{D}(\Gamma, X)$.

Usually, a $(\Gamma, X)$-diagram will be represented by a 3-valent graph immersed in the plane in such a way that, at every 3-valent vertex, the cyclic ordering is given by the orientation of the plane.

[2] This is an expanded and updated version of a 1995 preprint.

Example of a $(\Gamma, X)$-diagram where $\Gamma$ has two closed components and $X$ has two elements:

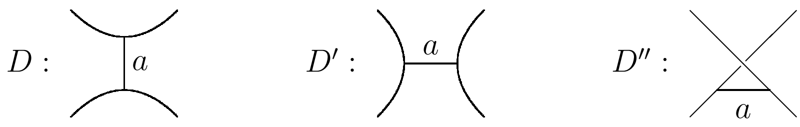


Let $\mathcal{C}$ be a subclass of $\mathcal{D}(\Gamma, X)$ which is closed under changing cyclic ordering. Let $k$ be a commutative ring. Denote by $\mathcal{A}_k(\mathcal{C})$ the quotient of the free $k$-module generated by the isomorphism classes of $(\Gamma, X)$-diagrams in $\mathcal{C}$ by the following relations:

— if $D$ is a $(\Gamma, X)$-diagram in $\mathcal{C}$, and $D'$ is obtained from $K$ by changing the cyclic ordering at one vertex, we have:

(AS) $$D' \equiv -D$$

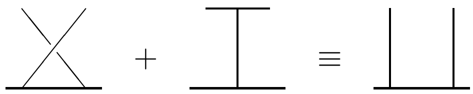
— if $D$, $D'$, $D''$ are three $(\Gamma, X)$-diagrams in $\mathcal{C}$ which differ only near an edge $a$ in the following way:




we have:

(IHX) $$D \equiv D' - D''.$$

**Remark:** If the edge meets the curve $\Gamma$ the relation (IHX) is called (STU) in [BN]:




The module $\mathcal{A}_k(\mathcal{C})$ is a graded $k$-module. The degree $\partial^\circ D$ of a $(\Gamma, X)$-diagram $D$ is $-\chi(D)$ where $\chi$ is the Euler characteristic.

By considering different classes of diagrams, we get the following examples of graded modules:

— the module $\mathcal{A}_k(\Gamma, X)$, if $\mathcal{C}$ is the class $\mathcal{D}'(\Gamma, X)$ of $(\Gamma, X)$-diagrams $D$ such that every connected component of $D$ meets $\Gamma$ or $X$

— the module $\mathcal{A}_k^c(\Gamma, X)$, if $\mathcal{C}$ is the class $\mathcal{D}^c(\Gamma, X)$ of $(\Gamma, X)$-diagrams $D$ such that $D \setminus \Gamma$ is connected and nonempty (*connected case*)

— the module $\mathcal{A}_k^s(\Gamma, X)$, if $\mathcal{C}$ is the class $\mathcal{D}^s(\Gamma, X)$ of $(\Gamma, X)$-diagrams $D$ such that $D \setminus \Gamma$ is connected and has at least one 3-valent vertex (*special case*)

— the module $\mathcal{A}_k(\Gamma) = \mathcal{A}_k(\Gamma, \emptyset)$

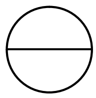
— the module $\mathcal{A}_k^c(\Gamma) = \mathcal{A}_k^c(\Gamma, \emptyset)$

— the module $F_k(X) = \mathcal{A}_k^c(\emptyset, X)$. If $X$ is the set $[n] = \{1, \ldots, n\}$, the module $F_k(X)$ will be denoted by $F_k(n)$

— the module ${}_X\Delta_{kY}$, where $X$ and $Y$ are finite sets and $\mathcal{C}$ is the class of all $(\emptyset, X \coprod Y)$-diagrams.

The most interesting case is $k = \mathbf{Q}$. So the modules $\mathcal{A}_{\mathbf{Q}}(\mathcal{C})$, $\mathcal{A}_{\mathbf{Q}}(\Gamma, X)$, $\mathcal{A}_{\mathbf{Q}}^c(\Gamma, X)$, $\mathcal{A}_{\mathbf{Q}}^s(\Gamma, X) \ldots$ will be simply denoted by $\mathcal{A}(\mathcal{C})$, $\mathcal{A}(\Gamma, X)$, $\mathcal{A}^c(\Gamma, X)$, $\mathcal{A}^s(\Gamma, X) \ldots$.

The module $\mathcal{A}_k(\Gamma)$ is strongly related to the theory of links. In the case of knots, the Kontsevich integral provides a universal Vassiliev invariant with values in a completion of the quotient of the module $\mathcal{A} = \mathcal{A}_{\mathbf{Q}}(S^1) = \mathcal{A}(S^1)$ by some submodule $I$ [BN]. The module $\mathcal{A}$ is actually a commutative and cocommutative Hopf algebra (the product corresponds to the connected sum of knots) and $I$ is the ideal generated by the following diagram of degree 1:



**Remark:** The definition of the module $\mathcal{A}_k(\Gamma)$ is slightly different from the classical one. The classical definition needs an orientation of $\Gamma$, but cyclic orderings near vertices in $\Gamma$ are not part of the data. The relationship between these two definitions come from the fact that, if $\Gamma$ is oriented, there is a canonical choice for the cyclic ordering of edges ending at each vertex in $\Gamma$.

Let $\mathcal{P}_k = \mathcal{A}_k^c(S^1)$ and $\mathcal{A}_k = \mathcal{A}_k(S^1)$. The inclusion $\mathcal{D}^c(S^1, \emptyset) \subset \mathcal{D}(S^1, \emptyset)$ induces a linear map from $\mathcal{P}_k$ to $\mathcal{A}_k$ and a morphism of Hopf algebras from $S(\mathcal{P}_k)$ to $\mathcal{A}_k$.

**1.1 Proposition:** *The morphism $S(\mathcal{P}_{\mathbf{Z}}) \to \mathcal{A}_{\mathbf{Z}}$ is surjective with finite kernel in each degree.*

**Proof:** For $n > 0$, denote by $\mathcal{E}_n$ the submodule of $\mathcal{A}_{\mathbf{Z}}$ generated by the diagrams $D$ such that $D \setminus S^1$ has at most $n$ components. Because of relation STU, it is easy to see that, mod $\mathcal{E}_n$, $\mathcal{E}_{n+1}$ is generated by connected sums $K_1 \sharp K_2 \ldots \sharp K_{n+1}$ where $K_i \setminus S^1$ are connected. That proves, by induction, that the canonical map from $S(\mathcal{P}_{\mathbf{Z}})$ to $\mathcal{A}_{\mathbf{Z}}$ is surjective. Because $S(\mathcal{P}_{\mathbf{Z}})$ and $\mathcal{A}_{\mathbf{Z}}$ are finitely generated over $\mathbf{Z}$ in each degree, it's enough to prove that the map from $S(\mathcal{P}_{\mathbf{Z}})$ to $\mathcal{A}_{\mathbf{Z}}$ is a rational isomorphism, and because $S(\mathcal{P}_{\mathbf{Z}})$ and $\mathcal{A}_{\mathbf{Z}}$ are commutative and cocommutative Hopf algebras, it is enough to prove that the map from $\mathcal{P} = \mathcal{P}_{\mathbf{Q}}$ to $\mathcal{A} = \mathcal{A}_{\mathbf{Q}}$ is an isomorphism from $\mathcal{P}$ to the module of primitives of $\mathcal{A}$.

Consider the module $\mathcal{C}_p$ of 3-valent diagrams with $p$ univalent vertices and the module $\mathcal{C}_p^c$ of connected 3-valent diagrams with $p$ univalent vertices. In [BN] Bar Natan constructs a rational isomorphism from $\mathcal{A}$ to the direct sum $\bigoplus_{p>0} \mathcal{C}_p$ which respects the comultiplication. In the same way we have a rational isomorphism from $\mathcal{P}$ to $\bigoplus_{p>0} \mathcal{C}_p^c$.

Therefore $\mathcal{P}$ is isomorphic to the module of primitives of $\mathcal{A}$. $\square$

Very few things are known about $\mathcal{A}$ and $\mathcal{P}$. They are finitely generated modules in each degree. The rank is known in degree $\leq 9$. For $\mathcal{P}$, this rank is: 1, 1, 1, 2, 3, 5, 8, 12, 18 [BN]. Some linear forms (called weight functions) on $\mathcal{A}$ (coming from Lie algebras) are known. Rationally the module $\mathcal{P}$ splits in a direct sum of modules of connected 3-valent diagrams $\mathcal{C}_n^c$ [BN]. Actually the module $\mathcal{C}_n^c$ is defined in the same way as $F(n) = F_{\mathbf{Q}}(n)$ except that the bijection from $[n]$ to the set of 1-valent vertices is forgotten. Hence this splitting may be written in the following manner:

**1.2 Proposition:** *There is an isomorphism:*

$$\bigoplus_{n>0} \mathrm{H}_0(\mathfrak{S}_n, F(n)) \xrightarrow{\sim} \mathcal{P}.$$

The last module ${}_X\Delta_{kY}$ defined above will be used later. Actually these modules define a $k$-linear monoidal category $\Delta_k$. The objects of $\Delta_k$ are finite sets, and the set of morphisms $\mathrm{Hom}(X,Y)$ is the module ${}_Y\Delta_{kX}$. The composition from ${}_X\Delta_{kY} \otimes {}_Y\Delta_{kZ}$ to ${}_X\Delta_{kZ}$ is obtained by gluing. In particular, for every finite set $X$, ${}_X\Delta_{kX}$ is a $k$-algebra.

The monoidal structure is the disjoint union of finite sets or diagrams.

For technical reasons we'll use a modified degree for modules $F_k(X)$ and ${}_X\Delta_{kY}$:

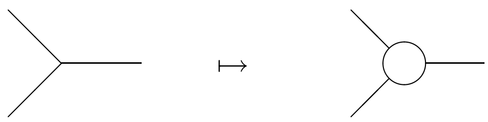
— the degree of an element $u \in F_k(X)$ represented by a diagram $D$ is $1 - \chi(D)$. So the degree of a tree is zero.

— the degree of an element $u \in {}_Y\Delta_{kX}$ represented by a diagram $D$ is $-\chi(D, X)$. This degree is compatible with the structure of $k$-linear monoidal category.

## 2. The transformation $t$.

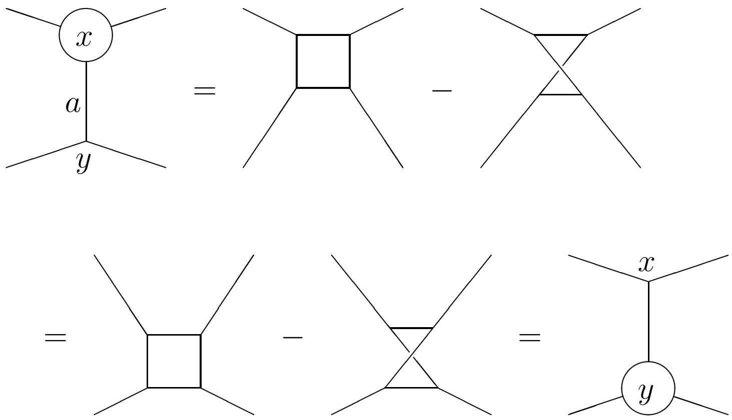
Let $\Gamma$ be a curve and $X$ be a finite set. We have three graded modules $\mathcal{A}_k(\Gamma, X)$, $\mathcal{A}_k^c(\Gamma, X)$ and $\mathcal{A}_k^s(\Gamma, X)$, and $\mathcal{A}_k^c(\Gamma, X)$ is isomorphic to $\mathcal{A}_k^s(\Gamma, X)$ except maybe in small degrees.

Let $D$ be a $(\Gamma, X)$-diagram in the class $\mathcal{D}_k^s(\Gamma, X)$. Take a 3-valent vertex outside of $\Gamma$. Then it is possible to modify $D$ near this vertex in the following way:



**2.1 Theorem:** *This transformation induces a well-defined endomorphism $t$ of the module $\mathcal{A}_k^s(\Gamma, X)$.*

**Proof:** Let $D$ be a diagram in the class $\mathcal{D}_k^s(\Gamma, X)$. Let $a$ be an edge of $D$ disjoint from the curve $\Gamma$. Denote vertices of $a$ by $x$ and $y$. Relations IHX imply the following:




Then transformations of $D$ at $x$ and $y$ produce the same element in the module $\mathcal{A}_k^s(\Gamma, X)$. Since the complement of $\Gamma$ in a diagram in $\mathcal{D}_s(\Gamma, X)$ is connected, the transformation $t$ is well defined from the class $\mathcal{D}_k^s(\Gamma, X)$ to $\mathcal{A}_k^s(\Gamma, X)$.

It is easy to see that $t$ is compatible with the AS relation. Consider an IHX relation:

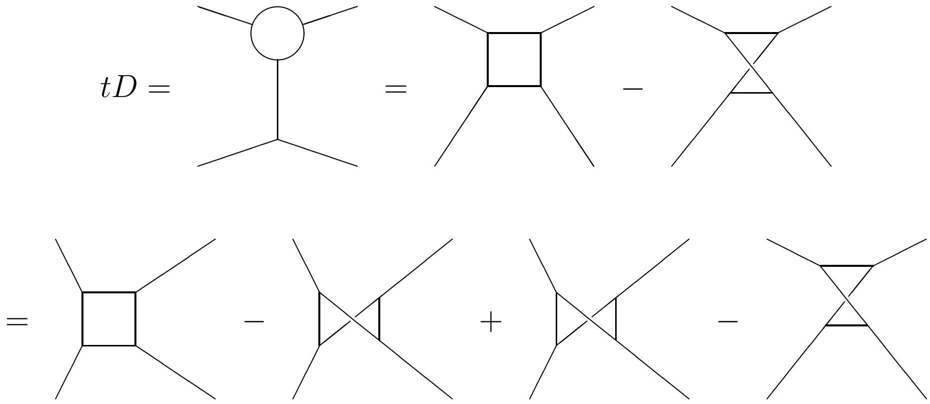
$$D \equiv D' - D''$$

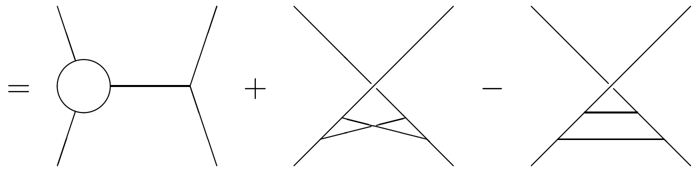
where $D$, $D'$ and $D''$ differ only near an edge $a$. If there is a 3-valent vertex in $D$ which is not in $a$ and not in the curve $\Gamma$, it is possible to define $tD$, $tD'$ and $tD''$ by using this vertex, and the relation

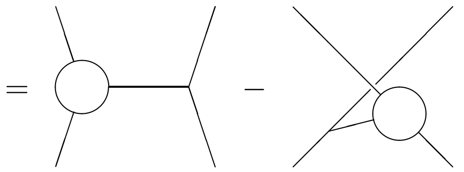
$$tD \equiv tD' - tD''$$

becomes obvious.

Suppose now $\Gamma \cup X \cup a$ contains every vertex in $D$. Then the edge $a$ is not contained in $\Gamma$, and that is true also for $D'$ and $D''$. Therefore $a$ doesn't meet $\Gamma$, and we have:





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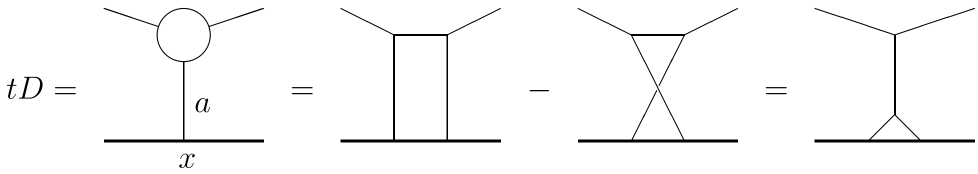
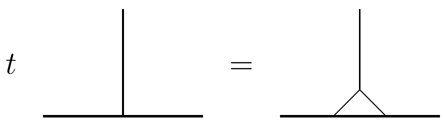
**2.2 Proposition:** *If* $\Gamma$ *is nonempty the transformation* $t$ *extends in a natural way to the module* $\mathcal{A}_k^c(\Gamma, X)$.

**Proof:** Let $D$ be a diagram in the class $\mathcal{D}_k^c(\Gamma, X)$. Let $x$ be a 3-valent vertex of $D$ contained in $\Gamma$. This vertex in contained in an edge $a$ in $D \setminus \Gamma$. If the diagram $D$ lies in the class $\mathcal{D}_k^s(\Gamma, X)$, $D$ has a vertex which is not in $\Gamma$. Therefore $a$ has a vertex outside of $\Gamma$ and we have:




Hence $t$ extends to the module $\mathcal{A}_k^c(\Gamma, X)$ by setting:

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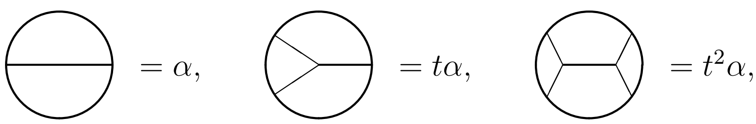
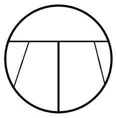
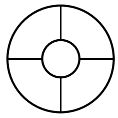
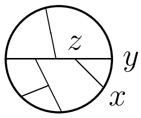
**Example:** The module $\mathcal{P}_k = \mathcal{A}_k^c(S^1) = \mathcal{A}_k^c(S^1, \emptyset)$ which is the module of primitives of the algebra of diagrams $\mathcal{A}_k$ has, in degree $\leq 4$ the following basis:



 $= t^3\alpha$ and 

**Corollary:** *Let $D$ be a planar $(S^1, \emptyset)$-diagram of degree $n$ such that the complement of $S^1$ in $D$ is a tree. Then the class of $D$ in the module $\mathcal{A}_k^c(S^1)$ is exactly $t^{n-1}\alpha$.*

**Proof:** The conditions satisfied by $D$ imply that $D$ contains a triangle $xyz$ with an edge $xy$ in the circle. By taking off the edge $xz$ we get a new diagram $D'$ such that the complement of the circle in $D'$ is still a planar tree. By induction, the class of $D'$ in $\mathcal{A}_k^c(S^1)$ is $t^{n-2}\alpha$ and the result follows.

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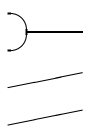
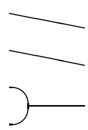
## 3. The algebra $\Lambda$.

In this section we construct an algebra of diagrams acting on many modules of diagrams. In particular this algebra acts in a natural way on the modules $\mathcal{A}_k^s(\Gamma, X)$. Actually the element $t$ is a particular element of $\Lambda$ of degree 1.

The module $F_k(X)$ is equipped with an action of the symmetric group $\mathfrak{S}(X)$. But we can also define natural maps from $F_k(X)$ to $F_k(Y)$ in the following way:

Let $D$ be a $(\emptyset, X \coprod Y)$-diagram such that every connected component of $D$ meets $X$ and $Y$. Then the gluing map along $X$ induces a graded linear map $\varphi_D$ from $F_k(X)$ to $F_k(Y)$. Actually the class $\mathcal{C}$ of $(\emptyset, X \coprod Y)$-diagrams satisfying this property induces a graded module ${}_X\Delta^c_{kY} = \mathcal{A}_k(\mathcal{C})$ and these modules give rise to a monoidal subcategory $\Delta_k^c$ of the category $\Delta_k$. For every finite set $X$ and $Y$ the gluing map is a map from $F_k(X) \otimes {}_X\Delta^c_{kY}$ to $F_k(Y)$.

In particular we have two maps $\varphi$ and $\varphi'$ from $F_k(3)$ to $F_k(4)$ induced by the following diagrams:

**3.1 Definition:** *$\Lambda_k$ is the set of elements $u \in F_k(3)$ satisfying the following conditions:*

$$\varphi(u) = \varphi'(u)$$

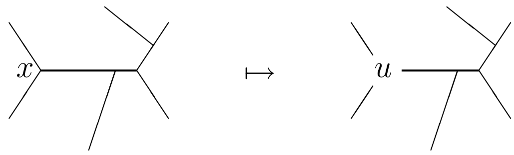
$$\forall \sigma \in \mathfrak{S}_3, \quad \sigma(u) = \varepsilon(\sigma)u$$

*where $\varepsilon$ is the signature homomorphism.*

*The module $\Lambda_{\mathbf{Q}}$ will be denoted by $\Lambda$.*

**3.2 Proposition:** *The module $\Lambda_k$ is a graded $k$-algebra acting on each module $\mathcal{A}_k^s(\Gamma, X)$.*

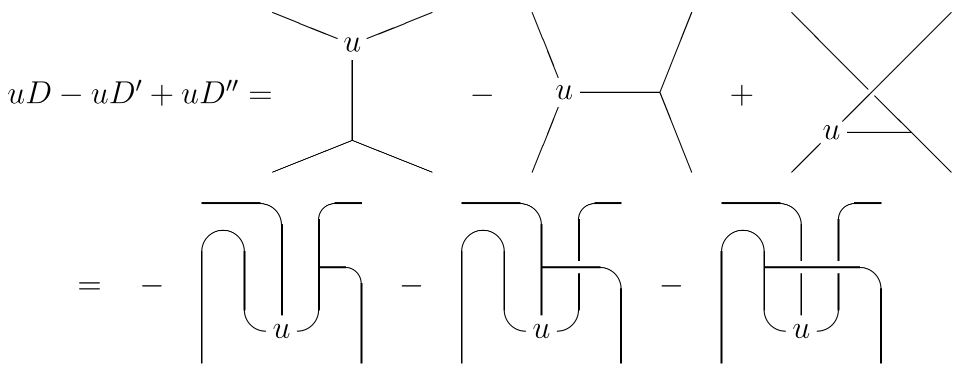
**Proof:** Let $\Gamma$ be a curve and $X$ be a finite set. Let $D$ be a $(\Gamma, X)$-diagram such that $D \setminus \Gamma$ is connected and has some 3-valent vertex $x$. If $u$ is an element of $\Lambda_k$, we can insert $u$ in $D$ near $x$ and we get a linear combination of diagrams and therefore an element $uD$ in $\mathcal{A}_k^s(\Gamma, X)$.




Since $u$ is completely antisymmetric with respect to the $\mathfrak{S}_3$-action, $uD$ doesn't depend on the given bijection from $[3] = \{1, 2, 3\}$ to the set of edges ending at $x$, but only on the cyclic ordering. Moreover, if this cyclic ordering is changed, $uK$ is multiplied by $-1$. The first condition satisfied by $u$ implies that the elements $uK$ constructed by two consecutive vertices are the same. Since the complement of $\Gamma$ in $D$ is connected, $uD$ doesn't depend on the choice of the vertex $x$, and $uD$ is well defined.

By construction, the rule $u \mapsto uD$ is a linear map from $\Lambda_k$ to $\mathcal{A}_k^s(\Gamma, X)$ of degree $\partial^\circ D$. Since the transformation $D \mapsto uD$ is compatible with the AS relations, the only thing to check is to prove that this transformation is compatible with the IHX relations.

Consider an IHX relation: $D \equiv D' - D''$ corresponding to an edge $a$ in $D$. If $D$ has a 3-valent vertex outside of $a$ and $\Gamma$, it is possible to make the transformation $? \mapsto u?$ by using a vertex which is not in $a$, and we get the equality: $uD = uD' - uD''$. Otherwise $a$ is outside of $\Gamma$ and we have:

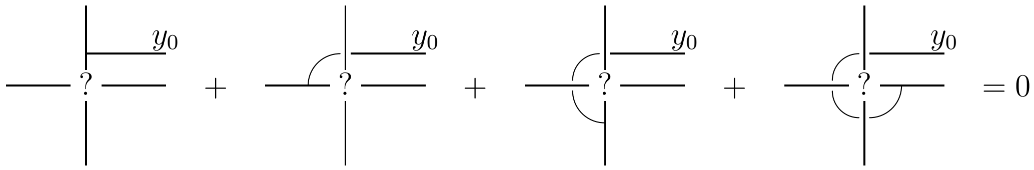



This last expression is trivial, because of Lemma 3.3 and the formula $uD = uD' - uD''$ is always true.

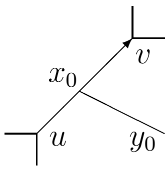
Therefore the transformation $? \mapsto u?$ is compatible with the IHX relation and induces a well-defined transformation from $\mathcal{A}_k^s(\Gamma, X)$ to itself. In particular, $\Lambda_k$ acts on itself. Therefore this module is a $k$-algebra and $\mathcal{A}_k^s(\Gamma, X)$ is a $\Lambda_k$-module. $\square$

**3.3 Lemma:** *Let $X$ be a finite set and $Y$ be the set $X$ with one extra point $y_0$ added. Let $D$ be a connected $(\emptyset, X)$-diagram. For every $x \in X$ denote by $D_x$ the $(\emptyset, Y)$-diagram obtained by adding to $D$ an extra edge from $y_0$ to a point in $D$ near $x$, the cyclic ordering near the new vertex being given by taking the edge ending at $y_0$ first, the edge ending at $x$ after and the last edge at the end.*

*Then the element $\sum\limits_x D_x$ is trivial in the module $F(Y)$.*

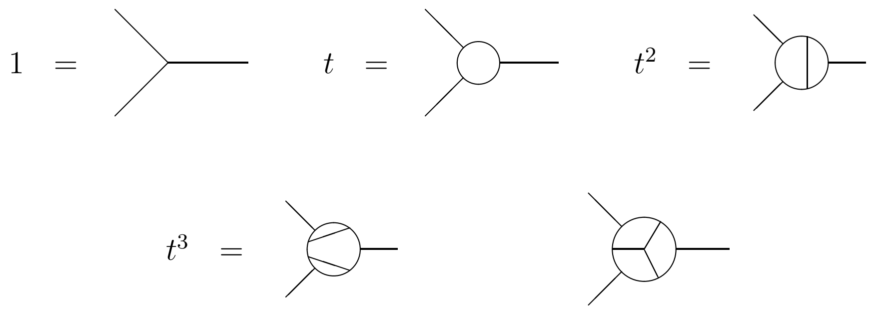



**Proof:** For every oriented edge $a$ in $D$ from a vertex $u$ to a vertex $v$, we can connect $y_0$ to $K$ by adding an extra edge from $y_0$ to a new vertex $x_0$ in $a$ and we get a $(\emptyset, Y)$-diagram $D_a$ where the cyclic ordering between edges ending at $x_0$ is $(x_0u, x_0y_0, x_0v)$.




It is clear that the expression $D_a + D_b$ is trivial if $b$ is the edge $a$ with the opposite orientation. Moreover if $a$, $b$ and $c$ are the three edges starting from a 3-valent vertex of $K$, the sum $D_a + D_b + D_c$ is also trivial. Therefore the sum $\Sigma D_a$ for all oriented edge $a$ in $D$ is trivial and is equal to the sum $\Sigma D_a$ for all oriented edge $a$ starting from a vertex in $X$. That proves the lemma. $\square$

In degree less to 4, the module $\Lambda_k$ is freely generated by the following diagrams:




## 4. Structure of modules $F(n)$ for small values of $n$.

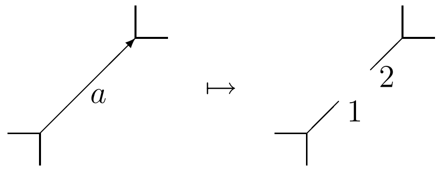
The module $F_k(n)$ is a $\Lambda_k$-module except for $n = 0, 2$. But the submodule $F'_k(n) = \mathcal{A}^s_k(\emptyset, [n])$ of $F_k(n)$ generated by diagrams having at least one 3-valent vertex is a $\Lambda_k$-

module. For $n \neq 0, 2$, $F'_k(n)$ is equal to $F_k(n)$ and for $n = 0, 2$, $F_k(n)$ is isomorphic to $k \oplus F'_k(n)$.

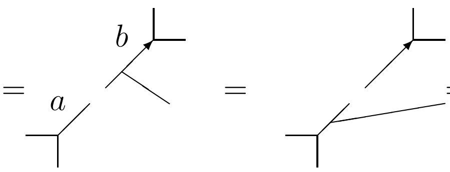
**4.1 Proposition:** *Connecting the elements of* [2] *by an edge induces an isomorphism from* $F_k(2)$ *to* $F_k(0)$.

**Proof:** This map is clearly surjective.

Let $D$ be a connected $(\emptyset, [0])$-diagram. Let $a$ be an oriented edge of $D$. We can cut off a part of $a$ and we get a $(\emptyset, [2])$-diagram $\varphi(D, a)$.



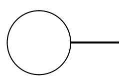

Let $a$ and $b$ be consecutive edges in $D$. Because of Lemma 3.3, we have:

$\varphi(D, a) =$  $= \varphi(D, b)$

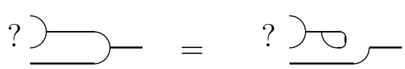

Therefore $\varphi(K, a)$ is independent of the choice of $a$ and induces a well-defined map from $F_k(0)$ to $F_k(2)$ which is obviously the inverse of the map above. $\square$

**4.2 Corollary:** *The action of the symmetric group* $\mathfrak{S}_2$ *on* $F_k(2)$ *is trivial.*

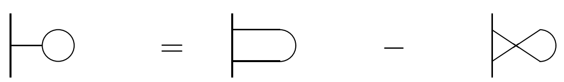
**4.3 Proposition:** *The module* $F_k(1)$ *is isomorphic to* $k/2$ *and generated by the following diagram:*



**Proof:** The diagram above is clearly a generator of $F_k(1)$ in degree 1, and the antisymmetric relation implies that this element is of order 2. Let $D$ be a $(\emptyset, [1])$-diagram of degree $> 1$. We have:

$D =$ 


and this last diagram contains the following diagram:

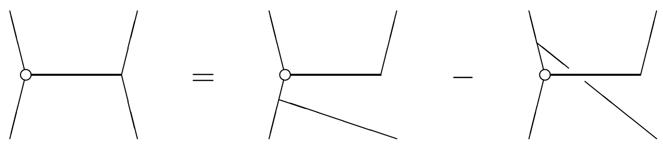
 $= 0$

□

**4.4 Proposition:** *The quotient map from* $[3]$ *to a point induces a surjective map from* $F_k(3) \underset{\mathfrak{S}_3}{\otimes} k^-$ *to* $\mathrm{F}'(0)$ *and its kernel is a* $k/2$*-module.*

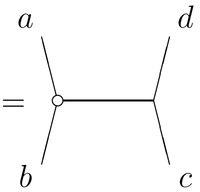
**Proof:** Here the group $\mathfrak{S}_3$ acts on $k = k^-$ via the signature. Actually, the module $F_k(3) \underset{\mathfrak{S}_3}{\otimes} k^-$ is isomorphic to the module $\mathcal{M}$ generated by connected 3-valent diagrams without univalent vertex, pointed by a vertex and equipped with a cyclic ordering near every vertex and where the relations are the AS relation everywhere and the IHS relation outside of the special vertex.

Because of Lemma 3.3, we have in $\mathcal{M}$:



Actually we have for every $n \geq 0$ a module $\widetilde{\mathrm{F}}(n)$ generated by connected diagrams $K$ with $\partial K = [n]$ and pointed by a 3-valent vertex. The relations are the antisymmetric relation AS everywhere and the relation IHX outside of the special vertex and the relation above.

If $\{a, b, c, d\} = [4]$, we can set:

$\varphi(a, b, c, d) =$ 

This diagram belongs to $\widetilde{\mathrm{F}}(4)$ and is antisymmetric with respect to the transpositions $a \leftrightarrow b$ and $c \leftrightarrow d$. Let $k^-$ be the maximal exterior power of the module generated by the elements of $[4]$. Define the element $\psi(a, b, c, d)$ in $k^- \otimes \widetilde{\mathrm{F}}(4)$ by: $\psi(a, b, c, d) = a{\wedge}b{\wedge}c{\wedge}d \otimes \varphi(a, b, c, d)$. By construction $\psi(a, b, c, d)$ depends only on the subset $\{c, d\}$ of $[4]$. So we set: $\psi(a, b, c, d) = f(c, d)$.

The relation obtained by Lemma 3.3 is:

$$\sum_{x \neq a} f(a, x) = 0$$

for every $a$ in $[4]$.

For $\{a, b, c, d\} = [4]$, set: $g(a, b) = f(a, b) - f(c, d)$. We have:

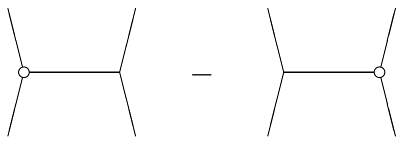
$$f(a, b) + f(a, c) + f(a, d) = 0 = f(b, a) + f(b, c) + f(b, d)$$

$$\Longrightarrow g(a, c) = g(b, c)$$

Then $u = g(a, b)$ doesn't depend on $\{a, b\}$ and we have:

$$u = g(a, b) = f(a, b) - f(c, d) = -g(c, d) = -u.$$

Therefore the diagram




is killed by 2 and invariant under the action of $\mathfrak{S}_4$.

Let $\alpha$ be an element in $F'_k(0)$ represented by a 3-valent diagram $D$. Take a vertex $x_0$ in $D$. The pair $(K, x_0)$ represents a well-defined element $\beta$ in the module $\mathcal{M} \simeq F_k(3) \underset{\mathfrak{S}_3}{\otimes} k^-$ and $2\beta$ doesn't depend on the choice of the vertex $x_0$. Hence the rule $\alpha \mapsto 2\beta$ is a well-defined map $\lambda$ from $F'_k(0)$ to $F_k(3) \underset{\mathfrak{S}_3}{\otimes} k^-$. Denote by $\mu$ the canonical map from $F_k(3) \underset{\mathfrak{S}_3}{\otimes} k^-$ to $F'(0)$. We have:

$$\mu\lambda = 2 \quad \text{and} \quad \lambda\mu = 2$$

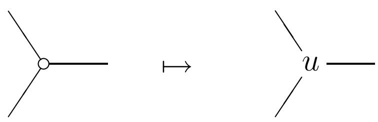
and Proposition 4.4 follows. $\square$

**4.5 Proposition:** *Let $F_k(3)^-$ be the submodule of $F_k(3)$ defined by:*

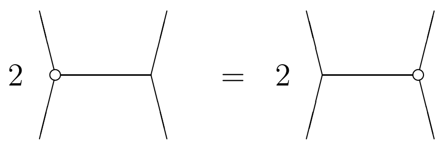
$$\forall u \in F_k(3), \qquad u \in F_k(3)^- \Leftrightarrow (\forall \sigma \in \mathfrak{S}_3, \quad \sigma(u) = \varepsilon(\sigma)u)$$

*where $\varepsilon$ is the signature homomorphism. Then $\Lambda_k$ is a submodule of $F(3)^-_k$ and the quotient $F(3)^-_k/\Lambda_k$ is a $k/2$-module.*

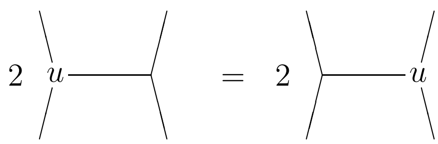
**Proof:** Let $u$ be an element of $F(3)^-_k$. If $v$ is an element of $\widetilde{F}_k(4)$ represented by a diagram $D$ equipped with a special vertex $x_0$, we can insert $u$ in $K$ near $x_0$ and we get a well-defined element $f(v)$ in the module $F_k(4)$.




But we have in $\widetilde{F}_k(4)$:

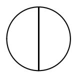



and that implies in $F_k(4)$:

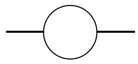


Therefore $2u$ lies in $\Lambda_k$. $\square$

**4.6 Corollary:** *Suppose* 6 *in invertible in* $k$. *Then the modules* $F'_k(0)$ *and* $F'_k(2)$ *are free* $\Lambda_k$*-modules of rank one generated by:*



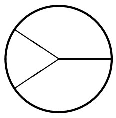
and



*respectively.*

**Proof:** Since $\mathfrak{S}_3$ is a group of order 6, the identity induces an isomorphism from $F_k(3)^-$ to $F_k(3) \underset{\mathfrak{S}_3}{\otimes} k^-$ and the corollary follows easily. $\square$

**4.7 Corollary:** *Let* $u$ *be the primitive element of* $\mathcal{A} = \mathcal{A}(S^1, \emptyset)$ *represented by the diagram:*



*Then the map* $\lambda \mapsto \lambda u$ *from* $\Lambda$ *to the module* $\mathcal{P}$ *of primitives of* $\mathcal{A}$ *is injective.*

**Proof:** That's a consequence of the fact that $\mathcal{P} = \mathcal{P}_{\mathbf{Q}}$ contains the module $F'(2) \underset{\mathfrak{S}_2}{\otimes} \mathbf{Q} = F'(2)$. $\square$

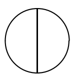
It is not clear that $\Lambda_k$ is commutative, but it's almost the case. If $\alpha$ and $\beta$ are elements in $\Lambda_k$, and $u$ an element of a module $\mathcal{A}^s_k(\Gamma, X)$ represented by a diagram with at least two 3-valent vertices outside of $\Gamma$, we may construct $\alpha\beta u$ by using $\alpha$ and $\beta$ modifications near two different vertices. Therefore: $\alpha\beta u = \beta\alpha u$.

**4.8 Proposition:** *The algebra* $\Lambda_k$ *has the following properties:*

$$\forall \alpha, \beta, \gamma \in \Lambda_k, \qquad \partial^\circ \gamma > 0 \Rightarrow \alpha\beta\gamma = \beta\alpha\gamma,$$

$$\forall \alpha, \beta \in \Lambda_k, \qquad 12\alpha\beta = 12\beta\alpha.$$

**Proof:** The first formula is a special case of the property explained above. For the second one, just use that property where $u$ is represented by the diagram

$\Theta =$ 

in $F'_k(0)$ and remark that the composite: $\Lambda_k \to F_k(3) \underset{\mathfrak{S}_3}{\otimes} k^- \to F'_k(0)$ has a kernel annihilated by $6 \times 2 = 12$. $\square$

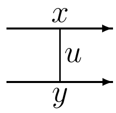
**4.9 Corollary:** *The algebra* $\Lambda$ *is commutative.*

**4.10 Proposition:** *Let* $\widehat{\Lambda}$ *be the algebra* $\Lambda$ *completed by the degree (i.e.* $\widehat{\Lambda} = \prod_i \Lambda_i$*). Let* $M$ *be a* 3*-dimensional homology sphere. Then there is a unique element* $\theta(M)$ *in* $\widehat{\Lambda}$ *such that the LMO invariant of* $M$ *is the exponential of the element* $\theta(M)\Theta$.

**Proof:** Let $u$ be the LMO-invariant of $M$ constructed by Le-Murakami-Ohtsuki [LMO]. Then $u$ is a group-like element in the completion of the module generated by 3-valent diagrams. Therefore its logarithm is primitive and lies in the completion of the module $F'(0)$. Since this module is a free $\widehat{\Lambda}$-module generated by $\Theta$ the result follows. $\square$

## 5. Constructing elements in $\Lambda$.

Let $\Gamma$ be a curve and $Z$ be a finite set. Let $D$ be a $(\Gamma, Z)$-diagram. Let $X$ be a finite set in $D$ outside the set of vertices of $D$. Suppose that $D$ is oriented near $X$. For each $x \neq y$ in $X$ we have a diagram $D_{xy}$ obtained from $D$ by adding an edge $u$ joining $x$ and $y$ in $D$. Cyclic orderings near $x$ and $y$ are chosen by an immersion from $D_{xy}$ to the plane which is injective on a neighborhood of $u$ and sends neighborhoods of $x$ and $y$ in $K$ to horizontal lines with the same orientation and $u$ to a vertical segment. This diagram $D_{xy}$ depends only on the subset $\{x, y\}$ in $X$.




The sum of the diagrams $D_{xy}$ for all subsets $\{x, y\} \subset X$ will be denoted by $D_X$.

**5.1 Lemma:** *Let* $\Gamma$ *and* $\Gamma'$ *be closed curves. Let* $X$, $Y$ *and* $Z$ *be finite disjoint sets. Let* $D$ *be a* $(\Gamma, X \cup Y)$*-diagram and* $D'$ *be a* $(\Gamma', X \cup Y \cup Z)$ *diagram. Suppose that the union* $H$ *of* $D$ *and* $D'$ *over* $X \cup Y$ *lies in* $\mathcal{D}^s(\Gamma \cup \Gamma', Z)$*. The diagram* $H$ *is oriented near* $X$ *and* $Y$ *by going from* $D'$ *to* $D$ *near* $X$ *and from* $D$ *to* $D'$ *near* $Y$*. Then we have the following formula in* $\mathcal{A}_k^s(\Gamma \cup \Gamma', Z)$*:*

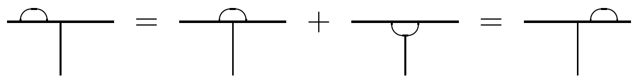
$$H_X - ptH = H_Y - qtH$$

*where* $p = \#X$, $q = \#Y$.

**Proof:** Set $\Gamma_1 = \Gamma \cup \Gamma'$. Let $u$ be a point in $H$ which is not a vertex. By adding one edge to $H$ near $u$ we get a new diagram $H_u$:


The class $[H_u]$ of $H_u$ in $\mathcal{A}_k^s(\Gamma_1, Z)$ will be denoted by $\varphi(u)$. If $u$ is not in $\Gamma_1$, $\varphi(u)$ is equal to $2t[H]$. Otherwise $\varphi(u)$ depends only on the component of $\Gamma_1$ which contains

$u$:



Consider a map $f$ from $H$ to the circle $S^1 = \mathbf{R} \cup \{\infty\}$ satisfying the following:

— $f$ is smooth and generic on $\Gamma_1$ and on each edge of $K$

— every singular value of $f$ is the image of a unique critical point in an open edge of $H \setminus \Gamma_1$ or a unique vertex of $H$

— a vertex in $H$ is never a local extremum of $f$

— each critical point of $f|\Gamma_1$ is not a vertex of $H$

— $f^{-1}(0) = X, \qquad f^{-1}(1) = Y, \qquad f^{-1}([0,1]) = D.$

Let $v$ be a regular value of $f$ and $V$ be the set $f^{-1}(v)$. The map $f$ induces an orientation of $H$ near each point of $V$. So $[H]_V$ is well defined in $\mathcal{A}_k^s(\Gamma_1, Z)$ and we can set:

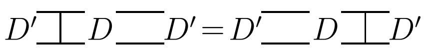
$$g(v) = [H_V] - 1/2 \sum_{u \in V} \varphi(u).$$

This expression is well defined because $V$ meets every component of $\Gamma_1$ in a even number of points.

By construction we have: $g(v) = [H_X] - pt[H]$ if $v$ is near 0 and $g(v) = [H_Y] - qt[H]$ if $v$ is near 1. Then the last thing to do is to prove that $g$ has no jump on the critical values of $f$.

If $v$ is the image of a critical point in an open edge in $H$, the jump of $f$ in $v$ is 0 because of the AS relations. If $v$ is the image of a vertex in $H$, the jump is also 0 because of the IHX relations. Therefore the map $g$ is constant and the lemma is proven. $\square$

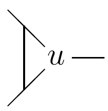
A special case of this lemma is the following equality:



in $\mathcal{A}_k^s(\Gamma, Z)$.

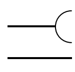
**5.2 Corollary:** *The element $t$ is central in $\Lambda_k$.*

**Proof:** For every $u \in \Lambda_k$, we have:

$ut =$  $= tu.$

$\square$

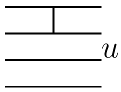
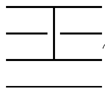
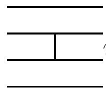
Let $\Gamma_4$ be the normal subgroup of order 4 of $\mathfrak{S}_4$. Consider the element $\delta \in {}_3\Delta_{k4}$ represented by the following diagram:



By gluing from the left or the right, we get a map $u \mapsto u\delta$ from $F_k(3)$ to $F_k(4)$ or a map $u \mapsto \delta u$ from $F_k(4)$ to $F_k(3)$. Denote by $E$ the submodule of $F_k(4)$ of all elements $u \in F_k(4)$ satisfying the following conditions:

$$\forall \sigma \in \mathfrak{S}_4, \quad \delta\sigma u \ \in \ \Lambda_k \qquad \text{and} \qquad \forall \sigma \in \Gamma_4, \quad \sigma u = u.$$

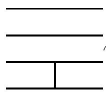
For every $u \in F_k(4)$, define elements $xu, yu, zu$ by:

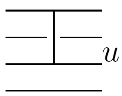
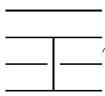
$xu =$  $u$ $\qquad$ $yu =$  $u$ $\qquad$ $zu =$  $u$

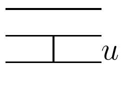
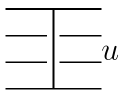
**5.3 Proposition:** *The module $E$ is a graded $\Lambda_k[\mathfrak{S}_4]$-submodule of $F_k(4)$ and for every $u \in E$ we have:*

$$xu, yu, zu \in E, \qquad xu + yu + zu = 2tu.$$

**Proof:** The fact that $E$ is a graded $\Lambda_k[\mathfrak{S}_4]$-submodule of $F_k(4)$ is obvious. Let $u$ be an element of $F_k(4)$. Because of Lemma 5.1, we have:

$xu =$  $u =$  $u$

$yu =$  $u =$  $u$

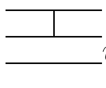
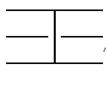
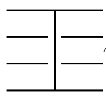
$zu =$  $u =$  $u$

Hence, if $\sigma$ is a permutation in $\mathfrak{S}_4$, there exists an element $\theta \in \{x, y, z\}$ such that $\sigma x u = \theta \sigma u$. More precisely $\mathfrak{S}_4$ acts on the set $\{x, y, z\}$ via an epimorphism $\sigma \mapsto \widehat{\sigma}$ from $\mathfrak{S}_4$ to $\mathfrak{S}_3$, and we have:

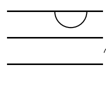
$$\sigma x u = \widehat{\sigma}(x)\sigma u, \qquad \sigma y u = \widehat{\sigma}(y)\sigma u, \qquad \sigma z u = \widehat{\sigma}(z)\sigma u.$$

The kernel of this epimorphism is $\Gamma_4$.

We have:

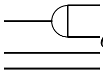
$xu + yu + zu =$  $u +$  $u +$  $u.$

Because of Lemma 3.3, we have:

$xu + yu + zu =$  $u = 2tu.$

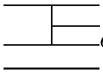
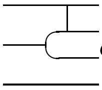
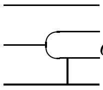
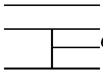
Moreover, if $u \in F_k(4)$ is $\Gamma_4$-invariant, $xu, yu, zu$ are $\Gamma_4$-invariant too, and the last thing to do is to prove that $\delta x\sigma u, \delta y\sigma u, \delta z\sigma u$ are in $\Lambda_k$ for every $u \in E$.

We have:

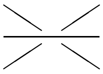
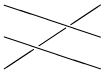
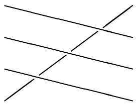
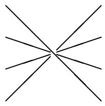
$\delta x\sigma u =$  $\sigma u = t\delta\sigma u \ \in \ \Lambda_k$

$$\delta y\sigma u = 2t\delta\sigma u - \delta x\sigma u - \delta z\sigma u$$

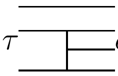
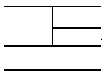
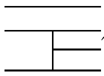
and it is enough to prove that $\delta z\sigma u$ belongs to $\Lambda_k$. Because of Lemma 5.1 we have:

$\delta z\sigma u =$  $\sigma u =$  $\sigma u =$  $\sigma u =$  $\sigma u.$

Let $s$, $\tau$, $\tau'$, $\theta$ be the permutations in $\mathfrak{S}_4$ or $\mathfrak{S}_3$ represented by the following diagrams:

$s =$  $\tau =$  $\tau' =$  $\theta =$ 

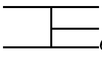
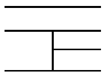
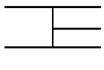
We have:

$\tau\delta z\sigma u = \tau$  $\sigma u =$  $\tau'\sigma u =$  $\tau'\sigma u$

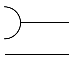
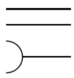
and then:

$$\tau\delta z\sigma u = \delta z\tau'\sigma u \qquad \Rightarrow \qquad \tau^2\delta z\sigma u = \delta z\tau'^2\sigma u.$$

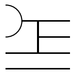
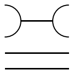
But $\tau'^2$ lies in $\Gamma_4$ and $\tau^2\delta z\sigma u = \delta z\sigma u$. Therefore $\delta z\sigma u$ is invariant under cyclic permutations. We have also:

$s\delta z\sigma u = s$  $\sigma u = -$  $\theta\sigma u = -$  $\theta\sigma u.$

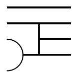
Since $\theta$ lies in $\Gamma_4$ also, $s\delta z\sigma u = -\delta z\sigma u$ and $\delta z\sigma u$ belongs to the submodule $F_k(3)^-$ of $F_k(3)$. Consider the following diagrams:

$\delta' =$  $\delta'' =$ 

We have to prove the last equality: $\delta'\delta z\sigma u = \delta''\delta z\sigma u$. Denote by $\sigma_{ij}$ the transposition $i \leftrightarrow j$. We have:

$\delta'\delta z\sigma u =$  $\sigma u =$  $z\sigma u = (1 - \sigma_{12})xz\sigma u$

and similarly:

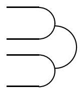
$$\delta''\delta z\sigma u = $$  $$\sigma u = (1-\sigma_{34})xz\sigma u.$$

But $\sigma_{12}$ and $\sigma_{34}$ are the same modulo $\Gamma_4$ and induce the transposition $y \leftrightarrow z$. Then we have:

$$\delta''\delta z\sigma u = xz\sigma u - xy\sigma_{34}\sigma u = xz\sigma u - xy\sigma_{12}\sigma u = \delta'\delta z\sigma u$$

and that finishes the proof. $\square$

Consider the following element of $F_k(4)$:

$$a = $$ 

For every $p > 0$ set: $x_p = \delta z^{p-1}a$. Because of the last result, $x_p$ is an element of degree $p$ in $\Lambda_k$. It is not difficult to check the following:

$$x_1 = 2t \qquad x_2 = t^2 \qquad 3x_4 = 4tx_3 + t^4$$

and $\Lambda_k$ is freely generated in degree $< 6$ by:

$$1,\ t,\ t^2,\ t^3,\ t^4,\ t^5,\ x_3,\ \frac{tx_3 - t^4}{3},\ \frac{t^2x_3 - t^5}{3},\ \frac{x_5 + t^2x_3}{2}.$$

Let $\tau$ be a permutation in $\mathfrak{S}_4$ inducing the cyclic permutation $x \mapsto y \mapsto z \mapsto x$. Set: $z_1 = x$, $z_2 = y$, $z_3 = z$, $\alpha_1 = a$, $\alpha_2 = \tau a$, $\alpha_3 = \tau^2 a$. The group $\mathfrak{S}_3$ acts on $E$ and for every $\sigma \in \mathfrak{S}_3$, every $i \in \{1,2,3\}$ and every $u \in E$ we have:

$$\sigma(z_i u) = z_{\sigma(i)}\sigma(u)$$

$$\sigma(\alpha_i) = \varepsilon(\sigma)\alpha_{\sigma(i)}$$

where $\varepsilon(\sigma)$ is the signature of $\sigma$. Denote also by $f_1$ the morphism $u \mapsto \delta u$ from $E$ to $\Lambda_k$. If $\sigma$ is the transposition keeping 1 fixed, on has for every $u \in E$:

$$f_1(\sigma(u)) = -f_1(u).$$

Therefore there are unique morphisms $f_2$ and $f_3$ from $E$ to $\Lambda_k$ such that:

$$f_{\sigma(i)}(\sigma(u)) = \varepsilon(\sigma)f_i(u)$$

for every $u \in E$, $\sigma \in \mathfrak{S}_3$ and $i \in \{1,2,3\}$. Moreover, if $\sigma$ is the transposition keeping $i$ fixed we have:

$$z_i(u - \sigma(u)) = f_i(u)\alpha_i$$

for every $u \in E$.

The set $\{1,2,3\}$ in canonically oriented and for every $i$, $j$ and $k$ distinct in $\{1,2,3\}$, there is a sign $i{\wedge}j{\wedge}k$ in $\{\pm 1\}$: the signature of the permutation $1 \mapsto i$, $2 \mapsto j$, $3 \mapsto k$.

**5.4 Proposition:** *Suppose 6 is invertible in $k$. Then there exist unique elements $e$, $\varepsilon_p$ and $\beta_{i,p}$ in $E$, for $i \in \{1,2,3\}$ and $p \geq 0$ and unique elements $\omega_p$ ($p \geq 0$) in $\Lambda_k$ such that the following holds for every $\sigma \in \mathfrak{S}_3$, every $i,j,k$ distinct in $\{1,2,3\}$ and every $p \geq 0$:*

$$\beta_{1,p} + \beta_{2,p} + \beta_{3,p} = 0$$

$$\sigma(e) = e, \quad \sigma(\varepsilon_p) = \varepsilon_p, \quad \sigma(\beta_{i,p}) = \varepsilon(\sigma)\beta_{\sigma(i),p}$$

$$f_i(\alpha_i) = 2t, \quad f_i(\beta_{i,p}) = 2\omega_p$$

$$f_i(\alpha_j) = -t, \quad f_i(\beta_{j,p}) = -\omega_p$$

$$f_i(e) = f_i(\varepsilon_p) = 0$$

$$z_i\alpha_i = t\alpha_i, \quad z_i\beta_{i,p} = \omega_p\alpha_i$$

$$z_i\alpha_j = i{\wedge}j{\wedge}k\ e + \frac{t}{3}(\alpha_j - \alpha_i)$$

$$z_i e = \frac{2t}{3}e + i{\wedge}j{\wedge}k\Big(\frac{10t^2}{9}(\alpha_j - \alpha_k) - \frac{1}{2}(\beta_{j,0} - \beta_{k,0})\Big)$$

$$z_i\beta_{j,p} = i{\wedge}j{\wedge}k\ \varepsilon_p + \frac{2t}{3}(\beta_{j,p} - \beta_{k,p}) + \omega_p\alpha_k$$

$$z_i\varepsilon_p = \frac{2t}{3}\varepsilon_p + i{\wedge}j{\wedge}k\Big(\frac{4t^2}{9}(\beta_{j,p} - \beta_{k,p}) - \frac{1}{2}(\beta_{j,p+1} - \beta_{k,p+1}) + \frac{2t\omega_p}{3}(\alpha_j - \alpha_k)\Big).$$

**Proof:** Consider formal elements $\omega'_p$, for $p \geq 0$ of degree $3+2p$. Then $R = k[t, \omega'_0, \omega'_1, \ldots]$ is a graded algebra. Let $E'$ be the $R$-module generated by elements $\alpha'_i$, $\beta'_{i,p}$, $e'$ and $\varepsilon'_p$ (for $p \geq 0$ and $i \in \{1,2,3\}$) with the following relations:

$$\sum_i \alpha'_i = 0, \qquad \forall p \geq 0,\ \sum_i \beta'_{i,p} = 0.$$

This module is graded by:

$$\partial^\circ \alpha'_i = 0, \quad \partial^\circ \beta'_{i,p} = 2+2p, \quad \partial^\circ e' = 1, \quad \partial^\circ \varepsilon'_p = 3+2p.$$

The symmetric group $\mathfrak{S}_3$ acts on $E'$ by:

$$\sigma(\alpha'_i) = \varepsilon(\sigma)\alpha'_{\sigma(i)}, \quad \sigma(\beta'_{i,p}) = \varepsilon(\sigma)\beta'_{\sigma(i),p}, \quad \sigma(e') = e', \quad \sigma(\varepsilon'_p) = \varepsilon'_p$$

and $E'$ is a graded $R[\mathfrak{S}_3]$-module.

Using relations above we have well-defined maps $u \mapsto z_i u$ from $E'$ to $E'$ and the sum of these maps is $2t$. We have also linear maps $f_i$ from $E'$ to $R$ sending $e'$ and $\varepsilon'_p$ to 0 and defined on the other generators by:

$$f_i(\alpha'_i) = 2t \quad f_i(\beta'_{i,p}) = 2\omega'_p$$

$$f_i(\alpha'_j) = -t \quad f_i(\beta'_{j,p}) = -\omega'_p$$

It is not difficult to check the formula:

$$\forall u \in E', \quad z_i(u - \sigma(u)) = f_i(u)\alpha'_i$$

where $\sigma$ is the transposition keeping $i$ fixed. The last thing to do is to construct an algebra homomorphism $\psi$ from $R$ to $\Lambda_k$ and a morphism $\varphi$ from $E'$ to $E$ which is linear over $\psi$ sending $\alpha'_i$ to $\alpha_i$ and $z_i$ to $z_i$.

Consider the elements $u(i,j,k) = z_i\alpha_j - t/3\alpha_j + t/3\alpha_i$ in $E$ (for $i$, $j$, $k$ distinct). One has:

$$u(i,j,k) - u(j,k,i) = z_i\alpha_j - z_j\alpha_k - t/3(\alpha_j - \alpha_i - \alpha_k + \alpha_j) = z_i\alpha_j - z_j\alpha_k - t\alpha_j$$

$$= z_i\alpha_j + (z_i + z_k - 2t)\alpha_k - t\alpha_j = z_i(\alpha_j + \alpha_k) + z_k\alpha_k - 2t\alpha_k - t\alpha_j$$

$$= -z_i\alpha_i + z_k\alpha_k + t\alpha_i - t\alpha_k = 0.$$

Then $u(i,j,k)$ is invariant under cyclic permutations. One has also:

$$u(i,j,k) + u(k,j,i) = (z_i + z_k)\alpha_j - 2t/3\alpha_j + t/3(\alpha_i + \alpha_k) = (2t - z_j)\alpha_j - t\alpha_j = 0.$$

Therefore $u(i,j,k)$ is totally antisymmetric in $i$, $j$, $k$ and $i{\wedge}j{\wedge}k\ u(i,j,k)$ is invariant under the action of $\mathfrak{S}_3$. So one can set:

$$e = \varphi(e') = i{\wedge}j{\wedge}k\ u(i,j,k).$$

The element $v(i,j,k) = i{\wedge}j{\wedge}k(z_je - z_ke)$ is clearly symmetric under the transposition $j \leftrightarrow k$. So it depends only on $i$ and we can set:

$$\beta_{i,0} = \frac{20t^2}{9}\alpha_i + \frac{2}{3}v(i,j,k).$$

Hence we have:

$$z_ie = \frac{1}{3}(2z_i - z_j - z_k + 2t)e = \frac{2t}{3}e + \frac{i{\wedge}j{\wedge}k}{3}(v(k,i,j) - v(j,k,i))$$

$$= \frac{2t}{3}e + i{\wedge}j{\wedge}k\Big(\frac{10t^2}{9}(\alpha_j - \alpha_k) - \frac{1}{2}(\beta_{j,0} - \beta_{k,0})\Big).$$

It is easy to see that the sum of the $\beta_{i,0}$ vanishes and we can set: $\varphi(\beta'_{i,0}) = \beta_{i,0}$. On the other hand we have:

$$f_i(-\beta_{k,0}) = -f_i(\beta_{j,0})$$

and $f_i(\beta_{j,0})$ depends only on $i$. But we have: $f_i(\beta_{j,0}) = f_j(\beta_{k,0})$ and $f_i(\beta_{j,0})$ doesn't depend on $i$. So we can set: $\omega_0 = -f_i(\beta_{j,0})$. Since $\beta_{i,0} + \beta_{j,0} + \beta_{k,0}$ is trivial, we have also: $f_i(\beta_{i,0}) = 2\omega_0$ and we can set: $\psi(\omega'_0) = \omega_0$.

Set: $w(i,j,k) = z_i\beta_{j,0} - \frac{2t}{3}(\beta_{j,0} - \beta_{k,0}) - \omega_0\alpha_k$. One has:

$$w(i,j,k) - w(j,k,i) = z_i\beta_{j,0} - z_j\beta_{k,0} - \frac{2t}{3}(-3\beta_{k,0}) - \omega_0(\alpha_k - \alpha_i)$$

$$= z_i\beta_{j,0} - (2t - z_i - z_k)\beta_{k,0} + 2t\beta_{k,0} - \omega_0(\alpha_k - \alpha_i)$$
$$= z_i(\beta_{j,0} + \beta_{k,0}) + z_k\beta_{k,0} - \omega_0(\alpha_k - \alpha_i)$$
$$= f_i(\beta_{j,0})\alpha_i + 1/2 f_k(\beta_{k,0})\alpha_k - \omega_0(\alpha_k - \alpha_i) = 0.$$

Then $w(i,j,k)$ is invariant under cyclic permutations. One has also:

$$w(i,j,k) + w(k,j,i) = z_i\beta_{j,0} + z_k\beta_{j,0} - \frac{2t}{3}(3\beta_{j,0}) - \omega_0(\alpha_k + \alpha_i)$$
$$= (2t - z_j)\beta_{j,0} - 2t\beta_{j,0} + \omega_0\alpha_j = -z_j\beta_{j,0} + \omega_0\alpha_j = 0.$$

Therefore $w(i,j,k)$ is totally antisymmetric in $i$, $j$, $k$ and $i{\wedge}j{\wedge}k\ w(i,j,k)$ is invariant under the action of $\mathfrak{S}_3$. So one can set:

$$\varepsilon_0 = \varphi(\varepsilon_0') = i{\wedge}j{\wedge}k\ w(i,j,k).$$

Let $p \geq 0$ be en integer. Suppose that $\beta_{i,q}$ and $\varepsilon_q$ are constructed for $q \leq p$ and $\varphi$ and $\psi$ are constructed in degree $\leq 3 + 2p$. Consider the element $u(i,j,k) = i{\wedge}j{\wedge}k(z_j - z_k)\varepsilon_p + \frac{4t^2}{3}\beta_{i,p} + 2t\omega_p\alpha_i$. This element is invariant under the transposition $j \leftrightarrow k$ and depends only on $i$. So we can set:

$$\beta_{i,p+1} = \frac{2}{3}u(i,j,k).$$

It is easy to check the following:

$$\beta_{1,p+1} + \beta_{2,p+1} + \beta_{3,p+1} = 0$$

$$z_i\varepsilon_p = \frac{2t}{3}\varepsilon_p + i{\wedge}j{\wedge}k\Big(\frac{4t^2}{9}(\beta_{j,p} - \beta_{k,p}) - \frac{1}{2}(\beta_{j,p+1} - \beta_{k,p+1}) + \frac{2t\omega_p}{3}(\alpha_j - \alpha_k)\Big)$$

and we can set: $\varphi(\beta'_{i,p+1}) = \beta_{i,p+1}$. On the other hand we have:

$$f_i(-\beta_{k,p+1}) = -f_i(\beta_{j,p+1})$$

and $f_i(\beta_{j,p+1})$ depends only on $i$. But we have: $f_i(\beta_{j,p+1}) = f_j(\beta_{k,p+1})$ and $f_i(\beta_{j,p+1})$ doesn't depend on $i$. So we can set: $\omega_{p+1} = -f_i(\beta_{j,p+1})$. Since $\beta_{i,p+1} + \beta_{j,p+1} + \beta_{k,p+1}$ is trivial, we have also: $f_i(\beta_{i,p+1}) = 2\omega_{p+1}$ and we can set: $\psi(\omega'_{p+1}) = \omega_{p+1}$.

Set: $w(i,j,k) = z_i\beta_{j,p+1} - \frac{2t}{3}(\beta_{j,p+1} - \beta_{k,p+1}) - \omega_{p+1}\alpha_k$. One has:

$$w(i,j,k) - w(j,k,i) = z_i\beta_{j,p+1} - z_j\beta_{k,p+1} - \frac{2t}{3}(-3\beta_{k,p+1}) - \omega_{p+1}(\alpha_k - \alpha_i)$$
$$= z_i\beta_{j,p+1} - (2t - z_i - z_k)\beta_{k,p+1} + 2t\beta_{k,p+1} - \omega_{p+1}(\alpha_k - \alpha_i)$$
$$= z_i(\beta_{j,p+1} + \beta_{k,p+1}) + z_k\beta_{k,p+1} - \omega_{p+1}(\alpha_k - \alpha_i)$$
$$= f_i(\beta_{j,p+1})\alpha_i + 1/2 f_k(\beta_{k,p+1})\alpha_k - \omega_{p+1}(\alpha_k - \alpha_i) = 0.$$

Then $w(i,j,k)$ is invariant under cyclic permutations. One has also:

$$w(i,j,k) + w(k,j,i) = z_i\beta_{j,p+1} + z_k\beta_{j,p+1} - \frac{2t}{3}(3\beta_{j,p+1}) - \omega_{p+1}(\alpha_k + \alpha_i)$$

$$= (2t - z_j)\beta_{j,p+1} - 2t\beta_{j,p+1} + \omega_{p+1}\alpha_j = -z_j\beta_{j,p+1} + \omega_{p+1}\alpha_j = 0.$$

Therefore $w(i,j,k)$ is totally antisymmetric in $i$, $j$, $k$ and $i{\wedge}j{\wedge}k\ w(i,j,k)$ is invariant under the action of $\mathfrak{S}_3$. So one can set:

$$\varepsilon_{p+1} = \varphi(\varepsilon'_{p+1}) = i{\wedge}j{\wedge}k\ w(i,j,k).$$

So $\varphi$ and $\psi$ are defined by induction and the result follows. $\square$

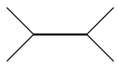
**Remark:** The subalgebra $\Lambda'$ of $\Lambda_k$ generated by the $x_i$'s is generated by $x_1, x_3, x_5, \ldots$ and also by $t, \omega_0, \omega_1, \ldots$. Then every $x_i$ can be expressed in term of $t$ and the $\omega_j$'s. In low degree we get:

$$x_1 = 2t, \quad x_2 = t^2, \quad x_3 = 4t^3 - \frac{3}{2}\omega_0, \quad x_4 = 5t^4 - 2t\omega_0,$$

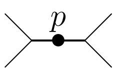
$$x_5 = 12t^5 - \frac{17}{2}t^2\omega_0 + \frac{3}{2}\omega_1, \quad x_6 = 21t^6 - 17t^3\omega_0 + 5t\omega_1 - \frac{3}{2}\omega_0^2,$$

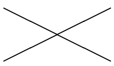
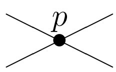
$$x_7 = 44t^7 - \frac{91}{2}t^4\omega_0 - \frac{7}{2}t\omega_0^2 + \frac{37}{2}t^2\omega_1 - \frac{3}{2}\omega_2.$$

Suppose that $\alpha_i \in E$ is represented by:

$\alpha_i =$ 

Then we set:



$= \beta_{i,p}$,  $= e$, 

$= \varepsilon_p$.

These diagrams are well defined in $F_k(4)$ if 6 is invertible in $k$. By gluing we are able to define new $(\Gamma, X)$-diagrams represented by a graph $D$ containing $\Gamma$ such that:

— the set $\partial D$ of 1-valent vertices of $D$ is the disjoint union of $\partial\Gamma$ and $X$

— each vertex of $D$ in $\Gamma \setminus \partial\Gamma$ is 3-valent

— each vertex of $D$ is 1-valent, 3-valent, or 4-valent

— each 3-valent vertex of $K$ is oriented (by a cyclic ordering)

— some 4-valent vertex is marked by a bullet and labeled by a nonnegative integer

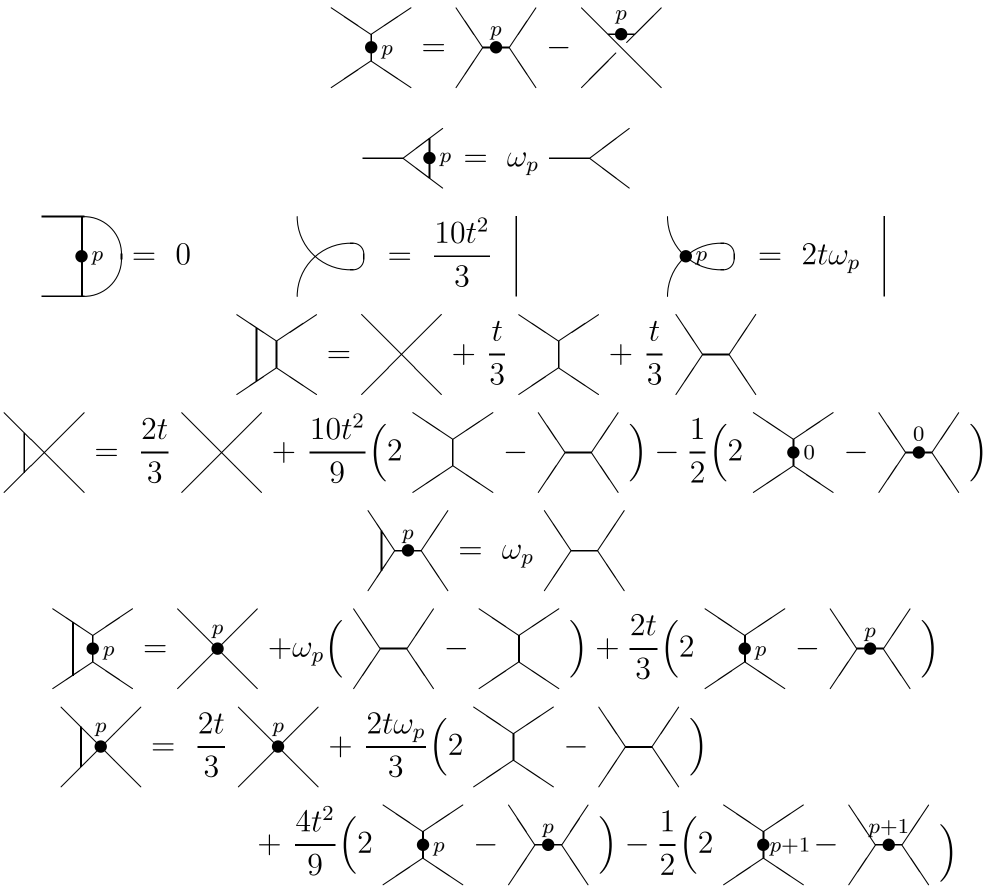
— some edge is marked by a bullet and labeled by a nonnegative integer

— each marked edge is outside of $\Gamma$ and its boundary has two 3-valent vertices

— the marked edges are pairwise disjoint.

Such a diagram will be called an extended $(\Gamma, X)$-diagram. Each extended $(\Gamma, X)$-diagram is a linear combination of usual $(\Gamma, X)$-diagrams. A marked diagram $D$ is an extended diagram with at least one marqued vertex. The sum of the markings is called the total marking of $D$.

**5.5 Proposition:** *Suppose* 6 *is invertible in* $k$*. Then we have the following formulas:*




*for every* $p \geq 0$.

**Proof:** This is essentially a graphical version of Proposition 5.4. $\square$

There are many relations in the algebra $\Lambda$. Kneissler [Kn] founded relations in term if the $x_i$'s. In term of the $\omega_i$'s Kneissler's result becomes the following:

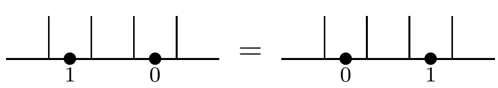
**5.6 Theorem:** *The following relations hold in* $\Lambda$*:*

$$\forall p, q \geq 0, \quad \omega_p \omega_q = \omega_0 \omega_{p+q}.$$

**5.7 Theorem:** *Let* $\Gamma$ *be a closed curve and* $X$ *be a finite set. Let* $u$ *be an element of* $\mathcal{A}^s(\Gamma, X)$ *represented by a marked diagram* $D$ *with total marking* $p$*. Let* $D_0$ *be the diagram* $D$ *where each marking is replaced by* 0*. Then* $u$ *depends only on* $p$ *and* $D_0$*. Moreover* $\omega_q u$ *depends only on* $p+q$ *and* $D_0$*.*

**Proof:** Here we are working over the rationals ($k = \mathbf{Q}$).

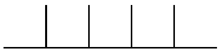
**5.7.1 Lemma:** *The following relation holds in* $F(6)$*:*

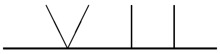


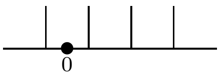
**Proof:** Let $E_n$ be the component of $F(6)$ of degree $n$. These modules can be determined by computer for $n \leq 6$. In this range the dimensions are:

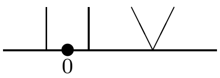
$$24 \quad 60 \quad 120 \quad 199 \quad 309 \quad 439 \quad 594$$

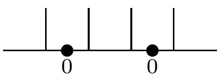
The desired relation lies in the module $E_6$ and can be checked directly. More precisely, $E_n$ has a decomposition in a direct sum of pieces corresponding to the Young diagrams of size 6. Using this decomposition and formulas in Proposition 5.5 we get:

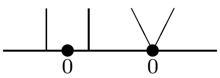
 $= A_0(4,2) + A_0(2,2,2) + A_0(3,1,1,1)$

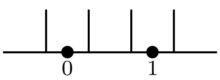
 $= A_1(4,2) + A_1(3,2,1)$

 $= A_2(4,2) + A_2(2,2,2) + A_2(3,1,1,1) + A_2(3,2,1)$

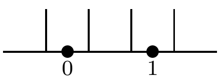
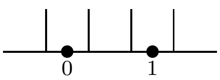
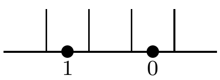
 $= A_3(4,2) + A_3(3,2,1) + A_3(5,1)$

 $= A_4(4,2) + A_4(2,2,2) + A_4(3,1,1,1) + A_4(3,2,1)$

 $= A_5(4,2) + A_5(5,1) + A_5(3,2,1)$

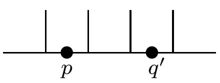
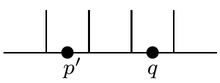
 $= A_6(4,2) + A_6(2,2,2) + A_6(3,1,1,1) + A_6(3,2,1)$.

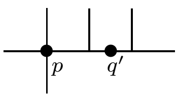
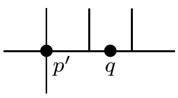
It is not difficult to see that the symmetry $\sigma$ along a vertical axis acts trivially on $A_6(4,2)$, $A_6(2,2,2)$, $A_6(3,1,1,1)$, $A_6(3,2,1)$ and then on the last diagram. So we have:

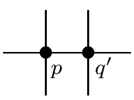
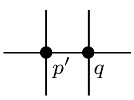
 $= \sigma$  $=$ 

and that proves the lemma. $\square$

**5.7.2 Lemma:** *For every $p$ and $q$ we have the following relations in $F(6)$:*

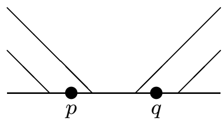
 $=$ 

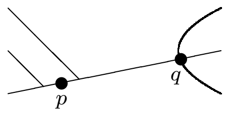
 $=$ 

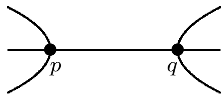
 $=$ 

*with $p' = p+1$ and $q' = q+1$.*

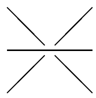
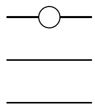
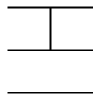
**Proof:** Consider the following diagrams:

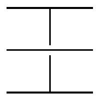
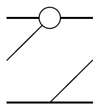
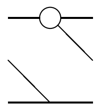
$A(p,q) =$ 


$B(p,q) =$ 


$C(p,q) =$ 


These diagrams are morphisms in the category $\Delta$.
Consider the following morphisms in this category:

$\tau =$  $\theta =$  $\alpha =$ 

$\beta =$  $u =$  $v =$ 

Then we set:

$$f = (\frac{1}{2}\alpha - \frac{1}{6}\theta) \circ (1 + \tau)$$

$$g = (1 + \tau) \circ (\frac{1}{2}\alpha - \frac{1}{6}\theta)$$

$$\varphi = \frac{2}{3} f \circ (\beta - \alpha) + \frac{2}{9}\theta^2 + \frac{2}{3}u$$

$$\psi = \frac{2}{3}(\beta - \alpha) \circ g + \frac{2}{9}\theta^2 + \frac{2}{3}v$$

Because of Proposition 5.5 we can check the following:

$$A(p,q) \circ f = B(p,q) \qquad A(p,q) \circ \varphi = A(p, q+1)$$

$$g \circ B(p,q) = C(p,q) \qquad \psi \circ A(p,q) = A(p+1, q)$$

Because of Lemma 5.7.1 we have: $A(0,1) = A(1,0)$. Therefore we get:

$$A(p, q+1) = \psi^p \circ A(0,1) \circ \varphi^q = \psi^p \circ A(1,0) \circ \varphi^q = A(p+1, q)$$

$$B(p, q+1) = A(p, q+1) \circ f = A(p+1, q) \circ f = B(p+1, q)$$

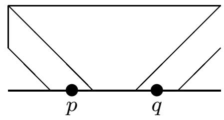
$$C(p, q+1) = g \circ B(p, q+1) = g \circ B(p+1, q) = C(p+1, q)$$

and that proves the lemma. $\square$

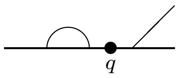
Then diagrams $A(p,q)$, $B(p,q)$, $C(p,q)$ depend only on $p+q$ and we can set:

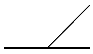
$$A(p,q) = A(p+q), \qquad B(p,q) = B(p+q), \qquad C(p,q) = C(p+q).$$

Let $u_0$ be the diagram represented in $F(3)$ by $1 \in \Lambda \subset F(3)$. By gluing $u_0$ on the diagram $A(p,q)$ we get the following diagram $v$ in $F(3)$:

$v =$ 


depending only on $p+q$. Because of Proposition 5.5, we get:

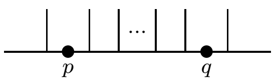
$v = \omega_p$ 

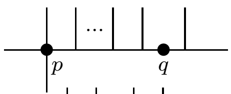
$= 2\omega_p\omega_q$ 

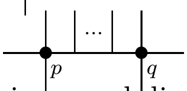
Therefore $\omega_p\omega_q$ depend only on $p+q$ and Theorem 5.6 follows.

Let $D$ be a marked diagram representing an element $U$ in $\mathcal{A}^s(\Gamma, X)$. Then $D \setminus \Gamma$ is connected. Let $Z$ be the set of all marked vertices or edges of $D$. We'll said that two elements $u$ and $v$ in $Z$ are related if there is a path $\gamma$ in $D$ connecting $u$ and $v$ such that $\gamma$ doesn't meet $\Gamma$ and meets $Z$ only in $u$ and $v$. This relation generates an equivalence relation $\equiv$. Since $D \setminus \Gamma$ is connected, $Z$ has only one equivalence class modulo $\equiv$. Therefore in order to prove the first part of Theorem 5.7 it is enough to prove the following: if $u$ and $v$ in $Z$ are related the class of $D$ in $\mathcal{A}^s(\Gamma, X)$ depends only on the sum of the marking of $u$ and $v$.

Consider the following diagrams in $F(6+n)$, for some integers $p$, $q$, $n$:

$A_n(p,q) =$ 


$B_n(p,q) =$ 


$C_n(p,q) =$ 


If $u$ and $v$ in $Z$ are related $D$ contains a subdiagram isomorphic to $A_n(p,q)$, $B_n(p,q)$ or $C_n(p,q)$. Then it is enough to prove that $A_n(p,q)$, $B_n(p,q)$ and $C_n(p,q)$ depend only on $n$ and $p+q$. Let $X$ be one of the symbol $A$, $B$, $C$. Because of Lemma 3.3, we can push away all strands in the middle of $X_n(p,q)$ through the marked edge (or the marked vertex) in the right part of the diagram and $X_n(p,q)$ is equivalent in $F(6+n)$ to a linear combination of diagrams containing $X(p,q)$. Then, because of Lemma 5.7.2, $X_n(p,q)$ depends only on $n$ and $p+q$ and the first part of Theorem 5.6 is proven.

The element $\omega_q U$ is represented by a diagram $D'$ obtained from $D$ by adding a new marked edge with marking $q$. Therefore $\omega_q U$ depends only on $D_0$ and the sum of $q$ and the total marking of $D$. $\square$

**Remark:** Consider the commutative $\mathbf{Q}$-algebra $R'$ defined by the following presentation:

— generators: $t$, $\omega_0$, $\omega_1$,...
— relations: $\omega_p\omega_q = \omega_0\omega_{p+q}$, for every $p, q$.

We have a canonical morphism from $R'$ to $\Lambda$. On the other hand there is a morphism $f : R' \longrightarrow \mathbf{Q}[t,\sigma,\omega]$ sending $t$ to $t$ and each $\omega_p$ to $\omega\sigma^p$. Is is easy to see that this morphism is injective with image $R_0 = \mathbf{Q}[t] \oplus \omega\mathbf{Q}[t,\sigma,\omega]$. Then the morphism $R' \longrightarrow \Lambda_k$ induces a morphism from $R_0$ to $\Lambda$:

**5.8 Proposition:** *Let $R$ be the polynomial algebra $\mathbf{Q}[t,\sigma,\omega]$ where $t$, $\sigma$ and $\omega$ are formal variables of degree 1, 2 and 3 respectively and $R_0$ be the subalgebra $\mathbf{Q}[t]\oplus\omega\mathbf{Q}[t,\sigma,\omega]$ of $R$. Then there is a unique graded algebra homomorphism $\varphi$ from $R_0$ to $\Lambda$ sending $t$ to $t$ and each $\omega\sigma^p$ to $\omega_p$.*

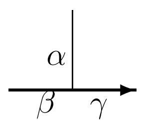
## 6. Detecting elements in $\Lambda$.

In this section we'll construct weight functions on modules of diagrams and characters on $\Lambda$ using Lie superalgebras.

Let $L$ be a finite dimensional Lie superalgebra over a field $K$ equipped with a nonsingular supersymmetric bilinear form $< \ , \ >$ invariant under the adjoint representation. Such a data will be called a quadratic Lie superalgebra and the bilinear form is called the inner form. Take a homogeneous basis $(e_j)$ of $L$ and its dual basis $(e'_j)$. The Casimir element $\Omega = \sum_j e_j \otimes e'_j \in L \otimes L$ is independent of the choice of the basis and its degree is zero.

Let $\Gamma$ be an closed oriented curve and $X = [n]$ be a finite set. Suppose that a $L$-representation $E_i$ is chosen for each component $\Gamma_i$ of $\Gamma$. We will say that $\Gamma$ is colored by $L$-representations. Then it is possible to construct a linear map from $\mathcal{A}(\Gamma, X)$ to $L^{\otimes n}$ in the following way:

Let $D$ be a $(\Gamma, X)$-diagram. Up to some changes of cyclic ordering we may as well suppose that, at each vertex $x$ in $\Gamma$ the cyclic ordering is given by $(\alpha,\beta,\gamma)$ where $\alpha$ is the edge which is not contained in $\Gamma$ and $\beta$ is the edge in $\Gamma$ ending at $x$ (with the orientation of $\Gamma$).




For each component $\Gamma_i$ we can take a basis $(e_{ij})$ of $E_i$ and its dual basis $(e'_{ij})$ of the dual $E'_i$ of $E_i$ and we get a Casimir element $\omega_i = \Sigma_j e_{ij} \otimes e'_{ij} \in E_i \otimes E'_i$. This element is of degree zero and is independent of the choice of the basis.

For each oriented edge $\alpha$ in $D$ denote by $V(\alpha)$ the module $L$ if $\alpha$ is not contained in $\Gamma$ and $E_i$ (resp. $E'_i$) if $\alpha$ is contained in the component $\Gamma_i$ of $\Gamma$ with a compatible (resp. not compatible) orientation. If $\alpha$ is an oriented edge in $D$ denote by $W(\alpha)$ the module $V(\alpha)\otimes V(-\alpha)$ where $-\alpha$ is the edge $\alpha$ equipped with the opposite orientation.

Let $a$ be an edge in $D$. Take an orientation of $a$ compatible with the orientation of $\Gamma$ if $a$ is contained in $\Gamma$. Denote also by $\omega(a)$ the Casimir element $\omega$ if $a$ is not contained in $\Gamma$ and the element $\omega_i$ if $a$ is contained in $\Gamma_i$. This element belongs to the module $W(a)$ and is independent on the orientation of $a$. If a numbering of the set of edges is chosen the tensor product $W = \bigotimes_a W(a)$ is well defined and the element $\Omega = \bigotimes_a \omega(a)$ is a well-defined element in $W$.

Let $x$ be a 3-valent vertex in $D$. There are three oriented edges $\alpha, \beta$ and $\gamma$ ending at $x$ (the ordering $(\alpha, \beta, \gamma)$ is chosen to be compatible with the cyclic ordering given at $x$ and, if $x$ is in $\Gamma$, $\alpha$ is supposed to be outside of $\Gamma$).


Then we get a module $H(x) = V(\alpha) \otimes V(\beta) \otimes V(\gamma)$. If a numbering of the set of 3-valent vertices of $D$ is chosen, the module $\bigotimes_x H(x)$ is well defined. We can permute (in the super sense) the big tensor product $W$ and we get an isomorphism $\varphi$ from $W$ to the module:

$$H = L^{\otimes n} \otimes \bigotimes_x H(x)$$

and $\varphi(\Omega)$ is an element of $H$.

Suppose that $x$ is not contained in $\Gamma$. Then the rule $u \otimes v \otimes w \mapsto < [u, v], w >$ induces a map $f_x$ from $H(x)$ to $K$. If $x$ is in $\Gamma$ the rule $u \otimes e \otimes f \mapsto (-1)^{\partial^\circ f \partial^\circ (u \otimes e)} f(ue)$ is a map $f_x$ from $H(x)$ to $K$. Hence the image of $\varphi(\Omega)$ under the tensor product of all $f_x$ is an element $\Phi_L(D) \in L^{\otimes n}$. Since elements $w$ and $w_i$ and maps $f_x$ are of degree zero, this element doesn't depend on these numberings.

Since the map $u \otimes v \otimes w \mapsto < [u, v], w >$ from $L \otimes L \otimes L$ to $K$ is totally antisymmetric (in the super sense), $\Phi_L(D)$ is multiplied by $-1$ if one cyclic ordering is changed in $D$. Moreover, the Jacobi identity and the property of the $L$-action on modules $E_i$ imply that the correspondence $D \mapsto \Phi_L(D)$ is compatible with the IHX relation. Therefore this correspondence induces a well-defined linear map $\Phi_L$ from $\mathcal{A}(\Gamma, X)$ to $L^{\otimes n}$.

**Definition:** *A Lie superalgebra $L$ over a field $K$ will be called quasisimple if it satisfies the two conditions:*

— *$L$ is not abelian*

— *every endomorphism of $L$ of degree 0 is the multiplication by a scalar.*

**Remark:** Every simple Lie superalgebra is quasisimple but the converse is not true.

**Lemma:** *A quasisimple quadratic Lie superalgebra has a trivial center and a surjective Lie bracket.*

**Proof:** Let $L$ be a quasisimple quadratic Lie superalgebra over a field $K$. Let $f$ be a morphism from $L$ to $K$. By duality we get a morphism $g$ from $K$ to $L$. The composite $g \circ f$ is an endomorphism of $L$ and there is a scalar $\lambda \in K$ such that: $g \circ f = \lambda \mathrm{Id}$.

Suppose $f \neq 0$. Then $f$ is surjective, $g$ is injective, $g \circ f$ is not trivial and $\lambda \neq 0$. Therefore $g \circ f$ is bijective and $f$ is bijective also. But that's impossible because $L$ is not abelian.

Then every morphism from $L$ to $K$ is zero and (by duality) every morphism from $K$ to $L$ is zero too. The result follows. $\square$

**6.1 Theorem:** *Let $K$ be a field and a $k$-algebra and $L$ be a quasisimple quadratic Lie superalgebra over $K$. Then there is a well-defined character $\chi_L : \Lambda_k \longrightarrow K$ such that:*

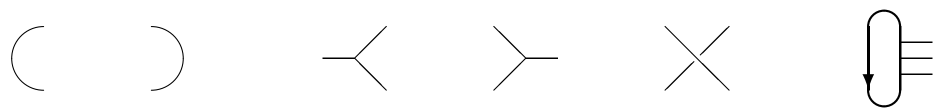
*for every closed oriented curve $\Gamma$ colored by $L$-representations and every finite set $X$, the map $\Phi_L$ satisfies the following property:*

$$\forall \alpha \in \Lambda_k, \quad \forall u \in \mathcal{A}^s(\Gamma, X), \quad \Phi_L(\alpha u) = \chi_L(\alpha)\Phi_L(u).$$

*Let $A$ be a $k$-subalgebra of $K$. Suppose $K$ is the fraction field of $A$ and $A$ is a unique factorization domain. Suppose also that $L$ contains a finitely generated $A$-submodule $L_A$ such that the Lie bracket and its dual are defined on $L_A$. Then $\chi_L$ takes values in $A$.*

**Proof:** First of all, it is possible to extend the map $\Phi_L$ to a functor between two categories $Diag(L)$ and $\mathcal{C}(L)$. The objects of these categories are the sets $[p], p \geq 0$. For $p, q \geq 0$ the set of morphisms in $\mathcal{C}(L)$ from $[p]$ to $[q]$ is the set of $L$-linear homomorphisms from $L^{\otimes p}$ to $L^{\otimes q}$, and the set of morphisms in $Diag(L)$ from $[p]$ to $[q]$ is the $k$-module generated by the isomorphism classes of $(\Gamma, [p] \cup [q])$-diagrams where $\Gamma$ is any $L$-colored oriented curve and where the relations are the AS and IHX relations.

These two categories are monoidal and $Diag(L)$ contains $\Delta_k$ as a subcategory. Moreover $Diag(L)$ is generated (as a monoidal category) by the following morphisms:



The last morphism is a morphism in $Diag(L)$ from $[p]$ to $[0]$ depending on an integer $p \geq 0$ and a $L$-representation $E$.

The map $\Phi_L$ associates to each $L$-colored $(\Gamma, [p] \cup [q])$-diagram $D$ an element $\Phi_L(K)$ in $L^{\otimes p} \otimes L^{\otimes q}$. But $L^{\otimes p}$ is isomorphic to its dual and $\Phi_L(K)$ may be seen as a linear map from $L^{\otimes p}$ to $L^{\otimes q}$.

It is not difficult to see that the image under $\Phi_L$ of the generators above are:
— the inner form from $L^{\otimes 2}$ to $L^{\otimes 0} = K$,
— the Casimir element consider as a morphism from $K = L^{\otimes 0}$ to $L^{\otimes 2}$,
— the Lie bracket from $L^{\otimes 2}$ to $L$,
— the dual of the Lie bracket (the Lie cobracket) from $L$ to $L^{\otimes 2}$,
— the map $x \otimes y \mapsto (-1)^{\partial^\circ x \partial^\circ y} y \otimes x$ from $L^{\otimes 2}$ to itself,
— the map $x_1 \otimes \ldots \otimes x_p \mapsto \tau_E(x_1 \ldots x_p)$ from $L^{\otimes p}$ to $L^{\otimes 0} = K$,
where $\tau_E(x_1 \ldots x_p)$ is the supertrace of the endomorphism $x_1 \ldots x_p$ of $E$.

All these maps are $L$-linear. Therefore $\Phi_L$ induces a functor still denoted by $\Phi_L$ from $Diag(L)$ to the category $\mathcal{C}(L)$.

Let $\Gamma$ be a $L$-colored oriented curve and $X = [n]$ be a finite set. Consider an element $\alpha \in \Lambda_k$ and an element $u \in \mathcal{A}^s(\Gamma, X)$ represented by a $(\Gamma, X)$-diagram $D$. Take a 3-valent vertex $x$ in $D$ and a bijection from [3] to the set of edges ending at $x$. By taking off a neighborhood of $x$ in $D$, we get a diagram $H$ inducing a morphism $v$ in $Diag(L)$ from [3] to $[n]$.

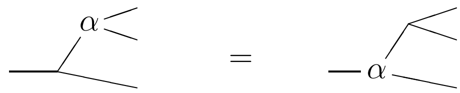
On the other hand, $\alpha$ induces a morphism $\beta$ in $Diag(L)$ from [0] to [3], and $1 \in \Lambda$ induces an element $\beta_0$ from [0] to [3]. Let $\widetilde{u}$ and $\widetilde{\alpha u}$ be the morphisms from [0] to $[n]$ induced by $u$ and $\alpha u$. We have:

$$\widetilde{u} = v \circ \beta_0, \qquad \widetilde{\alpha u} = v \circ \beta.$$

Hence:

$$\Phi_L(\widetilde{u}) = \Phi_L(v) \circ \Phi_L(\beta_0), \qquad \Phi_L(\widetilde{\alpha u}) = \Phi_L(v) \circ \Phi_L(\beta).$$

The elements $\alpha \in \Lambda_k$ and $1 \in \Lambda_k$ also induce morphisms $\gamma$ and $\gamma_0$ from [2] to [1]. Denote by $\varphi$ and $\varphi_0$ the morphisms $\Phi_L(\gamma)$ and $\Phi_L(\gamma_0)$. The morphism $\varphi_0$ is the Lie bracket and $\varphi$ is $L$-linear and antisymmetric. Since $\alpha$ belongs to $\Lambda_k$, we have the following:




and, for every $x, y, z$ in $L$, we have: $[\varphi(x \otimes y), z] = \varphi([x, y] \otimes z)$.

Denote by $u \mapsto [u]$ the Lie bracket from $L^{\otimes 2}$ to $L$. For every $u \in L^{\otimes 2}$ and every $z \in L$ we have: $[\varphi(u), z] = \varphi([u] \otimes z)$.

Suppose $[u] = 0$ then $[\varphi(u), z]$ vanishes for every $z \in L$ and $\varphi(u)$ lies in the center of $L$. Since this center is trivial, $\varphi(u)$ is trivial too. Therefore $\varphi(u)$ depends only on the image $[u]$ of $u$. Since the Lie bracket is surjective, there is a unique morphism $\psi$ from $L$ to $L$ such that:

$$\forall u \in L^{\otimes 2}, \;\; \varphi(u) = \psi([u])$$

and there is a unique $\lambda \in K$ such that:

$$\forall u \in L^{\otimes 2}, \;\; \varphi(u) = \lambda[u]$$

and we have:

$$\Phi_L(\beta) = \lambda \Phi_L(\beta_0), \qquad \Phi_L(\widetilde{\alpha u}) = \lambda \Phi_L(\widetilde{u}), \qquad \Phi_L(\alpha u) = \lambda \Phi_L(u).$$

Now it is easy to see that $\alpha \mapsto \lambda$ is a character depending only on $L$ and the Casimir element $\Omega$.

Suppose now that $L$ contains a finitely generated $A$-submodule $L_A$ such that the Lie bracket and the Lie cobracket (the dual of the Lie bracket) are defined on $L_A$. Let $\alpha$ be an element in $\Lambda_k$ represented by a $(\emptyset, [3])$-diagram $D$ and $u \in K$ be its image under $\chi_L$. Because this diagram is connected there exists a continuous map $f$ from $D$ to $[0, 1]$ such that:

— $f(1) = f(2) = 0 \;\; f(3) = 1$

— $f$ is affine and injective on each edge of $D$
— $f$ is injective on the set of 3-valent vertices of $D$
— $f$ has no local extremum.

Such a map can be constructed by induction on the number of edges of $D$. Using this map, the map from [2] to [1] represented by $D$ can be described by composition, tensor product, Lie bracket and Lie cobracket and we have:

$$\forall x, y \in L_A, \quad u[x, y] \in L_A.$$

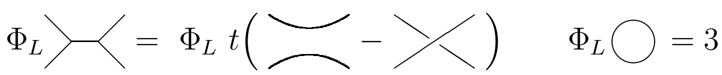
Let $w$ be a nonzero element in the image of the Lie bracket $L_A \otimes L_A \longrightarrow L_A$. By applying the formula above for each power of $\alpha$, we get:

$$\forall p \geq 0, \quad u^p w \in L_A.$$

Since $L_A$ is finitely generated the $A$-submodule of $K$ generated by the powers of $u$ is also finitely generated. Then $u$ lies in the integral closure of $A$ in $K$. Since $A$ is a unique factorization domain, $A$ is integrally closed and $u$ belongs to $A$. Therefore $\chi_L(\alpha)$ lies in $A$ for every $\alpha \in \Lambda_k$. $\square$

**Remark:** If every endomorphism of $L$ is the multiplication by a scalar, every invariant bilinear form of $L$ is a multiple of the given inner form. If we divide the inner form par some $c \in K$, we multiply the Casimir element $\Omega$ by $c$ and for every $\alpha \in \Lambda_k$ of degree $n$, $\chi_L(\alpha)$ is multiplied by $c^n$.

**6.2 Proposition:** *Let $L$ be the Lie algebra $sl_2$ (defined over $K$). Then the functor $\Phi_L$ satisfies the following properties:*

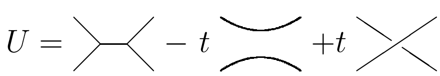



*Moreover there is a unique graded algebra homomorphism $\chi_{sl2}$ from $\Lambda_k$ to $k[t]$ sending $t$ to $t$ and each $\omega_n$ to $0$ such that the character $\chi_L$ is the composite:*

$$\Lambda_k \xrightarrow{\chi_{sl2}} k[t] \xrightarrow{\gamma} K$$

*where $\gamma$ is a $k$-algebra homomorphism. If the inner form on $L$ send $\alpha \otimes \beta$ to the trace of $\alpha\beta$, $\gamma$ sends $t$ to $2$.*

**Proof:** Since $L$ is defined over $\mathbf{Q}$ it is enough to consider the case $k = K = \mathbf{Q}$. Set:



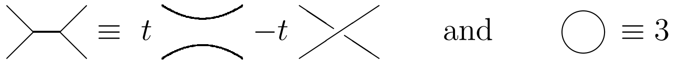

The element $\Phi_L(U)$ is a map from $L^{\otimes 2} = \Lambda^2(L) \oplus S^2(L)$ to itself. Since $U$ is antisymmetric on the source and the target, $\Phi_L(U)$ is trivial on $S^2(L)$ and its image is contained in $\Lambda^2(L)$. Since $L$ is 3-dimensional, the Lie bracket $\Lambda^2(L) \longrightarrow L$ is bijective. But $U$ composed with this bracket is zero. Therefore $U$ is killed by $\Phi_L$.

The fact that $\Phi_L$ sends the circle to 3 come from the fact that $L$ is 3-dimensional.

Denote by $\equiv$ the following relation:

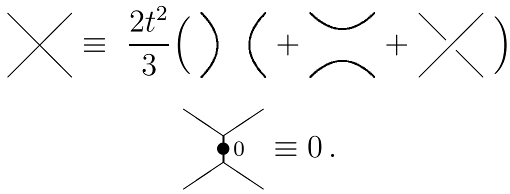
$$a \equiv b \Longleftrightarrow \Phi_L(a) = \Phi_L(b)$$

So we have:

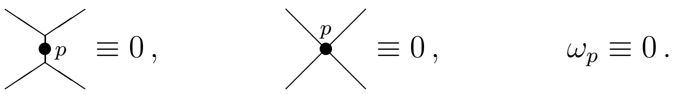



and it is easy to see by induction that every element $\alpha$ in $\Lambda_k$ is equivalent to some polynomial $P(t) \in k[t]$. Let $c$ be the scalar $\chi_L(t)$. Then we have: $\Phi_L(\alpha) = P(c)$. If $\alpha$ is homogeneous of degree $n$, we have: $P(t) = at^n$ and: $\Phi_L(\alpha) = ac^n$. Then $P(t)$ is completely determined by $\Phi_L(\alpha)$. Therefore $\alpha \mapsto P(t)$ is a well-defined algebra homomorhism $\chi_{sl2}$ from $\Lambda_k$ to $k[t]$ such that $\chi_L$ is the composite $\gamma \circ \chi_{sl2}$ where $\gamma$ sends $t$ to $c$. If the inner form is $\alpha \otimes \beta \mapsto \tau(\alpha\beta)$, we have $c = 2$ and $\gamma$ sends $t$ to 2.

A direct computation gives the following:




Then by induction we get the following for every $p \geq 0$:



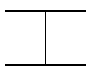

Therefore each $\omega_p$ is killed by $\chi_{sl2}$ and that finishes the proof. $\square$

Let $L$ be a quasisimple quadratic Lie superalgebra over a field $K$. Let $X$ be the kernel of the Lie bracket: $\Lambda^2(L) \longrightarrow L$ and $Y$ be the quotient of $S^2(L)$ by the Casimir element $\Omega$ of $L$. So we have exact sequences of $L$-modules:

$$0 \longrightarrow X \longrightarrow \Lambda^2(L) \longrightarrow L \longrightarrow 0$$

$$0 \longrightarrow K\Omega \longrightarrow S^2(L) \longrightarrow Y \longrightarrow 0$$

Let $\Psi_L$ be the endomorphism of $L^{\otimes 2}$ represented by the diagram:



Since this diagram is symmetric, $\Psi_L$ respects the decomposition: $L^{\otimes 2} = S^2(L) \oplus \Lambda^2(L)$. But $\Psi_L$ respects the exact sequences also and $\Psi_L$ acts on $X$ and $Y$. If $\alpha$ is a eigenvalue of $\Psi_L$ acting on $Y$, the corresponding eigenspace will be denoted by $Y_\alpha$.

**6.3 Theorem:** *Let $L$ be a quasisimple quadratic Lie superalgebra over a field $K$ which is not $sl_2$. Let $\Omega$, $X$, $Y$ and $\Psi_L$ defined as above. Let $P$ be the minimal polynomial of $\Psi_L$ acting on $Y$.*

*Suppose the following conditions hold:*

— $6$ *is invertible in* $K$,

— $\Psi_L$ *acts bijectively on* $Y$,

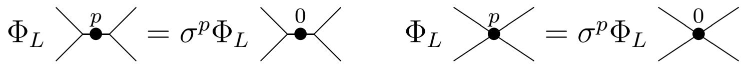
— $\chi_L$ *is nontrivial on some* $\omega_p$ *or* $\partial^\circ P \leq 3$.

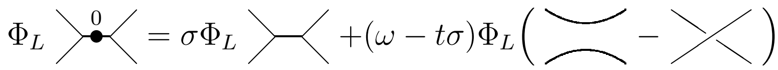
*Then the degree of* $P$ *is* $2$ *or* $3$ *and there exist three elements* $t$, $\sigma$, $\omega$ *in* $K$ *such that:*

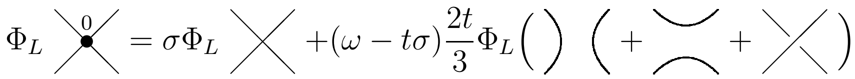
— $\chi_L(t) = t, \qquad \forall p \geq 0,\ \chi_L(\omega_p) = \omega\sigma^p$,

— $\Psi_L$ *is the multiplication by* $0$, $t$ *and* $2t$ *on* $X$, $\Lambda^2(L)/X \simeq L$ *and* $K\Omega$,

— *for every* $p \geq 0$ *we have the following:*

(1) 

(2) 

(3) 

*If* $P$ *is of degree* $2$ *(exceptional case),* $P$ *has* $2$ *nonzero roots* $\alpha$ *and* $\beta$ *in some algebraic extension of* $K$ *and we have:*

$$t = 3(\alpha+\beta), \qquad \sigma = (4\alpha+5\beta)(4\beta+5\alpha), \qquad \omega = 5(\alpha+\beta)(3\alpha+4\beta)(3\beta+4\alpha)$$

$$sdim(L) = -2\frac{(5\alpha+6\beta)(5\beta+6\alpha)}{\alpha\beta}$$

$$sdim(X) = 5\frac{(4\alpha+\beta)(4\beta+\alpha)(5\alpha+6\beta)(5\beta+6\alpha)}{\alpha^2\beta^2}$$

$$\alpha \neq \beta \implies sdim(Y_\alpha) = -90\frac{(\alpha+\beta)^2(6\alpha+5\beta)(3\alpha+4\beta)}{\alpha^2\beta(\alpha-\beta)}.$$

*If* $P$ *is of degree* $3$ *(regular case),* $P$ *has* $3$ *nonzero roots* $\alpha$, $\beta$, $\gamma$ *in some algebraic extension of* $K$ *and we have:*

$$t = \alpha+\beta+\gamma, \qquad \sigma = \alpha\beta+\beta\gamma+\gamma\alpha+2t^2, \qquad \omega = (t+\alpha)(t+\beta)(t+\gamma)$$

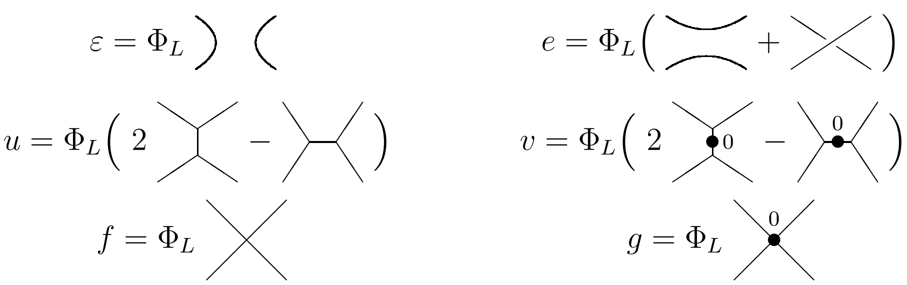
$$sdim(L) = -\frac{(2t-\alpha)(2t-\beta)(2t-\gamma)}{\alpha\beta\gamma}$$

$$sdim(X) = \frac{\omega(2t-\alpha)(2t-\beta)(2t-\gamma)}{\alpha^2\beta^2\gamma^2}$$

$$\alpha \neq \beta,\gamma \implies sdim(Y_\alpha) = \frac{t(2t-\beta)(2t-\gamma)(t+\beta)(t+\gamma)(2t-3\alpha)}{\alpha^2\beta\gamma(\alpha-\beta)(\alpha-\gamma)}.$$

**Remark:** In the exceptional case, we may add formally a new root $\gamma = 2t/3$ and a trivial corresponding eigenspace $Y_\gamma$. Then the formulas of the superdimensions are exactly the same in the exceptional case or the regular case except that $\gamma$ is possibly equal to 0.

**Proof:** Set: $\omega = \chi_L(\omega_0)$ and consider the following endomorphisms in $L^{\otimes 2}$:




These endomorphisms act on $S^2(L)$ and act trivially on $\Lambda^2(L)$.

**The degree of $P$:**

Suppose $\chi_L(\omega_p) \neq 0$. We have:

$$\chi_L(\omega_p^2) = \chi_L(\omega_0\omega_{2p}) = \omega\chi_L(\omega_{2p}) \neq 0 \implies \omega \neq 0.$$

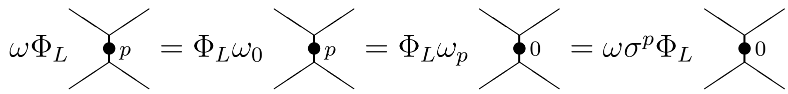
So we can set:

$$\sigma = \frac{\chi_L(\omega_1)}{\omega}$$

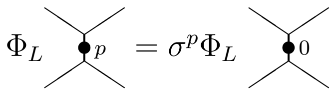
and we have for every $p > 0$:

$$\omega^{p-1}\chi_L(\omega_p) = \chi_L(\omega_0^{p-1}\omega_p) = \chi_L(\omega_1^p) = \omega^p\sigma^p \implies \chi_L(\omega_p) = \sigma^p\omega.$$

Because of theorem 5.7, we have also:




and this implies:




Similarly we get for every $p \geq 0$:


and formulas (1) are proven in this case.

Let $E$ be the vector space formally generated by $e$, $\varepsilon$, $u$, $v$, $f$ and $g$. Because of Proposition 5.5 the operator $\Psi_L$ induces an action $\psi$ on $E$ defined by:

$$\psi(\varepsilon) = 2t\varepsilon$$

$$\psi(e) = u$$
$$\psi(u) = \frac{t}{3}u + 2f$$
$$\psi(v) = -\omega u + \frac{4t}{3}v + 2g$$
$$\psi(f) = \frac{10t^2}{9}u - \frac{1}{2}v + \frac{2t}{3}f$$
$$\psi(g) = \frac{2t\omega}{3}u + \Big(\frac{4t^2}{9} - \frac{\sigma}{2}\Big)v + \frac{2t}{3}g.$$

It is easy to see that $\psi$ vanishes on the following element in $E$:

$$U = g - \sigma f - \frac{t}{3}(v - \sigma u) - t(\omega - t\sigma)e.$$

Since $\Psi_L$ acts bijectively on $S^2(L)/K\Omega$, $U$ induces the trivial endomorphism of $S^2(L)/K\Omega$ and there exists an element $\lambda \in K$ such that the following holds in $\mathrm{End}(L^{\otimes 2})$ (or in $\mathrm{Hom}(L^{\otimes 4}, K)$):

$$g - \sigma f - \frac{t}{3}(v - \sigma u) - t(\omega - t\sigma)e = \lambda\varepsilon.$$

The group $\mathfrak{S}_4$ acts on this equality and the invariant part of it is:

$$g = \sigma f + \Big(\frac{2t}{3}(\omega - t\sigma) + \frac{\lambda}{3}\Big)(e + \varepsilon).$$

Hence we have also:

$$t(v - \sigma u) = (t(\omega - t\sigma) - \lambda)(2\varepsilon - e).$$

By making a quarter of a turn and composing with the Lie bracket, we get:

$$t(3\omega - 3t\sigma) = 3(t(\omega - t\sigma) - \lambda)$$

which implies: $\lambda = 0$ and we get Formula (3):

$$g = \sigma f + \frac{2t}{3}(\omega - t\sigma)(e + \varepsilon)$$

and also the following:

$$t(v - \sigma u) = t(\omega - t\sigma)(2\varepsilon - e).$$

Let $E'$ be the quotient of $E$ by these two relations. It is easy to see that $\psi$ vanishes on $v - \sigma u - (\omega - t\sigma)(2\varepsilon - e) \in E'$.

For the same reason as above, there is an element $\mu \in K$ such that:

$$v - \sigma u - (\omega - t\sigma)(2\varepsilon - e) = \mu\varepsilon.$$

By making a quarter of a turn and composing with the Lie bracket, we get:

$$3\omega - 3t\sigma - (\omega - t\sigma)3 = \mu.$$

Hence $\mu$ is zero and we get the formula (2).

Denote by $\varphi$ the endomorphism of $Y$ induced by $\Psi_L$. In this endomorphism algebra we have:

$$\varepsilon = 0 \qquad e = 2$$

$$u = 2\varphi \qquad f = \varphi^2 - \frac{t}{3}\varphi$$

$$v = 2\Big(\frac{20t^2}{9}\varphi - \varphi \circ f + \frac{2t}{3}f\Big) = 2(-\varphi^3 + t\varphi^2 + 2t^2\varphi).$$

The relation $v = \sigma u + (\omega - t\sigma)(2\varepsilon - e)$ implies:

$$2(-\varphi^3 + t\varphi^2 + 2t^2\varphi) = 2\sigma\varphi - 2(\omega - t\sigma)$$

and then:

$$\varphi^3 - t\varphi^2 + (\sigma - 2t^2)\varphi - (\omega - t\sigma) = 0.$$

The minimal polynomial $P$ of $\varphi$ is then a divisor of the polynomial $Q(X) = X^3 - tX^2 + (\sigma - 2t^2)X - (\omega - t\sigma)$. Since $L$ is quasisimple $Y$ is nonzero and the degree of $P$ is 1, 2 or 3.

Therefore in any case the degree of $P$ is 1, 2 or 3.

Suppose $\partial^\circ P = 1$. Let $\alpha$ be the root of $P$. Then the endomorphism $v - \alpha e$ of $L^{\otimes 2}$ has its image contained in $K\Omega$ and there is some $\lambda \in K$ such that the following holds in $\mathrm{End}(L^{\otimes 2})$ (or in $\mathrm{Hom}(L^{\otimes 4}, K)$):

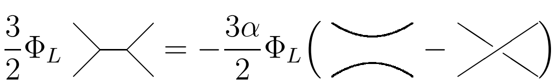
$$v = \alpha e + \lambda\varepsilon.$$

The group $\mathfrak{S}_4$ acts on this equality and the invariant part of this equality is:

$$0 = (\frac{2\alpha}{3} + \frac{\lambda}{3})(e + \varepsilon).$$

Then we get: $\lambda = -2\alpha$.

By making a quarter of a turn and composing with the projection: $L^{\otimes 2} \longrightarrow \Lambda^2(L) \subset L^{\otimes 2}$ we get the equality:



Since $\Psi_L$ acts bijectively on $Y$, $\alpha$ is not zero and $\Lambda^2(L)$ is contained in the image of the cobracket. Therefore the Lie bracket is bijective from $\Lambda^2(L)$ to $L$ . But that's impossible because $L$ is not isomorphic to $sl_2$. Therefore the degree of $P$ is 2 or 3.

**The exceptional case:**

Suppose: $\partial^\circ P = 2$ and denote by $\alpha$ and $\beta$ the roots of $P$. Since $\Psi_L$ acts bijectively on $Y$, $\alpha$ and $\beta$ are not zero. The endomorphism $(v-\alpha e)(v-\beta e)$ is trivial on $Y$ and $\Lambda^2(L)$. Then its image is contained in $K\Omega$ and there exists $\mu \in K$ such that:

$$(v-\alpha e)(v-\beta e) = \mu\varepsilon.$$

So we get:

$$4f + 2(\frac{t}{3} - \alpha - \beta)v + 2\alpha\beta e = \mu\varepsilon.$$

By taking the invariant part of this equation (under $\mathfrak{S}_4$) we get:

$$4f + \frac{4\alpha\beta}{3}(e+\varepsilon) = \frac{h}{3}(e+\varepsilon)$$

and then:

$$2(\frac{t}{3} - \alpha - \beta)v = \frac{2\alpha\beta + \mu}{3}(2\varepsilon - e).$$

Since $L$ is not $sl_2$, $v$ and $2\varepsilon - e$ are linearly independent and we get:

$$t = 3(\alpha+\beta) \qquad \mu = -2\alpha\beta$$

$$f = -\frac{\alpha\beta}{2}(e+\varepsilon).$$

By applying $\Psi_L$ to this equality we get:

$$-\frac{\alpha\beta}{2}(u + 2t\varepsilon) = -\frac{\alpha\beta t}{3}(e+\varepsilon) + \frac{10t^2}{9}u - \frac{v}{2}$$

$$\Longrightarrow \quad v = (4\alpha+5\beta)(4\beta+5\alpha)u + 2\alpha\beta(\alpha+\beta)(2\varepsilon - e)$$

and that implies in any case the formula (2) with: $\sigma = (4\alpha+5\beta)(4\beta+5\alpha)$ and $\omega = 5(\alpha+\beta)(3\alpha+4\beta)(3\beta+4\alpha)$. If $\omega = 0$ we still have: $\chi_L(\omega_p) = \omega\sigma^p$ and formulas (1) and (3) are consequences of (2).

Let $d$ be the superdimension of $L$ and $\tau$ be the supertrace operator. Since $\Psi_L$ acts by multiplication by $0$, $t$ and $2t$ on $X$, $L$ and $K\Omega$, we have:

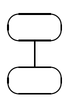
$$\tau(\varphi^0) = \frac{d(d+1)}{2} - 1 = \frac{(d-1)(d+2)}{2}$$

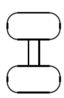
$$\tau(\Psi_L) = td + 2t + \tau(\varphi)$$

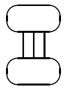
$$\tau(\Psi_L^2) = t^2 d + 4t^2 + \tau(\varphi^2)$$

$$\tau(\Psi_L^3) = t^3 d + 8t^3 + \tau(\varphi^3).$$

Using a simple graphical calculus, we get:

$\tau(\Psi_L) = \Phi_L$  $= 0$

$\tau(\Psi_L^2) = \Phi_L$  $= 4t^2 d$

$$\tau(\Psi_L^3) = \Phi_L$$  $$= 2t^3 d.$$

Hence we have:

$$\tau(\varphi^0) = \frac{(d-1)(d+2)}{2}$$
$$\tau(\varphi) = -t(d+2)$$
$$\tau(\varphi^2) = t^2(3d-4)$$
$$\tau(\varphi^3) = t^3(d-8).$$

Since $\varphi$ has $\alpha$ and $\beta$ as eigenvalues, we get:

$$t^2(3d-4) + t(\alpha+\beta)(d+2) + \frac{(d-1)(d+2)}{2}\alpha\beta = 0$$

$$t^3(d-8) - (\alpha+\beta)t^2(3d-4) - t(d+2)\alpha\beta = 0$$

and that implies the following:

$$(\alpha+\beta)(60(\alpha+\beta)^2 + (d+2)\alpha\beta) = 0$$

$$(d-1)(60(\alpha+\beta)^2 + (d+2)\alpha\beta) = 0.$$

Suppose $t = 0$. Since $(\Psi_L - \alpha)(\Psi_L - \beta)$ vanishes on $Y$, there exists $\mu \in K$ such that:

$$(\Psi_L - \alpha)(u - \beta e) = 2\mu\varepsilon$$
$$\Longrightarrow \quad (\frac{t}{3} - \alpha - \beta)u + 2f = \alpha\beta e + 2\mu\varepsilon.$$

Since the left hand side of this equation is invariant under $\mathfrak{S}_4$, it is the same for the other side and we get:

$$2f = \alpha\beta(e+\varepsilon).$$

By composing with the inner product, we get: $d+2=0$. Therefore $d-1$ is non zero and we have in any case:

$$60(\alpha+\beta)^2 + (d+2)\alpha\beta = 0.$$

Then it is not difficult to compute the superdimensions of $L$ and $X$ and we get the desired formula.

Suppose $\alpha \neq \beta$. Denote by $d_\alpha$ and $d_\beta$ the superdimensions of eigenspaces $Y_\alpha$ and $Y_\beta$. We have:

$$d_\alpha + d_\beta = \frac{(d-1)(d+2)}{2}$$
$$\alpha d_\alpha + \beta d_\beta = -t(d+2)$$

and $d_\alpha$ and $d_\beta$ are easy to compute.

**The regular case:**

Consider now the regular case: $P$ is of degree 3 and has 3 nonzero roots $\alpha$, $\beta$, $\gamma$. Since $(\Psi_L-\alpha)(\Psi_L-\beta)(\Psi_L-\gamma)$ acts trivially on $Y$, there exists $\mu\in K$ such that:

$$(\Psi_L-\alpha)(\Psi_L-\beta)(u-\gamma e)=2\mu\varepsilon.$$

After reduction we get:

$$(\frac{7t^2}{3}-\frac{t}{3}(\alpha+\beta+\gamma)+\alpha\beta+\beta\gamma+\gamma\alpha)u+2(t-\alpha-\beta-\gamma)f-v=\alpha\beta\gamma e+2\mu\varepsilon.$$

The invariant part of this formula is:

$$2(t-\alpha-\beta-\gamma)f=\frac{2}{3}(\alpha\beta\gamma+\mu)(e+\varepsilon).$$

Since the minimal polynomial of $\varphi$ has degree 3, $f$ is not a multiple of $e+\varepsilon$. Hence we get:

$$\alpha+\beta+\gamma=t \qquad\qquad \mu=-\alpha\beta\gamma$$

and also:

$$(2t^2+\alpha\beta+\beta\gamma+\gamma\alpha)u-v=\alpha\beta\gamma(e-2\varepsilon).$$

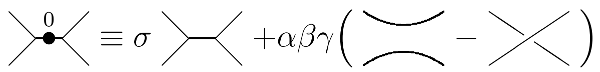
If $\omega$ is nonzero, $P$ is equal to $Q$ and we have:

$$\alpha\beta+\beta\gamma+\gamma\alpha=\sigma-2t^2 \qquad \alpha\beta\gamma=\omega-t\sigma.$$

Otherwise we can set: $\sigma=\alpha\beta+\beta\gamma+\gamma\alpha+2t^2$ and we have:

$$v=\sigma u+\alpha\beta\gamma(2\varepsilon-e)$$

and then:




By applying the Lie bracket, we get: $0=2\omega=2t\sigma+2\alpha\beta\gamma$. In this case we have: $\alpha\beta\gamma=\omega-t\sigma$ and the formula (2) follows. As above formulas (1) and (3) are easy to check.

In any case $t$, $\sigma$, $\omega$ can be expressed in term of $\alpha$, $\beta$, $\gamma$. As above we get the following:

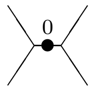
$$\tau(\varphi^0)=\frac{(d-1)(d+2)}{2}$$
$$\tau(\varphi)=-t(d+2)$$
$$\tau(\varphi^2)=t^2(3d-4)$$
$$\tau(\varphi^3)=t^3(d-8).$$

Since $\varphi$ has $\alpha$, $\beta$, $\gamma$ as eigenvalues, we get:

$$t^3(d-8)-t^3(3d-4)-t(\sigma-2t^2)(d+2)-\frac{(d-1)(d+2)}{2}\alpha\beta\gamma=0$$

$$\Longrightarrow \quad (d+2)(\alpha\beta\gamma d + (2t-\alpha)(2t-\beta)(2t-\gamma)) = 0.$$

Let $F$ be the endomorphism of $L^{\otimes 2}$ represented by the diagram:




Because of the formula (2), $F$ acts by $2\omega$ on $L$ and by $2(\omega - t\sigma)$ on $X$. It is trivial on $S^2(L)$. Therefore we get:

$$0 = \tau(F) = 2\omega d + 2(\omega - t\sigma)\frac{d(d-3)}{2} \quad \Longrightarrow \quad \omega d(d-1) = t\sigma d(d-3).$$

Suppose $d = -2$. Then we have:

$$3\omega = 5t\sigma$$

and this implies:

$$-\frac{(2t-\alpha)(2t-\beta)(2t-\gamma)}{\alpha\beta\gamma} = -\frac{4t^3 + 2t(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma}{\alpha\beta\gamma}$$

$$= -2 + \frac{3\alpha\beta\gamma - 2t\sigma}{\alpha\beta\gamma} = -2 + \frac{3\omega - 5t\sigma}{\alpha\beta\gamma} = -2.$$

Therefore in any case we have:

$$\alpha\beta\gamma d + (2t-\alpha)(2t-\beta)(2t-\gamma) = 0$$

and $d$ and the superdimension of $X$ are easy to compute.

If $\alpha$ is different from $\beta$ and $\gamma$, we have the following (with $d_\alpha = \mathrm{sdim} Y_\alpha$):

$$(\alpha-\beta)(\alpha-\gamma)d_\alpha = \tau(\varphi^2 - (\beta+\gamma)\varphi + \beta\gamma\varphi^0)$$

$$= t^2(3d-4) + t(d+2)(\beta+\gamma) + \frac{(d-1)(d+2)}{2}\beta\gamma$$

and that gives the value of $d_\alpha$. □

## 7. The eight characters.

**7.1 The $gl$ case.** Let $E$ be a supermodule of superdimension $m$. Take a homogeneous basis $\{e_i\}$ of $E$ and denote by $\{e_{ij}\}$ the corresponding basis of $gl(E)$. Let $sl(E) \subset gl(E)$ be the Lie superalgebra of endomorphisms of $E$ with zero supertrace. The map sending $\alpha \otimes \beta \in gl(E) \otimes gl(E)$ to the supertrace of $\alpha \circ \beta$ is an nonsingular invariant bilinear form on $gl(E)$ and $gl(E)$ is a quadratic Lie superalgebra. If $m$ is invertible, $sl(E)$ is also a quadratic Lie superalgebra. If $m = 0$, the inner form is singular on $sl(E)$, but the quotient of $sl(E)$ by its center is a quadratic Lie superalgebra $psl(E)$.

**7.2 Theorem:** *Let $[t, u]$ be the polynomial algebra generated by variables $t$ and $u$ of degree 1 and 2 respectively. For each $m \in \mathbf{Z}$, denote by $\gamma_m$ the ring homomorphism* *sending $t$ to $m$ and $u$ to 1. Then there exists a unique graded algebra homomorphism $\chi_{gl}$ from $\Lambda_k$ to $k[t,u]$ such that the following holds for every supermodule $E$ of superdimension $m$:*

*— for every closed oriented curve $\Gamma$ colored by $gl(E)$-representations, and every finite set $X$, we have:*

$$\forall \alpha \in \Lambda_k,\ \forall u \in \mathcal{A}_k(\Gamma, X),\quad \Phi_{gl(E)}(\alpha u) = \gamma_m \circ \chi_{gl}(\alpha)\Phi_{gl(E)}(u)$$

*— if $m$ is invertible in $k$ and $sl(E)$ is quasisimple, $\chi_{sl(E)}$ is the composite $\gamma_m \circ \chi_{gl}$*
*— if $m = 0$ and $psl(E)$ is quasisimple, $\chi_{psl(E)}$ is the composite $\gamma_0 \circ \chi_{gl}$.*
*Moreover $\chi_{gl}$ satisfies the following:*

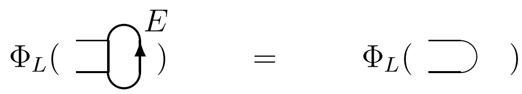
$$\chi_{gl}(t) = t \qquad \text{and} \qquad \forall p \geq 0,\ \ \chi_{gl}(\omega_p) = \omega\sigma^p$$

*with: $\omega = 2t(t^2 - 4u)$ and $\sigma = 2(t^2 - 2u)$.*

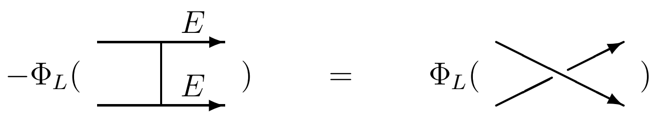
**Proof:** Let $E$ be a finite dimensional free $k$-supermodule of superdimension $m$. Let $\{e_i\}$ be a homogeneous basis of $E$ and $\{e_{ij}\}$ be the corresponding basis of $L = gl(E)$. Then the Casimir element is

$$\Omega = \sum_{ij} (-1)^{\partial^\circ e_j} e_{ij} \otimes e_{ji}.$$

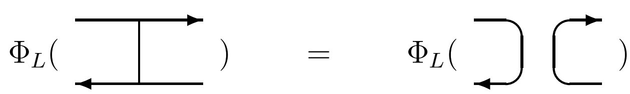
Since the inner product of $x$ and $y$ in $L$ is $< x, y >= \tau_E(xy)$ we have the following:



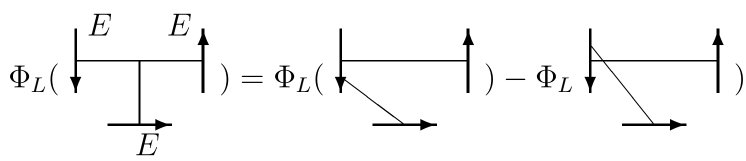

Moreover, it is not difficult to show the following:

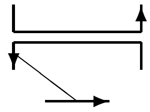
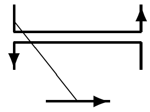
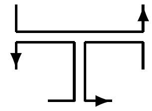
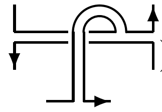



Whence:



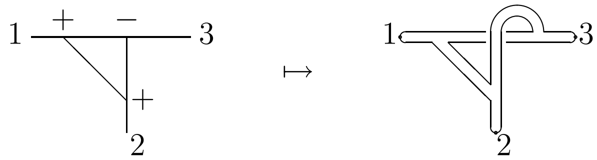
and we get:



$= \Phi_L($  $) - \Phi_L($  $) = \Phi_L($  $) - \Phi_L($  $)$

Therefore, to compute the image by $\Phi_L$ of a $(\emptyset, [n])$-diagram $K$, we may proceed as follows:

Let $S(K)$ be the set of functions $\alpha$ from the set of 3-valent vertices of $K$ to $\pm 1$. For every $\alpha \in S(K)$ denote by $\varepsilon(\alpha)$ the product of all $\alpha(x)$. If $\alpha \in S(K)$ is given we may construct a thickening of $K$ by using the given cyclic ordering of edges ending at a 3-valent vertex $x$ if $\alpha(x) = 1$ and the other one if not, and we get an oriented surface $\Sigma_\alpha(K)$ equipped with $n$ numbered points in its boundary.



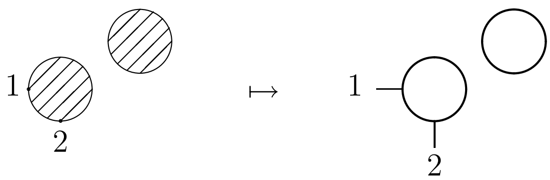

Denote by $S_n$ the set of isomorphism classes of oriented connected surfaces equipped with $n$ numbered points in its boundary. Under the connected sum, $S = S_0$ is a monoid and acts on $S_n$. This monoid is a graded commutative monoid freely generated by the disk $D$ of degree 1 and the torus $T$ of degree 2. The set $S_n$ is a graded $S$-set with $\dim \mathrm{H}_1$ as degree. Let $k[S_n]$ and $k[S]$ be the free modules generated by $S_n$ and $S$. They are graded modules, and $k[S]$ is a polynomial algebra acting on $k[S_n]$.

If $K$ is connected, the sum

$$s(K) = \sum_\alpha \varepsilon(\alpha)\Sigma_\alpha(K)$$

lies in $k[S_n]$. It is easy to check that $s$ is compatible with AS and IHX relations and induces a well-defined graded homomorphism from $F_k(n)$ to $k[S_n]$. Moreover, this homomorphism is $\Lambda_k[S]$-linear with respect to a character $\chi$ from $\Lambda_k$ to $k[S] = k[D, T]$.

On the other hand, for each $\Sigma \in S_n$ we have a diagram $\partial(\Sigma)$ in $\mathcal{D}(\Gamma, [n])$ where $\Gamma$ is colored by $E$: $\partial(\Sigma)$ is the boundary of $\Sigma$ colored by $E$ with intervals added near each marked point:




We can extend $\partial$ linearly and for every $\Sigma \in k[S_n]$, $\Phi_L(\partial(\Sigma))$ is well defined in $L^{\otimes n}$. Moreover we have:

$$\Phi_L(K) \sum_\alpha \varepsilon(\alpha)\Phi_L(\partial\Sigma_\alpha(K)) = \Phi_L(\partial s(K)).$$

Therefore if $a$ is an element of $\Lambda_k$, we have:

$$\Phi_L(aK) = \Phi_L(\partial s(aK)) = \Phi_L(\partial\chi(a)s(K)) = \Phi_L(\chi(a)\partial s(K))$$

$$= \gamma_m(\chi(a))\Phi_L(\partial s(K)) = \gamma_m(\chi(a))\Phi_L(K)$$

and the first part of the theorem is proven in the case $\Gamma = \emptyset$ (with $\chi_{gl} = \chi$). The general case follows.

If $m$ is invertible in $k$ and $sl(E)$ is quasisimple, $sl(E)$ is actually simple and $gl(E)$ is semisimple: $gl(E) = sl(E) \oplus k$. Since $\Phi_k$ is trivial, we have:

$$\Phi_{sl(E)}(aK) = \Phi_{gl(E)}(aK) = \gamma_m(\chi_{gl}(a))\Phi_{gl(E)}(K) = \gamma_m(\chi_{gl}(a))\Phi_{sl(E)}(K)$$

$$\Longrightarrow \ \chi_{sl(E)} = \gamma_m \circ \chi_{gl}.$$

Suppose now: $m = 0$ and $psl(E)$ is quasisimple. Since the Lie bracket on $gl(E)$ takes values in $sl(E)$, $\Phi_{gl(E)}(K)$ lies in $sl(E)^{\otimes n}$ and for every $a \in \Lambda_k$ the equality

$$\Phi_{gl(E)}(aK) = \gamma_m(\chi_{gl}(a))\Phi_{gl(E)}(K)$$

holds in $sl(E)^{\otimes n}$. Hence in the quotient $psl(E)$ we have:

$$\Phi_{psl(E)}(aK) = \gamma_m(\chi_{gl}(a))\Phi_{psl(E)}(K)$$

and we get:

$$\chi_{psl(E)}) = \gamma_0 \circ \chi_{gl}.$$

In order to prove the last part of the theorem, it is enough to determine $\chi_{sl(E)}(\omega_p)$ for $k = \mathbf{Q}$ and for infinitely many values of $m$. Suppose now $m > 2$ and $E$ has no odd part. Then $L = sl(E)$ is the classical Lie algebra $sl_m$. The morphism $\Psi = \Psi_L$ from $L^{\otimes 2}$ to itself is the morphism:

$$x \otimes y \mapsto \sum_{ij} [x, e_{ij}] \otimes [e_{ji}, y]$$

and because of Proposition 6.3 we have to determine eigenvalues of $\Psi$ acting on $Y = S^2(L)/\Omega$.

Denote by $\tau$ the trace operator. Let $f : L^{\otimes 2} \longrightarrow L$ be the following morphism:

$$f : x \otimes y \mapsto xy + yx - \frac{2}{m}\tau(xy)\mathrm{Id}.$$

Since $m > 2$, $f$ is surjective and $L$-linear. We have:

$$f\Psi(x \otimes y) = \sum_{ij}((xe_{ij} - e_{ij}x)(e_{ji}y - ye_{ji}) + (e_{ji}y - ye_{ji})(xe_{ij} - e_{ij}x))$$

$$-\sum_{ij} \frac{2}{m}\tau((xe_{ij} - e_{ij}x)(e_{ji}y - ye_{ji}))$$

$$= mxy+\tau(xy)+myx+\tau(yx)-\frac{2}{m}\tau(mxy+\tau(xy)) = mxy+myx-2\tau(xy) = mf(x\otimes y).$$

The map $f$ factorizes through $Y$ and there is an exact sequence:

$$0 \longrightarrow Z \longrightarrow Y \longrightarrow L \longrightarrow 0$$

compatible with the action of $\Psi$ and $\Psi$ induces the multiplication by $m$ on $L$.

The module $Z$ can be seen as a submodule of $L^{\otimes 2}$ and the morphisms sending $x \otimes y$ to $xy$, $yx$, $x \otimes y - y \otimes x$ are trivial on $Z$. If $z$ lies in $L$, denote by $z_{ij}$ the entries of $z$. We have:

$$\Psi(x \otimes y) = \sum (x_{ki}e_{kj} - x_{jk}e_{ik}) \otimes (y_{il}e_{jl} - y_{lj}e_{li})$$

$$= \sum (xy)_{kl}e_{kj} \otimes e_{jl} + (yx)_{lk}e_{ik} \otimes e_{li} - x_{ki}e_{kj} \otimes y_{lj}e_{li} - y_{il}e_{ik} \otimes x_{jk}e_{jl}.$$

Therefore the morphism $\Psi$ is equal on $Z$ to the morphism $\Psi'$ defined by:

$$\Psi'(x \otimes y) = -2 \sum xe_{ij} \otimes ye_{ji}$$

and we have:

$$\Psi'^2(x \otimes y) = 4 \sum xe_{ij}e_{kl} \otimes ye_{ji}e_{lk} = 4x \otimes y.$$

Therefore the minimal polynomial of $\Psi$ acting on $Y$ is of degree three with roots $m$, 2, $-2$.

Hence Theorem 6.3 applies and we get:

$$\chi_L(t) = m \qquad \forall p \geq 0, \ \ \chi_L(\omega_p) = 2m(m+2)(m-2)(2m^2-4)^p$$

and that finishes the proof. □

**7.3 The *osp* case.** Let $E$ be a supermodule of superdimension $m$ equipped with a supersymmetric nonsingular bilinear form $< \, , \, >$. We'll say that $E$ is a quadratic supermodule. For every endomorphism $\alpha$ of $E$, we have a endomorphism $\alpha^*$ defined by:

$$\forall x, y \in E \qquad < \alpha^*(x), y >= (-1)^{pq} < x, \alpha(y) >$$

where $p$ is the degree of $x$ and $q$ is the degree of $\alpha$. An endomorphism $\alpha$ is *antisymmetric* if $\alpha^* = -\alpha$. Let $L = osp(E)$ be the Lie superalgebra of antisymmetric endomorphisms of $E$. The superdimension of $L$ is $d = m(m-1)/2$. With the same notation as before, a Casimir element of $L$ is:

$$\Omega = \frac{1}{2}\sum_{i,j}(-1)^{\partial^\circ e_j}(e_{ij} - e_{ij}^*) \otimes (e_{ji} - e_{ji}^*)$$

and with this Casimir element, $t = m-2$. The bilinear form corresponding to $\Omega$ is half the supertrace of the product.

**7.4 Theorem:** *Let $k[t, v]$ be the polynomial algebra generated by variables $t$ and $v$ of degree 1. Then there exists a unique graded algebra homomorphism $\chi_{osp}$ from $\Lambda_k$ to $k[t, v]$ such that:*

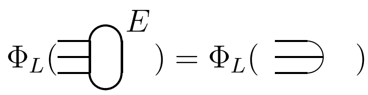
*— for every quadratic supermodule $E$, if $osp(E)$ is quasisimple, then $\chi_{osp(E)}$ is the composite $\gamma \circ \chi_{osp}$, where $\gamma$ is the ring homomorphism sending $t$ to sdim$(E) - 2$ and $v$ to 1.*

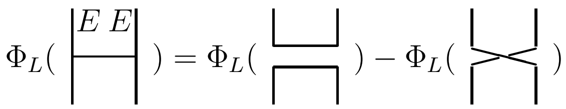
*Moreover $\chi_{osp}$ satisfies the following:*

$$\chi_{osp}(t) = t \qquad \text{and} \qquad \forall p \geq 0, \;\; \chi_{osp}(\omega_p) = \omega\sigma^p$$

*with: $\omega = 2(t - v)(t - 2v)(t + 4v)$ and $\sigma = 2(t - 2v)(t + 3v)$.*

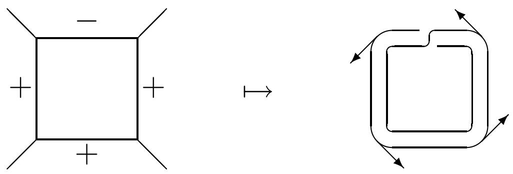
**Proof:** Let $E$ be a quadratic supermodule and $L$ be the Lie superalgebra $osp(E)$. Let $K$ be a $L$-colored diagram. If we change the orientation of a component colored by $E$, $\Phi_L(K)$ is unchanged. Therefore we may consider in $K$ unoriented components colored by $E$. On the other hand it is easy to see the following:




Therefore, to compute the image by $\Phi_L$ of a $(\emptyset, [n])$-diagram $K$, we may proceed as follows:

Let $S(K)$ be the set of functions from the set of edges of $K$ joining two 3-valent vertices of $K$ to $\pm 1$. For every $\alpha \in S(K)$ denote by $\varepsilon(\alpha)$ the product of all $\alpha(a)$. If $\alpha \in S(K)$ is given we may construct a thickening of $K$ by using the given cyclic ordering of edges ending at each 3-valent vertex and making a half-twist near every edge $a$ with negative $\alpha(a)$. So we get an unoriented surface $\Sigma_\alpha(K)$ equipped with $n$ numbered points in its boundary and a local orientation of $\partial\Sigma_\alpha(K)$ near each of these points.




Denote by $US_n$ the set of isomorphism classes of connected surfaces $\Sigma$ equipped with $n$ numbered points in its boundary and an orientation of $\partial\Sigma$ near each of these points. Under the connected sum, $US = US_0$ is a monoid and acts on $US_n$. This monoid is a graded commutative monoid generated by the disk $D$, the projective plane $P$ and the torus $T$ and the only relation is: $PT = P^3$.

Let $k(US_n)$ be the $k$-module generated by the elements of $US_n$ with the following relations:

If $\Sigma'$ is obtained from $\Sigma$ by changing the local orientation near one point, $\Sigma + \Sigma'$ is trivial in $k(US_n)$.

Then $k[US]$ is a commutative algebra and $k(US_n)$ is a graded $k[US]$-module.

If $K$ is connected, the sum $s(K) = \sum_\alpha \varepsilon(\alpha)\Sigma_\alpha(K)$ lies in $k[US_n]$. It is easy to check that $s$ is compatible with AS and IHX relations and induces a well-defined graded homomorphism from $F_k(n)$ to $k[US_n]$. Moreover this homomorphism is $\Lambda_k[US]$-linear with respect to a character $\chi$ from $\Lambda_k$ to $k[US] = k[D,P,T]/(PT-P^3)$.

On the other hand, we have a map $\partial$ from $US_n$ and $k(US_n)$ to $F_k(n)$ by sending each surface $\Sigma$ with numbered points in $\partial\Sigma$ to the boundary $\partial\Sigma$ colored by $E$ with intervals added near each marked point. If $K$ is a diagram, $\Phi_L(K)$ is equal to the sum $\sum_\alpha \varepsilon(\alpha)\Phi_L(\partial\Sigma_\alpha(K)) = \Phi_L(\partial s(K))$. Therefore if $u$ is an element of $\Lambda$, we have $\chi_L(u) = \chi_L(\partial\chi(u))$. Since $\chi_L \circ \partial$ is a ring homomorphism sending $D$ to $m = \mathrm{sdim}E$ and $P$ and $T$ to 1, the character $\chi_L$ factorizes through $k[D,P] = k[US]/(T-P^2)$ and the first part of the theorem is proven (with $t = D-2P$, $v = P$).

To prove the last part of the theorem, it is enough to consider the case where $E$ is a classical module over $k = \mathbf{Q}$ of large dimension $m$. Then the second symmetric power $S^2(L)$ decomposes into four simple $L$-modules $E_0, E_1, E_2, E_3$ of dimensions $1$, $(m-1)(m+2)/2$, $m(m-1)(m-2)(m-3)/4!$, $m(m+1)(m+2)(m-3)/12$. Therefore we have the decomposition: $Y = E_1 \oplus E_2 \oplus E_3$. Moreover the Casimir homomorphism acts on $E_1$, $E_2$, $E_3$ by multiplication by $2m$, $4m-16$, $4m-4$. On the other hand, this homomorphism is equal to $4t - 2\Psi_L$. Therefore $\Psi_L$ acts on $E_1$, $E_2$, $E_3$ by multiplication by $m-4$, $4$, $-2$.

The rest of the proof is an straightforward consequence of Theorem 6.3. $\square$

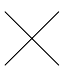
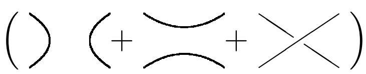
**Remark:** The use of surfaces in the *gl*- and *osp*-cases was introduced in a slightly different way by Bar-Natan to produce weight functions [BN].

**7.5 The exceptional case.** Consider a quasisimple quadratic Lie superalgebra $L$ over a field $K$ of characteristic 0. This Lie superalgebra $L$ is said to be exceptional if it satisfies the following condition:

— the square of the Casimir generates in degree 4 the center of the enveloping algebra $\mathcal{U}$ of $L$.

Exceptional Lie algebras $\mathrm{E}_6$, $\mathrm{E}_7$, $\mathrm{E}_8$, $\mathrm{F}_4$, $\mathrm{G}_2$ satisfy this property. But it is also the case for $sl_2$, $sl_3$, $osp(E)$ with $\mathrm{sdim}(E) = 2$ or $8$, $psl(E)$ with $\mathrm{sdim}(E) = 0$ and the exceptional Lie superalgebras $G(3)$ and $F(4)$.

Consider the following elements in $F_K(4)$:

$u = \Phi_L$  $v = \Phi_L$ 

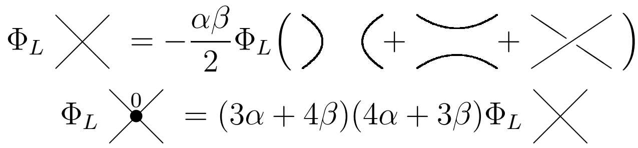
These elements are invariants elements in $S^4(L)$. But the condition satisfied by $L$ implies that the invariant part of $S^4(L)$ is generated by $v$. Therefore $u$ is a multiple of $v$ and the homomorphism $\Psi_L$ has only two eigenvalues on $S^2(L)/\Omega$. Hence we may apply Theorem 6.3 in the exceptional case and we get:

**7.6 Theorem:** *Let $L$ be an exceptional quasisimple quadratic Lie superalgebra over a field $K$ of characteristic zero. Then there exist $\sigma$ and $\omega$ in $K$ and two elements $\alpha$ and $\beta$ in some extension of $K$ such that:*

$$t = 3(\alpha+\beta) \qquad \sigma = (4\alpha+5\beta)(5\alpha+4\beta) \qquad \omega = 5(\alpha+\beta)(3\alpha+4\beta)(4\alpha+3\beta)$$

$$\chi_L(t) = t \qquad \forall p \geq 0,\ \chi_L(\omega_p) = \omega\sigma^p$$

$$sdim(L) = -2\frac{(5\alpha+6\beta)(6\alpha+5\beta)}{\alpha\beta}$$




**Remark:** In this theorem, we may consider the Casimir $\Omega$, and then $\alpha$ and $\beta$ up to a scalar. So $\alpha$ and $\beta$ may be consider as degree 1 variables related by some linear relation.

Case by case we get the following:

| $L$ | sdim($L$) | $\alpha/\beta$ | $\sigma$ | $\omega$ |
|---|---|---|---|---|
| $E_6$ | 78 | $-3$ | $\frac{77}{36}t^2$ | $\frac{25}{12}t^3$ |
| $E_7$ | 133 | $-4$ | $\frac{176}{81}t^2$ | $\frac{520}{243}t^3$ |
| $E_8$ | 248 | $-6$ | $\frac{494}{225}t^2$ | $\frac{98}{45}t^3$ |
| $F_4$ | 52 | $-5/2$ | $\frac{170}{81}t^2$ | $\frac{480}{243}t^3$ |
| $G_2$ | 14 | $-5/3$ | $\frac{65}{36}t^2$ | $\frac{55}{36}t^3$ |
| $sl_2$, $G(3)$ | 3 | $-4/3$ | $\frac{8}{9}t^2$ | 0 |
| $sl_3$, $F(4)$ | 8 | $-3/2$ | $\frac{14}{9}t^2$ | $\frac{10}{9}t^3$ |
| $osp(8)$ | 28 | $-2$ | $2t^2$ | $\frac{50}{27}t^3$ |
| $osp(2)$ | 1 | $-5/4$ | 0 | $-\frac{40}{27}t^3$ |
| $psl(E)$ | $-2$ | $-1$ | | 0 |

In this table, $osp(n)$ means any simple Lie superalgebra $osp(E)$ where $E$ is a supermodule with $\text{sdim}(E) = n$.

In the case $sl_2$ or $G(3)$ or $psl(E)$ (with $\mathrm{sdim}(E) = 0$), the induced character kills every $\omega_p$ and the value of $\sigma$ is useless. In the case $psl(E)$, the character is determined by any nonabelian $gl(F)$. Then $\chi_{psl(E)}$ is determined by $gl(1|1)$. But in this Lie superalgebra every double bracket $[[x, y], z]$ vanishes. Therefore $\Phi_{gl(1|1)}$ is trivial on $\Lambda$ in positive degree and the character $\chi_{psl(E)}$ is the trivial character.

**Remark:** The characters $\chi_{G(3)}$ and $\chi_{F(4)}$ are equal to $\chi_{sl2}$ and $\chi_{sl3}$ on the algebra generated by $t$ and the $\omega_p$'s. These characters are actually equal to $\chi_{sl2}$ and $\chi_{sl3}$ on $\Lambda$. This result was proven by Patureau-Mirand [Pa].

**Conjecture:** *Let $R$ be the subalgebra $\mathbf{Q}[\alpha + \beta, \alpha\beta]$ of $\mathbf{Q}[\alpha, \beta]$ where $\alpha$ and $\beta$ are two formal parameters of degree* 1. *Then there exists a unique graded algebra homomorphism $\chi_{exc}$ from $\Lambda$ to $R$ such that:*

$$\chi_{exc}(t) = 3(\alpha + \beta)$$

$$\forall p \geq 0, \quad \chi_{exc}(\omega_p) = 5(\alpha + \beta)(3\alpha + 4\beta)(4\alpha + 3\beta)(4\alpha + 5\beta)^p(5\alpha + 4\beta)^p.$$

**Remark:** This conjecture is actually equivalent to a conjecture of Deligne [D]. If Deligne's conjecture is true, there exist a monoidal category $\mathcal{C}$ which is linear over an algebra $\mathbf{Q}(\lambda)$ and looks like the category of representations of some virtual exceptional Lie algebra. It is not difficult to construct a functor from the category $\Delta$ to $\mathcal{C}$ and we get an algebra homomorphism from $\Lambda$ to the coefficient algebra $\mathbf{Q}(\lambda)$. But this morphism is equivalent to a graded homomorphism $\chi$ from $\Lambda$ to $R$ and the desired properties of $\chi$ are easy to check.

Conversely if such a morphism $\chi$ exists, we get an algebra homomorphism $\chi'$ from $\Lambda[d]$ to the localized algebra $R' = R[\frac{1}{\alpha\beta}]$ by:

$$\chi'(d) = -2\frac{(5\alpha + 6\beta)(6\alpha + 5\beta)}{\alpha\beta}.$$

Then we may force $\Lambda[d]$ to act on morphisms in the category $\Delta$ (and not only on special diagrams). So we get a new category $\Delta_1$ which is linear over $\Lambda[d]$, where $d$ represents the circle. By tensoring $\Delta_1$ over $\Lambda[d]$ by $R'$, we get a category $\Delta_2$ which is linear over $R'$. If we kill every morphism $f : X \longrightarrow Y$ in $\Delta_2$ such that the trace of $f \circ g$ vanishes for every $g : Y \longrightarrow X$, we get a category $\Delta_3$ which satisfies all Deligne properties. Hence we have a positive answer to Deligne's conjecture.

**Remark:** Suppose the conjecture is true. Let $\lambda$ be any element in $\Lambda_{\mathbf{Z}}$ and $P = P(\alpha, \beta)$ be its image under $\chi_{exc}$. The expression $P(\alpha, \beta)$ is known if $\alpha/\beta$ lies in the set $E = \{-3, -4, -6, -5/2, -5/3, -4/3, -3/2, -2, -5/4, -1\}$. Therefore $P$ is well defined modulo the following polynomial $\Pi$:

$$\Pi = (\alpha + 2\beta)(\beta + 2\alpha)(\alpha + 3\beta)(\beta + 3\alpha)(\alpha + 4\beta)(\beta + 4\alpha)(\alpha + 6\beta)(\beta + 6\alpha)\times$$

$$(2\alpha + 5\beta)(2\beta + 5\alpha)(3\alpha + 5\beta)(3\beta + 5\alpha)(3\alpha + 4\beta)(3\beta + 4\alpha)\times$$

$$(2\alpha+3\beta)(2\beta+3\alpha)(2\alpha+\beta)(2\beta+\alpha)(4\alpha+5\beta)(4\beta+5\alpha)(\alpha+\beta)$$

By looking carefully at each character corresponding to the exceptional Lie algebras we can check that $P(\alpha,\beta)$ is an integer if $\alpha/\beta$ or $\beta/\alpha$ lies in $E$ and $\alpha+\beta$ and $\alpha\beta/2$ are integers. So we may ask the following:

**Question:** *Suppose the character $\chi_{exc}$ exists. Is $\chi_{exc}(\Lambda_{\mathbf{Z}})$ contained in the subalgebra $\mathbf{Z}[\alpha+\beta,\alpha\beta/2]$?*

**7.7 The super case.** There exists an interesting Lie superalgebra depending on a parameter $\alpha$ called D(2,1,$\alpha$). This algebra is simple and has a nonsingular bilinear supersymmetric invariant form and a Casimir element. Therefore it produces a character on $\Lambda$ depending on the parameter $\alpha$. Actually this algebra produces a graded character from $\Lambda_{\mathbf{Z}}$ to a polynomial algebra $\mathbf{Z}[\sigma_2,\sigma_3]$.

Consider 2-dimensional free oriented $\mathbf{Z}$-modules $E_1$, $E_2$, $E_3$ and denote by $X$ the module $E_1\otimes E_2\otimes E_3$. This module $X$ is a module over the Lie algebra $L' = sl(E_1)\oplus sl(E_2)\oplus sl(E_3)$.

Since $E_i$ is oriented, there is a canonical isomorphism $x\otimes y\mapsto x{\wedge}y$ from $\Lambda^2(E_i)$ to $\mathbf{Z}$. On the other hand, we have a map from $S^2(E_i)$ to $sl(E_i)$ sending $x\otimes y$ to the endomorphism $x.y : z\mapsto x\ y{\wedge}z + y\ x{\wedge}z$.

For each $i\in\{1,2,3\}$ take an element $f_i\in sl(E_i)$ which is congruent to the identity mod 2. Let $A$ be the polynomial algebra $\mathbf{Z}[a,b,c]$ divided by the only relation $a+b+c=0$. Then we can define a Lie superalgebra $L$ over $A$ by the following:

— the even part $L_0$ of $L$ is the $A$-submodule of $A[1/2]\otimes(\oplus_i sl(E_i))$ generated by $sl(E_1)$, $sl(E_2)$, $sl(E_3)$ and $(af_1+bf_2+cf_3)/2$

— the odd part $L_1$ of $L$ is the $A$-module $A\otimes X$

— the Lie bracket on $L_0\otimes L_0$ is the standard Lie bracket on $sl(E_i)\otimes sl(E_i)$ and vanishes on $sl(E_i)\otimes sl(E_j)$ for $i\neq j$

— the Lie bracket on $L_0\otimes L_1$ is the standard action of $\oplus_i sl(E_i)$ on $X$

— the Lie bracket on $L_1\otimes L_0$ is the opposite of the standard action of $\oplus_i sl(E_i)$ on $X$

— the Lie bracket on $X\otimes X$ is defined by:

$$[x\otimes y\otimes z, x'\otimes y'\otimes z'] = \frac{1}{2}(a\ x.x'\ y{\wedge}y'\ z{\wedge}z' + b\ x{\wedge}x'\ y.y'\ z{\wedge}z' + c\ x{\wedge}x'\ y{\wedge}y'\ z.z').$$

It is not difficult to see that $L$ is a Lie superalgebra over $A$ with superdimension $9-8=1$. The Jacobi relation holds because $a+b+c=0$. If we take a character from $A$ to $\mathbf{C}$, we get a complex Lie superalgebra. Up to isomorphism, this algebra depends only on one parameter $\alpha$ and is called $\mathrm{D}(2,1,\alpha)$. Here this algebra $L$ will be denoted by $\widetilde{\mathrm{D}}(2,1)$.

In order to define a Casimir element in $\widetilde{\mathrm{D}}(2,1)$, we need some notations. Consider for each $i=1,2,3$ a direct basis $\{\varepsilon_{ij}\}$ of $E_i$ and the dual basis $\{\varepsilon'_{ij}\}$ with respect to the form $\wedge$:

$$\forall x\in E_i \qquad \sum_j \varepsilon_{ij}(\varepsilon'_{ij}{\wedge}x) = \sum_j (x{\wedge}\varepsilon_{ij})\varepsilon'_{ij} = x.$$

For each $i$, the trace of the product is an invariant form on $sl(E_i)$, and, corresponding to this form, we have a Casimir type element $\Omega_i = \sum_j e_{ij} \otimes e'_{ij}$. This element belongs to $L \otimes L \otimes \mathbf{Z}[1/2]$, but $2\omega_i$ lies in $L \otimes L$. We have also a Casimir element $\pi \in X \otimes X$ defined by:

$$\pi = \sum_{ijk} (\varepsilon_{1i} \otimes \varepsilon_{2j} \otimes \varepsilon_{3k}) \otimes (\varepsilon'_{1i} \otimes \varepsilon'_{2j} \otimes \varepsilon'_{3k}).$$

**7.7.1 Lemma:** *For each $i \in \{1, 2, 3\}$ and $x \in E_i$, we have the following:*

$$\sum_j \varepsilon_{ij} \otimes x.\varepsilon'_{ij} = 2 \sum_j e_{ij}(x) \otimes e'_{ij},$$

$$\sum_j x.\varepsilon_{ij} \otimes \varepsilon'_{ij} = -2 \sum_j e_{ij} \otimes e'_{ij}(x).$$

**Proof:** Denote by $\tau$ the trace map. For every $\alpha \in$End$(E_i)$ we have:

$$\sum_j \varepsilon_{ij} \tau((x.\varepsilon'_{ij})\alpha) = \sum_j \varepsilon_{ij}(\varepsilon'_{ij} \wedge \alpha(x)) + \sum_j \varepsilon_{ij}(x \wedge \alpha(\varepsilon'_{ij}))$$

$$= \alpha(x) - \sum_j \varepsilon_{ij}(\alpha(x) \wedge \varepsilon'_{ij}) = 2\alpha(x) = 2 \sum_j e_{ij}(x) \tau(e'_{ij}\alpha)$$

and that gives the first formula. The second one is obtained in the same way. □

**7.7.2 Lemma:** *Let $K$ be the fraction field of $A$. Then $\widetilde{\mathrm{D}}(2, 1) \otimes K$ has an invariant bilinear form and the corresponding Casimir element is:*

$$\Omega = -a\Omega_1 - b\Omega_2 - c\Omega_3 + \pi.$$

*Moreover the cobracket induced by $\Omega$ is a morphism from $\widetilde{\mathrm{D}}(2, 1)$ to $\widetilde{\mathrm{D}}(2, 1) \otimes \widetilde{\mathrm{D}}(2, 1)$.*

**Proof:** Let $x \otimes y \otimes z$ be an element of $X$. We have:

$$x \otimes y \otimes z(\pi) = \sum_{ijk} [x \otimes y \otimes z, \varepsilon_{1i} \otimes \varepsilon_{2j} \otimes \varepsilon_{3k}] \otimes (\varepsilon'_{1i} \otimes \varepsilon'_{2j} \otimes \varepsilon'_{3k})$$

$$- \sum_{ijk} (\varepsilon_{1i} \otimes \varepsilon_{2j} \otimes \varepsilon_{3k}) \otimes [x \otimes y \otimes z, \varepsilon'_{1i} \otimes \varepsilon'_{2j} \otimes \varepsilon'_{3k}]$$

$$= \frac{1}{2}(aZ_1 + bZ_2 + cZ_3)$$

with:

$$Z_1 = \sum_{ijk} x.\varepsilon_{1i}\ y \wedge \varepsilon_{2j}\ z \wedge \varepsilon_{3k} \otimes (\varepsilon'_{1i} \otimes \varepsilon'_{2j} \otimes \varepsilon'_{3k}) - \sum_{ijk} (\varepsilon_{1i} \otimes \varepsilon_{2j} \otimes \varepsilon_{3k}) \otimes x.\varepsilon'_{1i}\ y \wedge \varepsilon'_{2j}\ z \wedge \varepsilon'_{3k}$$

$$Z_2 = \sum_{ijk} x \wedge \varepsilon_{1i}\ y.\varepsilon_{2j}\ z \wedge \varepsilon_{3k} \otimes (\varepsilon'_{1i} \otimes \varepsilon'_{2j} \otimes \varepsilon'_{3k}) - \sum_{ijk} (\varepsilon_{1i} \otimes \varepsilon_{2j} \otimes \varepsilon_{3k}) \otimes x \wedge \varepsilon'_{1i}\ y.\varepsilon'_{2j}\ z \wedge \varepsilon'_{3k}$$

$$Z_3 = \sum_{ijk} x{\wedge}\varepsilon_{1i}\ y{\wedge}\varepsilon_{2j}\ z.\varepsilon_{3k} \otimes (\varepsilon'_{1i} \otimes \varepsilon'_{2j} \otimes \varepsilon'_{3k}) - \sum_{ijk} (\varepsilon_{1i} \otimes \varepsilon_{2j} \otimes \varepsilon_{3k}) \otimes x{\wedge}\varepsilon'_{1i}\ y{\wedge}\varepsilon'_{2j}\ z.\varepsilon'_{3k}.$$

Using Lemma 7.7.1, $Z_1$ is easy to compute:

$$Z_1 = \sum_{ijk} x.\varepsilon_{1i} \otimes (\varepsilon'_{1i} \otimes y \otimes z) - \sum_{ijk} (\varepsilon_{1i} \otimes y \otimes z) \otimes x.\varepsilon'_{1i}$$

$$= -2\sum_{ijk} e_{1i} \otimes (e'_{1i}(x) \otimes y \otimes z) - 2\sum_{ijk} (e_{1i}(x) \otimes y \otimes z) \otimes e'_{1i} = 2x \otimes y \otimes z(\Omega_1)$$

and similarly we get: $Z_2 = 2x \otimes y \otimes z(\Omega_2)$, $Z_3 = 2x \otimes y \otimes z(\Omega_3)$. Therefore we have:

$$x \otimes y \otimes z(\Omega) = x \otimes y \otimes z(-a\Omega_1 - b\Omega_2 - c\Omega_3) + \frac{1}{2}(aZ_1 + bZ_2 + cZ_3) = 0$$

and $\Omega$, which is clearly invariant under the even part of $\widetilde{\mathrm{D}}(2,1)$, is $\widetilde{\mathrm{D}}(2,1)$-invariant.

Since $\Omega$ is symmetric and invariant, it corresponds to an invariant symmetric bilinear form on $\widetilde{\mathrm{D}}(2,1) \otimes K$ which is clearly non singular.

It is easy to see the following congruence modulo $\widetilde{\mathrm{D}}(2,1) \otimes \widetilde{\mathrm{D}}(2,1)$:

$$\Omega \equiv \frac{1}{2}(af_1 \otimes f_1 + bf_2 \otimes f_2 + cf_3 \otimes f_3)$$

and the cobracket takes values in $\widetilde{\mathrm{D}}(2,1) \otimes \widetilde{\mathrm{D}}(2,1)$. $\square$

**7.8 Theorem:** *Let $\mathbf{Z}[\sigma_2, \sigma_3]$ be the graded subalgebra of $A = \mathbf{Z}[a,b,c]/(a+b+c)$ generated by $\sigma_2 = ab + bc + ca$ of degree 2 and $\sigma_3 = abc$ of degree 3. Then the character $\chi_{sup}$ induced by $\widetilde{\mathrm{D}}(2,1)$ equipped with the Casimir $\Omega$ is a graded algebra homomorphism from $\Lambda_{\mathbf{Z}}$ to $\mathbf{Z}[\sigma_2, \sigma_3]$.*

*Moreover $\chi_{sup}$ satisfies the following:*

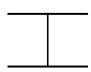
$$\chi_{sup}(t) = 0 \qquad \text{and} \qquad \forall p \geq 0, \quad \chi_{sup}(\omega_p) = \omega\sigma^p$$

*with: $\sigma = 4\sigma_2$, $\omega = 8\sigma_3$.*

**Proof:** Since $A$ is a unique factorization domain, we can apply Theorem 6.1 and the character induces by $\widetilde{\mathrm{D}}(2,1)$ is an algebra homomorphism $\chi_{sup}$ from $\Lambda_{\mathbf{Z}}$ to $A = \mathbf{Z}[a,b,c]/(a+b+c)$. There is an action of $\mathfrak{S}_3$ on $\widetilde{\mathrm{D}}(2,1)$. This action permutes the modules $E_i$ and the coefficients $a$, $b$, $c$. Therefore $\chi_{sup}$ takes values in the fixed part of $A$ under the action of $\mathfrak{S}_3$ and $\chi_{sup}$ is an algebra homomorphism from $\Lambda_{\mathbf{Z}}$ to $\mathbf{Z}[\sigma_2, \sigma_3]$.

On the other hand, $\widetilde{\mathrm{D}}(2,1)$ is a graded algebra: elements in $sl(E_i)$ are of degree 0, elements in $X$ are of degree 1 and $a$, $b$, $c$ are of degree 2. With this degree the degree of the Lie bracket is 0 and the degree of the cobracket is 2. Hence it is easy to see that each element $u \in \Lambda_{\mathbf{Z}}$ of degree $p$ is sent by $\chi_{sup}$ to an element of degree $2p$. Thus, after dividing degrees in $A$ by 2, $\chi_{sup}$ becomes a graded character. In particular $\chi_D(t)$ is trivial because $\mathbf{Z}[\sigma_2, \sigma_3]$ has no degree 1 element.

As above denote by $\Psi$ the morphism defined by the diagram



**7.8.1 Lemma:** *The endomorphism $\Psi$ satisfies the following:*

$$\Psi(\Omega_1) = -4a\Omega_1 + \frac{3}{2}\pi, \qquad \Psi(\Omega_2) = -4b\Omega_2 + \frac{3}{2}\pi, \qquad \Psi(\Omega_3) = -4c\Omega_3 + \frac{3}{2}\pi,$$
$$\Psi(\pi) = -4(a^2\Omega_1 + b^2\Omega_2 + c^2\Omega_3).$$

**Proof:** We have:

$$\Psi(\Omega_1) = -a\sum_{ij}[e_{1i}, e_{1j}] \otimes [e_{1j}, e_{1i}] + \sum_{ijlk}[e_{1i}, \varepsilon_{1j} \otimes \varepsilon_{2k} \otimes \varepsilon_{3l}] \otimes [\varepsilon'_{1j} \otimes \varepsilon'_{2k} \otimes \varepsilon'_{3l}, e'_{1i}].$$

The coefficient of $-a$ in this formula is the image of the Casimir of $sl_2$ under the corresponding homomorphism $\Psi_{sl2}$. Then it is equal to $2\chi_{sl2}(t)\Omega_1 = 4\Omega_1$, and:

$$\Psi(\Omega_1) = -4a\Omega_1 - \sum_{ijkl}(e_{1i}(\varepsilon_{1j}) \otimes \varepsilon_{2k} \otimes \varepsilon_{3l}) \otimes (e'_{1i}(\varepsilon'_{1j}) \otimes \varepsilon'_{2k} \otimes \varepsilon'_{3l}).$$

Because of Lemma 7.7.1, we have:

$$\begin{aligned}
\sum_{ij} e_{1i}(\varepsilon_{1j}) \otimes e'_{1i}(\varepsilon'_{1j}) &= \frac{1}{2}\sum_{ij} \varepsilon_{1i} \otimes \varepsilon_{1j}.\varepsilon'_{1i}(\varepsilon'_{1j}) \\
&= \frac{1}{2}\sum_{ij} \varepsilon_{1i} \otimes (\varepsilon_{1j}\ \varepsilon'_{1i}{\wedge}\varepsilon'_{1j} + \varepsilon'_{1i}\ \varepsilon_{1j}{\wedge}\varepsilon'_{1j}) \\
&= \frac{1}{2}\sum_{j} \varepsilon'_{1j} \otimes \varepsilon_{1j} + \frac{1}{2}\sum_{i} \varepsilon_{1i} \otimes \varepsilon'_{1i} \sum_{j} \varepsilon_{1j}{\wedge}\varepsilon'_{1j} \\
&= -\frac{1}{2}\sum_{j} \varepsilon_{1j} \otimes \varepsilon'_{1j} - \sum_{i} \varepsilon_{1i} \otimes \varepsilon'_{1i} = -\frac{3}{2}\sum_{j} \varepsilon_{1j} \otimes \varepsilon'_{1j}
\end{aligned}$$

and that implies the first formula. For computing $\Psi(\Omega_2)$ and $\Psi(\Omega_3)$, just apply a cyclic permutation.

Since $\Omega$ is the Casimir and $t$ is zero in this case, we have:

$$0 = \Psi(\Omega) = 4a^2\Omega_1 + 4b^2\Omega_2 + 4c^2\Omega_3 + \Psi(\pi)$$

and that proves the lemma. □

**7.8.2 Lemma:** *The module $S^2\widetilde{\mathrm{D}}(2,1) \otimes K$ decomposes into a direct sum $U_0 \oplus U_1 \oplus U_2 \oplus U_3$. The module $U_0$ is isomorphic to $K$ and generated by the Casimir. The homomorphism $\Psi$ respects this decomposition. It acts on $U_0$, $U_1, U_2, U_3$ by multiplication by $0, 2a, 2b, 2c$ respectively.*

**Proof:** Set: $L = \widetilde{\mathrm{D}}(2,1) \otimes K$. Let $V_0$ be the $K$-submodule of $S^2L$ generated by $\Omega_1$, $\Omega_2$, $\Omega_3$, $\pi$. The morphism $\Psi$ induces an endomorphism of $V_0$. The matrix of this endomorphism in the basis $(2\Omega_1, 2\Omega_2, 2\Omega_3, \pi)$ is:

$$\begin{pmatrix} -4a & 0 & 0 & -2a^2 \\ 0 & -4b & 0 & -2b^2 \\ 0 & 0 & -4c & -2c^2 \\ 3 & 3 & 3 & 0 \end{pmatrix}$$

The eigenvalues of this matrix are 0, $2a$, $2b$, $2c$ and corresponding eigenvectors are:

$$\Omega = -a\Omega_1 - b\Omega_2 - c\Omega_3 + \pi$$

$$2a(b-c)\Omega_1 + 6b^2\Omega_2 - 6c^2\Omega_3 - 3(b-c)\pi$$
$$2b(c-a)\Omega_2 + 6c^2\Omega_3 - 6a^2\Omega_1 - 3(c-a)\pi$$
$$2c(a-b)\Omega_3 + 6a^2\Omega_1 - 6b^2\Omega_2 - 3(a-b)\pi.$$

Let $L_0$ be the even part of $L$. Let $F_p$ be the simple $sl_2$-module of dimension $p+1$. This module is the symmetric power $S^pF_1$ and $F_2 = sl_2$. Denote by $[p,q,r]$ the isomorphism class of the $L_0$-module $F_p \otimes F_q \otimes F_r$. These elements form a basis of the Grothendieck algebra $\mathrm{Rep}(L_0)$ of representations of $L_0$. In this algebra we have:

$$[L_0] = [2,0,0] + [0,2,0] + [0,0,2] \qquad [X] = [1,1,1]$$

$$[S^2L_0] = 3[0,0,0] + [4,0,0] + [0,4,0] + [0,0,4] + [2,2,0] + [2,0,2] + [0,2,2]$$
$$[\Lambda^2 X] = [0,0,0] + [2,2,0] + [2,0,2] + [0,2,2]$$
$$[L_0 \otimes X] = 3[1,1,1] + [3,1,1] + [1,3,1] + [1,1,3].$$

The module $V_0$ is the submodule $3[0,0,0]+[0,0,0]$ of $S^2L$. Set $V_0' = V_0$ and define by induction submodules $V_p'$ to be the image of $X \otimes V_{p-1}'$ under the action map. Then set: $V_p = V_0' + \ldots + V_p'$. For every $p \geq 0$, $V_p$ is a $L_0$-module. It is not difficult to prove the following:

$$[V_0] = 4[0,0,0] \qquad [V_1] = 4[0,0,0] + 3[1,1,1]$$

$$[V_2] = 4[0,0,0] + 3[1,1,1] + [2,2,0] + [2,0,2] + [0,2,2] \implies \Lambda^2 X \subset V_2$$
$$[V_3] = 4[0,0,0] + 3[1,1,1] + [2,2,0] + [2,0,2] + [0,2,2] + [3,1,1] + [1,3,1] + [1,1,3]$$
$$\implies \Lambda^2 X \oplus L_0 \otimes X \subset V_3.$$

Then there is a unique $L_0$-submodule $W$ of $S^2L_0 \subset S^2L$ such that $V_3 \oplus W = S^2L$. If $V$ is the $L$-submodule of $S^2L$ generated by $V_0$, the module $S^2L/V$ is a quotient of $W$ and then has no odd degree component. Therefore this module is trivial and $S^2L$ is generated by $V_0$ as a $L$-module, and that implies that $S^2L$ is the direct sum of $L$-modules generated by the eigenvectors above and the lemma is proven. □

Now we are able to apply Theorem 6.3 and we get the desired result. □

**Remark:** There exist an extra Lie superalgebra equipped with a Casimir element: the Hamiltinian algebra $H(n)$ for $n > 4$ and $n$ even and that's a complete list of simple quadratic Lie superalgebras [Kc]. For $n > 4$ the Hamiltonian algebra $L = H(n)$ has the following property: it has a $\mathbf{Z}$-graduation compatible with the Lie bracket, and the Casimir has a nonzero degree. Therefore for any element $u \in \Lambda$ of positive degree, the induced element $\chi_L(u)$ has a nonzero degree. But it is an element of the coefficient field. Hence the character $\chi_L$ is the augmentation character.

## 8. Properties of the characters.

In the last section, we constructed eight characters $\chi_i$, $i = 1 \ldots 8$ corresponding to families $gl$, $osp$, $E_6$, $E_7$, $E_8$, $F_4$, $G_2$ and $\widetilde{D}(2,1)$. These characters are graded algebra homomorphisms from $\Lambda$ to $A_i$, where $A_1 = \mathbf{Q}[t,u]$, $A_2 = \mathbf{Q}[t,v]$, $A_3 = A_4 = A_5 = A_6 = A_7 = \mathbf{Q}[t]$, $A_8 = \mathbf{Q}[\sigma_2, \sigma_3]$.

Consider the subalgebra $R_0 = \mathbf{Q}[t] \oplus \omega\mathbf{Q}[t,\sigma,\omega]$ of $R = \mathbf{Q}[t,\sigma,\omega]$. This algebra is sent to $\Lambda$ by a morphism $\varphi$ defined by:

$$\varphi(t) = t, \qquad \forall p \geq 0, \quad \varphi(\sigma^p\omega) = \omega_p$$

For each $i = 1, \ldots, 8$ there is a unique character $\chi'_i$ from $R$ to $A_i$ which restricts on $R_0$ to $\chi_i \circ \varphi$. These morphisms are defined by:

$$\chi'_1(t) = t \qquad \chi'_1(\sigma) = 2(t^2 - 2u) \qquad \chi'_1(\omega) = 2t(t^2 - 4u)$$

$$\chi'_2(t) = t \qquad \chi'_2(\sigma) = 3(t - 2v)(t + 3v) \qquad \chi'_2(\omega) = 2(t - v)(t - 2v)(t + 4v)$$

$$\chi'_3(t) = t \qquad \chi'_3(\sigma) = \frac{77}{36}t^2 \qquad \chi'_3(\omega) = \frac{25}{12}t^3$$

$$\chi'_4(t) = t \qquad \chi'_4(\sigma) = \frac{176}{81}t^2 \qquad \chi'_4(\omega) = \frac{520}{243}t^3$$

$$\chi'_5(t) = t \qquad \chi'_5(\sigma) = \frac{494}{225}t^2 \qquad \chi'_5(\omega) = \frac{98}{45}t^3$$

$$\chi'_6(t) = t \qquad \chi'_6(\sigma) = \frac{170}{81}t^2 \qquad \chi'_6(\omega) = \frac{480}{243}t^3$$

$$\chi'_7(t) = t \qquad \chi'_7(\sigma) = \frac{65}{36}t^2 \qquad \chi'_7(\omega) = \frac{55}{36}t^3$$

$$\chi'_8(t) = 0 \qquad \chi'_8(\sigma) = 4\sigma_2 \qquad \chi'_8(\omega) = 8\sigma_3.$$

The kernels of these characters are:

$$I_1 = \mathrm{Ker}\chi'_1 = (P_{gl})$$

$$I_2 = \mathrm{Ker}\chi'_2 = (P_{osp})$$

$$I_3 = \mathrm{Ker}\chi'_3 = (P_{exc}, 77t^2 - 36\sigma)$$

$$I_4 = \mathrm{Ker}\chi'_4 = (P_{exc}, 176t^2 - 81\sigma)$$

$$I_5 = \mathrm{Ker}\chi'_5 = (P_{exc}, 494t^2 - 225\sigma)$$

$$I_6 = \mathrm{Ker}\chi'_6 = (P_{exc}, 170t^2 - 81\sigma)$$

$$I_7 = \mathrm{Ker}\chi'_7 = (P_{exc}, 65t^2 - 36\sigma)$$

$$I_8 = \mathrm{Ker}\chi'_8 = (t)$$

with:

$$P_{gl} = \omega - 2t\sigma + 2t^3$$

$$P_{osp} = 27\omega^2 - 72t\sigma\omega + 40t^3\omega + 4\sigma^3 + 29t^2\sigma^2 - 24t^4\sigma$$
$$P_{exc} = 27\omega - 45t\sigma + 40t^3.$$

Using the inclusion $\mathbf{Q}[t,\sigma,\omega] \subset \mathbf{Q}[\alpha,\beta,\gamma]$ we check the following:

$$P_{gl} = (\alpha - t)(\beta - t)(\gamma - t) = -(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$$

$$P_{osp} = (\alpha+2\beta)(2\alpha+\beta)(\beta+2\gamma)(2\beta+\gamma)(\gamma+2\alpha)(2\gamma+\alpha)$$
$$P_{exc} = (3\alpha - 2t)(3\beta - 2t)(3\gamma - 2t).$$

Since characters $\chi'_i$ are surjective each character $\chi_i$ may be consider as a graded algebra homomorphism from $\Lambda$ to a quotient of $R$. These characters are related. The complete relations between them are given by the following result of Patureau-Mirand:

**8.1 Theorem [Pa]:** *Let $I$ be the following ideal in $R$:*

$$I = t\omega P_{gl}P_{osp}(P_{exc}, (77t^2 - 36\sigma)(176t^2 - 81\sigma)(494t^2 - 225\sigma)(170t^2 - 81\sigma)(65t^2 - 36\sigma))$$

*Then there is a unique graded algebra homomorphism $\chi$ from $\Lambda$ to $R_0/I$ such that:*

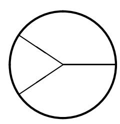
$$\chi_{sl2} \equiv \chi \ mod \ \omega R$$

$$\forall i = 1 \ \dots \ 8, \quad \chi_i \equiv \chi \quad mod \quad I_i$$

**Remark:** It was conjectured in [BN] that every element in $\mathcal{A}$ is detected by invariants coming from Lie algebras in series A, B, C, D. This conjecture is false. There is a weaker conjecture saying that invariants coming from simple Lie algebras detect every element in $\mathcal{A}$. That's also false because of the Lie superalgebra $\widetilde{\mathrm{D}}(2,1)$. Actually we have the following result:

**8.2 Theorem:** *There exists a primitive element of degree 17 in $\mathcal{A}$ which is rationally nontrivial and killed by every weight function obtained by a semisimple Lie (super)algebra and a finite dimensional representation.*

**Proof:** Let $u$ be the following primitive element of $\mathcal{A}$ of degree 2:



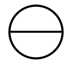
The map $\lambda \mapsto \lambda u$ is a rational injection from $\Lambda$ to the module $\mathcal{P}$ of primitives of $\mathcal{A}$ (see Corollary 4.7). Let $U$ be the image of $P = \omega P_{gl}P_{osp}P_{exc}$ under the morphism $\varphi : R_0 \longrightarrow \Lambda$. This element is detected by $\chi_8$ and is rationally non trivial. Then $Uu$ is an element rationally non trivial in $\mathcal{A}$ of degree 17.

Let $L$ be a simple Lie superalgebra equipped with a Casimir element. If $L$ is of type $i \neq 8$ we have:

$$\Phi_L(Uu) = \chi'_i(P)\Phi_L(u) = 0.$$

If $L$ is of type 8 (i.e. $L = \widetilde{\mathrm{D}}(2,1)$), we have:

$$\Phi_L(Uu) = \chi'_8(P)\Phi_L(u) = \chi'_8(P)\chi_8(t)\Phi_L(\quad).$$



But $\chi_8(t) = 0$. Therefore $Uu$ is killed by $\Phi_L$.

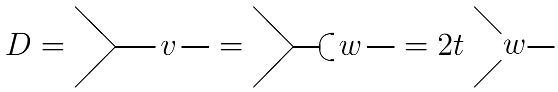
If $L = \oplus L_i$ is semisimple, $\Phi_L(Uu) = \sum \Phi_{L_i}(Uu) = 0$ because $Uu$ is primitive. $\square$

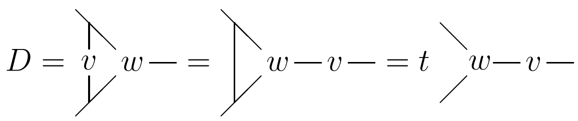
**8.3 Theorem:** *Let $u$ be an element in $\Lambda$ killed by all characters $\chi_i$. Let $L$ be a quadratic Lie superalgebra over a field of characteristic 0. Then $u$ is killed by $\Phi_L$.*

**Proof:** Let $D$ be a connected diagram in $\mathcal{D}(\emptyset, [3])$ representing some element $u'$ in $F(3)$. Let $D_0$ be the union of closed edges meeting $\partial K$ and $D_1$ be the complement of $D_0$ in $K$. We'll said that $D$ is reduced if $D_1$ is connected.

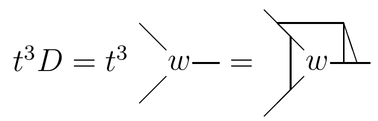
**8.3.1 Lemma:** *Every connected diagram in $\mathcal{D}(\emptyset, [3])$ of degree $> 2$ is equivalent in $F(3)$ to a multiple of a reduced diagram.*

**Proof:** Let $d$ be the degree of a connected diagram $D$. If $d$ is positive and $D$ is not reduced, we have the following possibilities in $F(3)$ (up to some cyclic permutation in $\mathfrak{S}_3$):




Therefore $D$ is equivalent in $F(3)$ to a multiple of $t^i D_1$, with $i < 3$ and $D_1$ reduced or $i = 3$. But it is easy to see the following:




Since a reduced diagram multiply by $t$ is represented by a reduced diagram, the result follows. $\square$

Since $\chi_{gl}$ detects every element in $\Lambda$ in degree $< 6$, we may suppose that $u$ is an element in $\Lambda$ of degree $d \geq 6$. Consider the category of diagrams $\Delta$. Any element in $F(m)$ may be seen as a morphism in $\Delta$ from $\emptyset$ to $[m]$. Let $\beta$ be the bracket from $[2]$ to $[1]$ ($\beta$ is represented by a tree). Because of the lemma, there is an element $v \in F(6)$ such that:

$$u = \beta^{\otimes 3} \circ v.$$

Moreover the degree of $v$ is $d - 3 > 2$.

Consider now a quadratic Lie superalgebra $L$ over a field in characteristic zero and a central extension $E$ of $L$. Denote by $K$ the kernel of $E \longrightarrow L$. The Lie bracket $E \otimes E \longrightarrow E$ is trivial on $K \otimes E + E \otimes K$ and induces an extended bracket $\psi : L \otimes L \longrightarrow E$. So we can set:

$$\Phi_{E,L}(u) = \psi^{\otimes 3}(\Phi_L(v)) \in E^{\otimes 3}.$$

**8.3.2 Lemma:** *Let $I$ be an ideal in $L$ and $I^\perp$ be its orthogonal. Let $E_1$ and $E_2$ be the pullback in $E$ of $I$ and $I^\perp$. Suppose that the inner form is nonsingular on $I$. Then we have:*

$$\Phi_{E,L}(u) = \Phi_{E_1,I}(u) + \Phi_{E_2,I^\perp}(u).$$

**Proof:** The modules $I$ and $I^\perp$ are Lie superalgebras. Since the form is nonsingular on $I$, $L$ is the direct sum $I \oplus I^\perp$. It is easy to see that $I$ and $I^\perp$ are quadratic Lie superalgebras and $E_1 \longrightarrow I$ and $E_2 \longrightarrow I^\perp$ are central extensions. Then we have:

$$\Phi_{E,L}(u) = \psi^{\otimes 3}(\Phi_L(v)) = \psi^{\otimes 3}(\Phi_I(v) + \Phi_{I^\perp}(v)) = \Phi_{E_1,I}(u) + \Phi_{E_2,I^\perp}(u).$$

□

**8.3.3 Lemma:** *Let $I$ be an isotropic ideal of $L$ and $I^\perp$ be its orthogonal. Let $J$ be the quotient $I^\perp/I$. Suppose that the form on $J$ induced by the inner form on $L$ is nonsingular on the center of $J$. Let $E_1$ be the pullback in $E$ of the module $[I^\perp, I^\perp] \subset L$. Then $E_1$ is a central extension of $J_1 = [J,J]$ and we have:*

$$\Phi_{E,L}(u) = \Phi_{E_1,J_1}(u).$$

**Proof:** Since $I$ is a $L$-module, $I^\perp$ and $J = I^\perp/I$ are $L$-modules too. Moreover for any $(x,y,z) \in I \times I^\perp \times L$ we have:

$$< [x,y], z > = < x, [y,z] > = 0$$

and $[x,y]$ is orthogonal to every $z \in L$. Then $[x,y] = 0$ for every $x \in I$ and $y \in I^\perp$ and the bracket is trivial on $I \otimes I^\perp$. Therefore the Lie bracket and the inner form induce a quadratic Lie superalgebra structure on $J = I^\perp/I$.

The central extension $I^\perp \longrightarrow J$ is determined by a 2-cocycle $\varphi : \Lambda^2 J \longrightarrow I$. The cohomology class of $\varphi$ is determined by a morphism $\mathrm{H}_2(J) \longrightarrow I$ and it is possible to modify $\varphi$ by a coboundary in such a way that $\varphi$ and $\mathrm{H}_2(J) \longrightarrow I$ have the same image. Then $I^\perp$ can be identify to $I \oplus J$ and the Lie bracket $[\ ,\ ]_1$ on $I \oplus J$ is given by:

$$\forall \alpha, \beta \in I, \quad \forall x, y \in J, \quad [\alpha + x, \beta + y]_1 = [x,y] + \varphi(x \otimes y)$$

where $\varphi$ is a cocycle satisfying: $\varphi(\Lambda^2(J)) = \varphi(\mathrm{Ker}(\Lambda^2 J \to J))$. The central extension induces an extended bracket $\psi'$ from $\Lambda^2(J)$ to $I^\perp$.

Since the form is non singular on $J$, it is nonsingular on its orthogonal $J^\perp$. Then there exists a module $I^* \subset L$ such that the form is trivial on $I^*$ and $J^\perp$ is the module $I \oplus I^*$. Therefore the Casimir element $\Omega$ decomposes in a sum: $\Omega = \Omega_0 + \Omega_+ + \Omega_-$, where $\Omega_0$, $\Omega_+$ and $\Omega_-$ are in $J \otimes J$, $I \otimes I^*$ and $I^* \otimes I$ respectively.

Suppose $v \in F(6)$ is represented by a connected diagram $D$ such that the edges of $D$ meeting $\partial D$ are disjoint. Therefore there exists a diagram $D'$ representing an element $w$ in $\mathcal{A}(\emptyset, [12])$ such that:

$$v = \beta^{\otimes 6} \circ w.$$

Actually every element in $F(6)$ of positive degree is a linear combination of such diagrams.

Set $\partial D = \{v_i\}$, $i = 1 \dots 6$ and denote by $e_i$ the oriented edge in $D$ starting from $v_i$. Let $U$ be the set of oriented subgraphs of $D$ and $\Gamma$ be an oriented graph in $U$. For each oriented edge $a \in D$ define the module $V_\Gamma(a)$ by:

$$V_\Gamma(a) = \begin{cases} I^* & \text{if } a \in \Gamma \\ I & \text{if } -a \in \Gamma \\ J & \text{otherwise} \end{cases}$$

Let $e$ be an edge in $D$ and $a$ and $-a$ be the corresponding oriented edges. Set $\Omega_\Gamma(e)$ be the component of the Casimir element $\Omega$ in $V_\Gamma(a) \otimes V_\Gamma(-a)$ and denote by $\Omega(\Gamma)$ the tensor product:

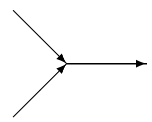
$$\Omega(\Gamma) = \bigotimes_e \Omega_\Gamma(e).$$

For each 3-valent vertex $x$ in $D$ the alternating form $< \ , \ , \ >$ induces a linear form on $V_\Gamma(a) \otimes V_\Gamma(b) \otimes V_\Gamma(c)$ where $a$, $b$, $c$ are the three oriented edges in $D$ starting from $x$. By applying all these forms to $\Omega(\Gamma)$ we get an element $\Phi(\Gamma)$ in $\otimes_i V_\Gamma(e_i)$. It is not difficult to see that $\Phi_L(v)$ is the sum of all $\Phi(\Gamma)$.

Let $x$ be a 3-valent vertex in $D$ and $a$, $b$, $c$ be the oriented edges in $D$ ending at $x$. Since the alternating form $< x, y, z >$ vanishes for $x \in I$ and $y \in I \oplus J$, $\Omega(\Gamma)$ is zero if $a$ is in $\Gamma$ and $-b$ (or $-c$) is not in $\Gamma$.

Denote by $U_+$ the set of all $\Gamma$ in $U$ such that:

for every 3-valent vertex $v$ in $D$, if one oriented edge starting from $v$ is in $\Gamma$ the two other edges ending at $v$ are in $\Gamma$ too.



Then we have:

$$\Phi_L(v) = \sum_{\Gamma \in U_+} \Phi(\Gamma).$$

Let $\Gamma$ be an oriented graph in $U_+$. Suppose $\Gamma$ contains some oriented edge $e$ disjoint from $\partial D$. Since $D \setminus \{e_i\}$ is a connected 3-valent graph, there is a long oriented path $(f_1, f_2, \dots, f_p = e)$ in $D$ such that each oriented edge $f_j$ is in $\Gamma$. Therefore $\Gamma$

contains an oriented cycle $C$. Since the degree of $v$ is at least 2, there exist an edge $e'$ outside of $\{e_i\}$ and meeting $C$ in some vertex $v$. Then there is a long oriented path $(g_1, g_2, \ldots, g_q)$ such that $g_q$ is the edge $e'$ ending at $v$. But this path is necessary included in $D$ because $\Gamma$ is in $U_+$ and that's impossible. Hence $\Gamma$ has to be included in $\{e_i\}$ with the right orientation.

Then we have:

$$\Phi_L(v) = \psi'^{\otimes 6}(\Phi_J(w)).$$

Let $J_2$ be the center of $J$. Since the form is nonsingular, $J$ is the direct sum: $J = J_1 \oplus J_2$. The center of $J_1$ is trivial and then: $J_1 = [J_1, J_1]$. Since $J_2$ is abelian, we have:

$$\Phi_L(v) = \psi'^{\otimes 6}(\Phi_J(w)) = \psi'^{\otimes 6}(\Phi_{J_1}(w) + \Phi_{J_2}(w)) = \psi'^{\otimes 6}(\Phi_{J_1}(w)).$$

Let $I_1$ be the image of $\varphi$. Then the module $[I^\perp, I^\perp]$ is $[I \oplus J, I \oplus J]_1 = I_1 \oplus J_1$. Denote by $\varphi_1$ a 2-cocycle on $\Lambda^2 L$ which determines the extension $E \longrightarrow L$. Let $\alpha \in I_1$ and $x$ and $y$ in $J$. Since $\varphi_1$ is a cocycle, we have:

$$\varphi_1(\alpha \otimes [x, y]_1) = -\varphi_1(x \otimes [y, \alpha]_1) - \varphi_1(y \otimes [\alpha, x]_1) = 0$$

$$\Longrightarrow \quad \varphi_1(\alpha, [x, y] + \varphi(x \otimes y)) = 0.$$

Then if $w$ is in $\mathrm{Ker}(\Lambda^2 J \to J)$ we have: $\varphi_1(\alpha, \varphi(w)) = 0$ and $\varphi_1$ is trivial on $I_1 \otimes I_1$ and therefore on $I_1 \otimes J_1$. Hence the cocycle on $[I^\perp, I^\perp]$ comes from a cocycle on $J_1$ and the extension $E_1 \longrightarrow J_1$ is central. This extension induces an extended bracket $\psi'' : \Lambda^2 J_1 \longrightarrow E_1$ and we have for every $x_1$, $x_2$, $x_3$, $x_4$ in $J_1$:

$$\psi(\psi'^{\otimes 2}(x_1 \otimes x_2 \otimes x_3 \otimes x_4)) = \psi(\psi'(x_1 \otimes x_2) \otimes \psi'(x_3 \otimes x_4)) = \psi''([x_1, x_2] \otimes [x_3, x_4])$$

$$\Longrightarrow \psi \circ \psi'^{\otimes 2} = \psi'' \circ \beta^{\otimes 2}$$

where $\beta$ is the Lie bracket on $L_1$. Therefore we have:

$$\Phi_{E,L}(u) = \psi^{\otimes 3}(\Phi_L(v)) = \psi^{\otimes 3}(\psi'^{\otimes 6}(\Phi_{J_1}(w)))$$

$$= \psi''^{\otimes 3}(\beta^{\otimes 3}(\Phi_{J_1}(w))) = \psi''^{\otimes 3}(\Phi_{J_1}(v)) = \Phi_{E_1,J_1}(u).$$

□

Now we are able to prove that $\Phi_{E,L}(u)$ is zero by induction on $\dim(E)+\dim(L)$.

Let $E$ be a central extension of a quadratic Lie superalgebra $L$. Suppose there is some nontrivial ideal in $L$ contained in its orthogonal. Let $I$ be such a maximal ideal. Set: $J = I^\perp / I$. Since $I$ is maximal, $J$ doesn't contain any nontrivial isotropic ideal and the inner form on $J$ is nonsingular on the center of $J$. Hence $\Phi_{E,L}(u)$ is trivial by induction, because of Lemma 8.3.3.

Suppose $L$ has some nontrivial simple submodule $I$. The inner form is now nonsingular on $I$ and $\Phi_{E,L}(u)$ is trivial by induction, because of Lemma 8.3.2.

So we have to suppose that $L$ is simple. If $L$ is isomorphic to some $sl(E)$, $osp(E)$, $E6$, $E7$, $E8$, $F4$, $G2$, $G(3)$, $F(4)$ or $D(2, 1, \alpha)$, the cohomology of $L$ is isomorphic to the cohomology of some semisimple Lie algebra [G] and $\mathrm{H}^2(L)$ is trivial. Therefore the extension $E \longrightarrow L$ is trivial and there is a section $s$ of it. So we have:

$$\Phi_{E,L}(u) = \psi^{\otimes 3}(\Phi_L(v)) = s^{\otimes 3} \circ \beta^{\otimes 3}(\Phi_L(v)) = s^{\otimes 3}(\Phi_L(u))$$

and this element is trivial because $u$ is killed by each character $\chi_i$.

If $L$ is isomorphic to some $psl(E)$, $\mathrm{H}^2(L)$ is a 1-dimensional module generated by the central extension $sl(E) \longrightarrow psl(E)$. Then there is a morphism $s : sl(E) \longrightarrow psl(E) = L$ and then this extension factorizes through $E$. So we have:

$$\Phi_{E,L}(u) = s^{\otimes 3}(\Phi_{sl(E),L}(u))$$

and $\Phi_{E,L}(u)$ is detected by $\Phi_{sl(E),L}(u)$ and then by $\Phi_{gl(E)}(u)$. Therefore $\Phi_{E,L}(u)$ is trivial because $\Phi_{gl(E)}(u)$ is detected by $\chi_1 = \chi_{gl}$.

In the last possibility $L$ is isomorphic to an Hamiltonian Lie superalgebra $H(n)$ with $n = 2p > 4$. Consider the Hamiltonian Lie superalgebra $E_0 = \widehat{H}(n)$ and its commutator $E_1 = [\widehat{H}(n), \widehat{H}(n)]$ (see the Appendix). Since $\mathrm{H}^2(H(n))$ is 1-dimensional and generated by the central extension $E_1 \longrightarrow H(n)$, there is a morphism $s : E_1 \longrightarrow E$ and this extension factorizes through $E$. So we have:

$$\Phi_{E,L}(u) = s^{\otimes 3}(\Phi_{E_1,L}(u))$$

and $\Phi_{E,L}(u)$ is detected by $\Phi_{E_1,L}(u)$ and then by $\Phi_{E_0}(u)$. But $E_0 = \widehat{H}(n)$ is $\mathbf{Z}$-graded and the degree of its cobracket is $n-4$. Then $\Phi_{E_0}(u)$ is an element in $\Lambda^3 E_0$ of degree $d(n-4)$. On the other hand $E_0$ is concentrated in degrees $-2, -1, \dots, n-2$ and $\Lambda^3 E_0$ is concentrated in degrees $-5, -4, \dots, 3n-7$. If $\Phi_{E_0}(u)$ is nonzero we have:

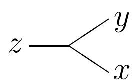
$$d(n-4) \leq 3n-7 \implies (d-3)(n-4) \leq 5 \implies 2(d-3) \leq 5 \implies d \leq 5.$$

But that's not true and $\Phi_{E,L}(u)$ is trivial. $\square$

**8.4 Theorem:** *Let $J$ be the ideal of $R$ generated by $t\omega P_{gl} P_{osp} P_{exc}$. Then $J$ is killed by the morphism $\varphi : R_0 \longrightarrow \Lambda$.*

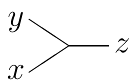
**Proof:** Let $\Delta'$ be the monoidal subcategory of $\Delta$ generated by diagrams where each component meets source and target. Let $X$ be a finite set. If $x$ and $y$ are distinct points in $X$ we may define three morphisms in the category $\Delta'$ in the following way:

Denote by $Y$ the complement: $Y = X \setminus \{x, y\}$. Take a point $z$ (outside of $Y$) and set: $Z = Y \cup \{z\}$. So we define a morphism $\Phi_z^{x,y}$ from $X$ to $Z$ by:

$\Phi_z^{x,y} =$  $\otimes 1_Y$

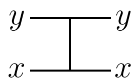


We have also a morphism $\Phi_{x,y}^z$ from $Z$ to $X$ defined by:

$\Phi_{x,y}^x =$  $\otimes 1_Y$



and a morphism $\Psi_{x,y}$ from $X$ to $X$ defined by:

$\Psi_{x,y} =$  $\otimes 1_Y$

The set of all these morphisms will be denoted by $\mathcal{M}$.

Let $f$ be one of these morphisms. The set $\{x, y, z\}$ in the first two cases or the set $\{x, y\}$ in the last case will be called the support of $f$. Using this terminology we have the following relations:

R1: if $f$ and $g$ are two composable morphisms in $\mathcal{M}$ with disjoint support they commute.

R2: $\Psi_{x,y} = \Phi^{z,y}_{y} \circ \Phi^{x}_{x,z}$

R3: $\Psi_{x,y} - \Psi_{x,y} \circ \tau_{x,y} = \Phi^{z}_{x,y} \circ \Phi^{x,y}_{z}$, where $\tau_{x,y}$ is the transposition $x \leftrightarrow y$.

Let $X$ be a finite set and $x$ be an element in $X$. Denote by $Y$ the complement $Y = X \setminus \{x\}$. We have the following morphisms:

$$\Phi_x = \sum_{y \in Y} \Phi^{y}_{x,y}, \qquad \Phi^x = \sum_{y \in Y} \Phi^{x,y}_{y}, \qquad \Psi_x = \sum_{y \in Y} \Psi_{x,y}.$$

They are morphisms from $X$ to $Y$, $Y$ to $X$ and $X$ to $X$ respectively.

The collection of modules $F'(X) = \mathcal{A}^s(\emptyset, X)$ define a $\Delta'$-module $F$. Because of Lemma 3.3 it is easy to see that $\Phi_x$ and $\Phi^x$ act trivially on $F$ and $\Psi_x$ acts on $F$ by multiplication by $2t$. So we may define a new category $\widetilde{\Delta}$: the objects in this category are nonempty finite sets and the morphisms are $\mathbf{Q}[t]$-modules defined by generators and relations where the generators are the bijections in finite sets and the elements in $\mathcal{M}$ and the relations are the following:

— relations R1,R2,R3

— $\Phi_x = 0$, $\Phi^x = 0$ and $\Psi_x = 2t$ for each point $x$ in some finite set.

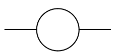
This category contains the category $\mathfrak{S}$ of finite sets and bijections and $F$ is a $\widetilde{\Delta}$-module.

Let $n > 1$ be an integer. Denote by $\Delta_n$ the category of finite sets with cardinal in $\{1, 2, \ldots, n\}$ and morphisms defined by generators and relations:

— generators: bijections and elements in $\mathcal{M}$ involving only sets of cardinal $\leq n$

— relations: relations in $\widetilde{\Delta}$ involving only sets of cardinal $\leq n$.

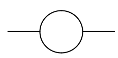
By restriction $F$ induces a $\Delta_n$-module $F_n$. For example $F_2(X)$ is trivial if $\#X \neq 2$ and is the free module generated by:

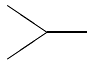


otherwise.

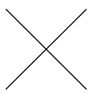
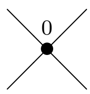
Define the $\Delta_4$-module $G_4$ by:

— $G_4(X) = 0$ if $\#X = 1$

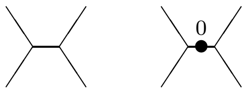
— if $\#X = 2$, $G_4(X)$ is the free $R_0$-module generated by 

— if $\#X = 3$, $G_4(X)$ is the free $R_0$-module generated by 

— if $\#X = 4$, $G_4(X)$ is a direct sum $R_0 \otimes U_1 \oplus R \otimes U_2 \oplus R_0 \otimes V_1 \oplus R \otimes V_2$, where $V_1$ and $V_2$ are 1-dimensional modules generated by the following diagrams:

and $U_1$ and $U_2$ are 2-dimensional simple $\mathfrak{S}_4$-modules generated by the following diagrams:




The action of the category $\widetilde{\Delta}_4$ on this module is defined by Proposition 5.5.

For each $n > 4$ define the module $G_n$ by scalar extension:

$$G_n = \Delta_n \underset{\Delta_{n-1}}{\otimes} G_{n-1}.$$

These modules can be determined by computer for small values of $n$. For every Young diagram $\alpha$ of size $n$ denote by $V(\alpha)$ a $\mathfrak{S}_n$-module corresponding to $\alpha$. If $X$ is a finite set of cardinal $p$, $G_n(X)$ is a $\mathfrak{S}_p$-module and we get the following:

— if $p \leq 4$, $G_4(X) \simeq G_5(X) \simeq G_6(X)$

— if $p = 5$, $G_5(X)$ and $G_6(X)$ are isomorphic to

$$(R_0 \oplus R \oplus R) \otimes V(3,1,1) \oplus R \otimes V(2,1,1,1)$$

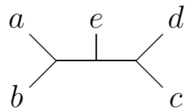
— if $p = 6$, $G_6(X)$ is isomorphic to

$$(R_0 \oplus R^5) \otimes (V(4,2) \oplus V(2,2,2)) \oplus (R_0 \oplus R^3) \otimes V(6)$$

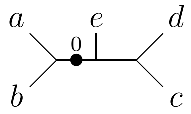
$$\oplus R^2 \otimes (V(3,2,1) \oplus V(2,1,1,1,1)) \oplus R \otimes (V(3,1,1,1) \oplus V(5,1)).$$

For a complete description of $G_6(X)$ in the case $p = 5$ we may proceed as follows:

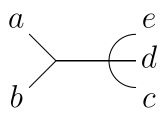
Let $E(X)$ be the $\mathbf{Q}$-vector space generated by the elements of $X$ with the single relation: $\sum_{x \in X} x = 0$. Then $\Lambda^2 E(X)$ and $\Lambda^3 E(X)$ are simple modules corresponding to Young diagrams $(3,1,1)$ and $(2,1,1,1)$ and we can set: $V(3,1,1) = \Lambda^2 E(X)$ and $V(2,1,1,1) = \Lambda^3 E(X)$. So with the identification $G_6(X) = (R_0 \oplus R \oplus R) \otimes V(3,1,1) \oplus R \otimes V(2,1,1,1)$, we have the following:



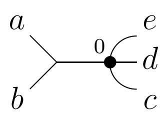
$= A \otimes (a-b) \wedge (d-c),$



$= C \otimes (a-b) \wedge (d-c) + D \otimes (a-b) \wedge d \wedge c,$



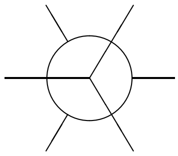
$= (\frac{10}{3} t A + B) \otimes a \wedge b,$



$= (\frac{10}{3} t C + \sigma B) \otimes a \wedge b,$

where $A$ generates a free $R_0$-module and $B$, $C$, $D$ generate free $R$-modules.

Let $X$ be a set of cardinal 6. Consider the element $U$ in $F(X)$ represented by the following diagram:



This element is not in $G_6$ but it corresponds to en element $U_0$ in $G_7$. With the following idempotent in $\mathbf{Q}[\mathfrak{S}_X]$:

$$\pi = \frac{1}{6!} \sum_{\sigma \in \mathfrak{S}_X} \varepsilon(\sigma)\sigma$$

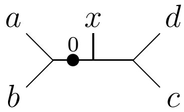
we can set: $V = \pi U$ and $V_0 = \pi U_0$.

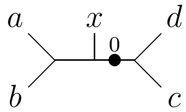
Let $x$ and $y$ be two distinct elements in $X$. Set: $Y = X \setminus \{y\}$ and $Z = X \setminus \{x, y\}$. We have:

$$tV_0 = \frac{1}{2}\Psi_x V_0 = \frac{1}{4}\sum_{z \neq x} \Phi^x_{x,z} \circ \Phi^{x,z}_x V_0 = \frac{\pi}{4}\sum_{z \neq x} \Phi^x_{x,z} \circ \Phi^{x,z}_x V_0 = \frac{5\pi}{4}\Phi^x_{x,y} \circ \Phi^{x,y}_x V_0.$$

It is not difficult to see that $\Phi^{x,y}_x V_0$ is an element in $G_6(Y)$ completely antisymmetric in $Z$.

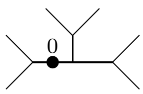
Let $a$, $b$, $c$, $d$ be the elements in $Z$. It is easy to see that every element in $V(3,1,1)$ completely antisymmetric in $a, b, c, d$ is trivial and any element in $V(2,1,1,1) = \Lambda^3 E(Y)$ completely antisymmetric in $a, b, c, d$ is a multiple of $a{\wedge}b{\wedge}c - a{\wedge}b{\wedge}d + a{\wedge}c{\wedge}d - b{\wedge}c{\wedge}d$. Therefore there is an element $P$ in $R$ such that:

$\Phi^{x,y}_x V_0 = P\Big($ 

$-$ 

$\Big).$

But for a diagram like this:




there is a double transposition in $\mathfrak{S}_6$ which acts on it by multiplication by $-1$ and its antisymmetrization is trivial. Therefore $V_0$ and $V$ are killed by $t$.

On the other hand there is a pairing on each $F'(X)$ with values in $\Lambda$:

if $u$ and $u'$ are two elements in $F'(X)$ represented by diagrams $D$ and $D'$, we can glue $D$ and $D'$ along $X$ and we get a connected diagram $D_1$. The class of $D_1$ in $F(0)$ is the multiple of the Theta diagram by some element $\lambda \in \Lambda$. So we set: $< u, u' >= \lambda$.

Consider the element $P =< U, V >$ in $\Lambda$. This element is of degree 15. Since $V$ is killed by $t$, we have in $\Lambda$ the relation: $tP = 0$.

On the other hand we can check by computer that the morphism $G_6(X) \longrightarrow G_7(X)$ is surjective for $\#X < 6$. So $P$ lies in a quotient of $R_0$ and $P$ can be seen as an element in $R_0$.

Since $tP = 0$, $P$ is killed by $\chi_{sl2}$, $\chi_{gl}$ and $\chi_{osp}$. So we have:

$$P = \omega P_{gl} P_{osp} Q$$

for some $Q \in R$ of degree 3. But $Q$ is also killed by the exceptional characters $\chi_i$ and $Q$ is a multiple of $P_{exc}$. At the end we get:

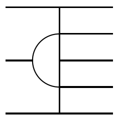
$$P = k\, \omega P_{gl} P_{osp} P_{exc}$$

for some rational $k$. A direct computation (by computer) gives the following result:

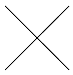
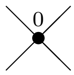
$$P = 2^{-10} \omega P_{gl} P_{osp} P_{exc} \quad \Longrightarrow \quad t\omega P_{gl} P_{osp} P_{exc} = 0 \in \Lambda$$

One can also determine $P$ by using the Lie superalgebra $\widetilde{D}(2,1)$.

Consider the morphism $A$ in $\Delta_6$ defined by the diagram:



The morphism $B = 1 \otimes A$ may be consider as a morphism from $X$ to a set $Z$ of cardinal 4. Let $\pi'$ be the sum of all elements in $\mathfrak{S}_Z$ divided by 4!. Since $B$ lies in $\Delta_7$ the element $\pi' \circ B.V_0$ belongs to $\Delta_7(Z)$ and can be seen as an element $W$ in $G_6(Z) = G_4(Z)$. Since $\mathfrak{S}_Z$ acts trivially on $W$, there is two elements $Q \in R_0$ and $Q' \in R$ such that $W = QH + Q'H'$ with:

$H =$  $\qquad H' =$ 


Degrees of $Q$ and $Q'$ are 12 and 10 respectively. Since $tV_0$ is trivial $W$ is killed by $t$ in $F(Z)$ and $W$ is killed by $\Phi_{sl2}$, $\Phi_{sl_n}$ and $\Phi_{o_n}$.

The functor $\Phi_{sl2}$ kills $H'$ but not $H$. Then $Q$ is killed by $\chi_{sl2}$.

For $n$ big enough the vectors $\Phi_{sl_n}(H)$ and $\Phi_{sl_n}(H')$ are linearly independant. Then $Q$ and $Q'$ are killed by $\chi_{gl}$.

The same holds for $\Phi_{o_n}$ and $Q$ and $Q'$ are killed by $\chi_{osp}$.

Thus there exist $c$ and $c'$ in $\mathbf{Q}$ with: $Q = c\omega P_{gl} P_{osp}$ and $Q' = c' t P_{gl} P_{osp}$.

Let $L$ be an exceptional Lie algebra. Then we have:

$$\Phi_L(H') = \frac{3\omega}{5t} \Phi_L(H) \quad \Longrightarrow \quad \chi_L(5tQ) + \chi_L(3Q') = 0 \quad \Longrightarrow \quad c' = -5/3c.$$

On the other hand we have:

$$P = < H, W > = c\omega P_{gl} P_{osp} < H, H > -5/3ctP_{gl} P_{osp} < H, H' >$$

$$= cP_{gl} P_{osp} (\omega < H, H > -5/3t < H, H' >)$$

and for every $p \geq 0$:

$$0 = < \sigma^p H', tW > = tcP_{gl} P_{osp} (\omega \sigma^p < H', H > -5/3t\sigma^p < H', H' >).$$

Since $P$ is non zero $c$ is non zero too. So we have:

$$t\sigma^p P_{gl} P_{osp}(\omega < H', H > -5/3t < H', H' >) = 0.$$

A direct computation gives:

$$< H', H >= -\frac{3}{2}\sigma\omega + \frac{10}{3}t^2\omega, \qquad < H', H' >= -\frac{3}{2}\sigma^2\omega + \frac{4}{3}t^2\sigma\omega + 2t\omega^2$$

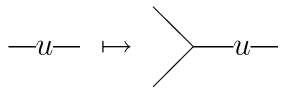
and that implies:

$$0 = t\sigma^p P_{gl} P_{osp}(-\frac{3}{2}\sigma\omega^2 + \frac{5}{2}t\sigma^2\omega - \frac{20}{9}t^3\sigma\omega) = -\frac{1}{18}t\sigma^{p+1}\omega P_{gl} P_{osp} P_{exc}.$$

Therefore $t\sigma^p\omega P_{gl} P_{osp} P_{exc}$ is zero in $\Lambda$ for every $p \geq 0$ and that finishes the proof. □

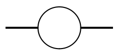
A particular consequence of this result is the fact that a cobracket morphism is not necessarily injective:

**8.5 Proposition:** *The morphism:*

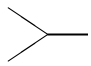



*from $F(2)$ to $F(3)$ is not injective.*

**Proof:** Denote this morphism by $f$. Let $U$ be the image of $\omega P_{gl} P_{osp} P_{exc}$ under the morphism $\varphi : R_0 \longrightarrow \Lambda$. We have: $U \neq 0$ and $tU = 0$. Consider the following element in $F(2)$:

$u = U$ 

Because of Corollary 4.6 $u$ is nonzero. But its image under $f$ is:

$2tU$  $= 0$

and the result follows. □

**Conjecture:** *Let $J$ be the ideal of $R$ generated by $t\omega P_{gl} P_{osp} P_{exc}$. Then the morphism $\varphi : R_0 \longrightarrow \Lambda$ induces an isomorphism from $R_0/J$ to $\Lambda$.*

## 9. Appendix: The Hamiltonian Lie superalgebra $\widehat{H}(n)$.

This section is devoted to the construction of the Lie superalgebra $\widehat{H}(n)$ considered in the proof of Theorem 8.3.

Let $x_1, x_2, \ldots, x_n$ be formal variables (with $n > 0$). Let $E$ be the exterior algebra on these variables. This algebra is graded by considering each $x_i$ as a degree 1 variable. For each $i$ there is a derivation $\partial_i$ sending $x_i$ to 1 and the other variables to 0. So we can define a bracket on $E$ by:

$$[u, v] = \sum_i (-1)^{|u|} \partial_i(u) \wedge \partial_i(v)$$

where $|u|$ is the degree of $u$. Let $f$ be the linear form on $E$ of degree $-n$ sending $x_1{\wedge}x_2{\wedge}\ldots{\wedge}x_n$ to 1.

**9.1 Proposition:** *Let $\widehat{H}(n)$ be the module $E$ with the degree shifted by $-2$. Then the bracket $[\ ,\ ]$ induces on $\widehat{H}(n)$ a structure of Lie superalgebra. Moreover the form:*

$$u \otimes v \mapsto < u, v >= f(u{\wedge}v)$$

*is a nonsingular invariant supersymmetric form on $\widehat{H}(n)$ of degree $4-n$.*

*The center of $\widehat{H}(n)$ is generated by 1. The derived algebra $[\widehat{H}(n), \widehat{H}(n)]$ is the kernel of $f$. Moreover the quotient of $\widehat{H}(n)$ by its center is isomorphic to the Hamiltonian Lie superalgebra $\widetilde{H}(n)$.*

**Proof:** See [Kc] for a description of Hamiltonian algebras $H(n)$ and $\widetilde{H}(n)$. The morphism from $\widehat{H}(n)$ to $\widetilde{H}(n)$ is given by:

$$x_{i_1}{\wedge}x_{i_2}{\wedge}\ldots{\wedge}x_{i_p} \mapsto \sum_{1\le k\le p} (-1)^{k-1}\theta_{i_1}\theta_{i_2}\ldots\widehat{\theta_{i_k}}\ldots\theta_{i_p}\frac{\partial}{\partial\theta_{i_k}}$$

and the proposition is easy to check. □

**Remark:** The algebra $\widehat{H}(n)$ can be determined for small values of $n$. One has:

$$\widehat{H}(1) \simeq osp(1|1) \qquad \widehat{H}(2) \simeq gl(1|1)) \qquad \widehat{H}(3) \simeq osp(2|2) \qquad \widehat{H}(4) \simeq gl(2|2)$$

**9.2 Proposition:** *For $n = 1$ or $n$ even the module $H^2(\widehat{H}(n))$ is trivial and $\widehat{H}(n)$ has no central extension. If $n$ is odd and bigger then 2, $H^2(\widehat{H}(n))$ is 1-dimensional and generated by the cocycle $u \otimes v \mapsto f(u)f(v)$.*

**Proof:** Let $\varphi$ be a 2-cocycle. In order to determine $\varphi$ we'll need some notations:

— A vector in $E$ is called basic if it is a product of distinct $x_i$'s (up to sign).
— The degree of a basic vector $u$ is denoted by $|u|$.
— The support of a basic vector $e = \pm x_{i_1}{\wedge}x_{i_2}{\wedge}\ldots{\wedge}x_{i_p}$ is the set $\{x_{i_1}, \ldots, x_{i_p}\}$.
— $\mathcal{B}$ is the set of collections of basic vectors with disjoint supports.

So we have the following:

$$\forall(u, v, w) \in \mathcal{B}, \quad [u{\wedge}v, u{\wedge}w] = \begin{cases} (-1)^{|u|+|v|}v{\wedge}w & \text{if } |u| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Since $\varphi$ is a 2-cocycle, the following condition

$$(*) \qquad (-1)^{|u||w|}\varphi([u, v] \otimes w) + (-1)^{|v||u|}\varphi([v, w] \otimes u) + (-1)^{|w||v|}\varphi([w, u] \otimes v) = 0$$

holds for every basic vectors $u$, $v$, $w$.

Consider three basic vectors $u$, $v$, $w$. There exist $(e, \alpha, \beta, \gamma, x, y, z)$ in $\mathcal{B}$ such that:

$$u = e{\wedge}\beta{\wedge}\gamma{\wedge}x \qquad v = e{\wedge}\gamma{\wedge}\alpha{\wedge}y \qquad w = e{\wedge}\alpha{\wedge}\beta{\wedge}z$$

and the only possibilities for which $[u,v]$ or $[v,w]$ or $[w,u]$ is nonzero are the following (up to a cyclic permutation):

$$|e| = 1, \quad |\alpha| = |\beta| = |\gamma| = 0$$

$$|e| = 1, \quad |\alpha| = |\beta| = 0, \quad |\gamma| > 0$$
$$|e| = 1, \quad |\alpha| = 0, \quad |\beta| > 0, \quad |\gamma| > 0$$
$$|e| = 0, \quad |\alpha| = 1, \quad |\beta| = |\gamma| = 0$$
$$|e| = 0, \quad |\alpha| = 1, \quad |\beta| > 1, \quad |\gamma| = 0$$
$$|e| = 0, \quad |\alpha| = 1, \quad |\beta| > 1, \quad |\gamma| > 1$$
$$|e| = 0, \quad |\alpha| = |\beta| = 1, \quad |\gamma| = 0$$
$$|e| = 0, \quad |\alpha| = |\beta| = 1, \quad |\gamma| > 1$$
$$|e| = 0, \quad |\alpha| = |\beta| = |\gamma| = 1.$$

By applying the condition $(*)$ to all these cases we get the following relations:

$$(R1) \quad (-1)^{|x||z|}\varphi(x{\wedge}y \otimes z{\wedge}e) + (-1)^{|y||x|}\varphi(y{\wedge}z \otimes x{\wedge}e) + (-1)^{|z||y|}\varphi(z{\wedge}x \otimes y{\wedge}e) = 0$$

$$(R2) \quad \varphi(\gamma{\wedge}z{\wedge}y \otimes x{\wedge}e{\wedge}\gamma) = (-1)^{|x||y|+|\gamma|+1}\varphi(\gamma{\wedge}z{\wedge}x \otimes y{\wedge}e{\wedge}\gamma)$$

$$(R3) \quad \varphi(\beta{\wedge}\gamma{\wedge}y{\wedge}z \otimes \beta{\wedge}\gamma{\wedge}e{\wedge}x) = 0$$

$$(R4) \quad \varphi(y{\wedge}z \otimes x) = 0$$

$$(R5) \quad \varphi(\beta{\wedge}y{\wedge}z \otimes \beta{\wedge}x) = 0$$

$$(R6) \quad \varphi(\beta{\wedge}\gamma{\wedge}y{\wedge}z \otimes \beta{\wedge}\gamma{\wedge}x) = 0$$

$$(R7) \quad (-1)^{|x||y|+|x|+|y|}\varphi(\beta{\wedge}y{\wedge}z \otimes \beta{\wedge}x) = (-1)^{|y||z|}\varphi(\alpha{\wedge}z{\wedge}x \otimes \alpha{\wedge}y)$$

$$(R8) \quad (-1)^{|x||y|+(|\gamma|+1)(|x|+|y|)}\varphi(\beta{\wedge}\gamma{\wedge}y{\wedge}z \otimes \beta{\wedge}\gamma{\wedge}x) = (-1)^{|y||z|+|\gamma|}\varphi(\gamma{\wedge}\alpha{\wedge}z{\wedge}x \otimes \gamma{\wedge}\alpha{\wedge}y)$$

$$(R9) \quad (-1)^{|x||z|}\varphi(\alpha{\wedge}\beta{\wedge}x{\wedge}y \otimes \alpha{\wedge}\beta{\wedge}z) + (-1)^{|y||x|}\varphi(\beta{\wedge}\gamma{\wedge}y{\wedge}z \otimes \beta{\wedge}\gamma{\wedge}x) + (-1)^{|z||y|}\varphi(\gamma{\wedge}\alpha{\wedge}z{\wedge}x \otimes \gamma{\wedge}\alpha{\wedge}y) = 0.$$

Using relations (R4) and (R1) with $|x| = |y| = 0$ and $|z| = n-1$ we get:

$$\forall (u,v) \in \mathcal{B}, \quad \varphi(u \otimes v) = 0.$$

Using relation (R5) we get:

$$\forall (u,v,w) \in \mathcal{B}, \quad |u| > 1, |u| + |v| + |w| < n \implies \varphi(u{\wedge}v \otimes u{\wedge}w) = 0.$$

With the relation (R3) we get:

$$\forall (u,v,w) \in \mathcal{B}, \quad 1 < |u| < n, |u| + |v| + |w| = n \implies \varphi(u{\wedge}v \otimes u{\wedge}w) = 0.$$

The relation (R2) implies:

$$\forall(u,v,w)\in\mathcal{B},\quad |u|=1,\ |w|>0 \quad\Longrightarrow\quad \varphi(u{\wedge}v\otimes w{\wedge}u)=\varphi(u\otimes v{\wedge}w{\wedge}u)$$

and since $\varphi$ is antisymmetric:

$$\forall(u,v,w)\in\mathcal{B},\quad |u|=1 \quad\Longrightarrow\quad \varphi(u{\wedge}v\otimes w{\wedge}u)=\varphi(u\otimes v{\wedge}w{\wedge}u).$$

Finally the relation (R7) implies:

$$\forall(u,v,w)\in\mathcal{B},\quad |u|=|v|=1 \quad\Longrightarrow\quad \varphi(u\otimes w{\wedge}u)=\varphi(v\otimes w{\wedge}v)$$

and $\varphi(u\otimes w{\wedge}u)$ depends only on $w$ (if $|u|=1$) and $\varphi(u{\wedge}v\otimes w{\wedge}u)$ depends only on $[u{\wedge}v, w{\wedge}u]$. Therefore there exist a linear morphism $g$ and a scalar $c$ such that:

$$\varphi(u\otimes v)=g([u,v])+cf(u)f(v)$$

for every $u$ and $v$ in $\widehat{H}(n)$.

On the other hand $\varphi$ is antisymmetric and: $c(1+(-1)^n)=0$.

If $n$ is even, $c=0$ and $\varphi$ is a coboundary. Then $H^2(\widehat{H}(n))$ is trivial.

If $n=1$, $u\otimes v\mapsto f(u)f(v)$ is a coboundary and $H^2(\widehat{H}(n))$ is also trivial.

If $n>2$ is odd, $H^2(\widehat{H}(n))$ is 1-dimensional and generated by the cocycle $u\otimes v\mapsto f(u)f(v)$. □

**9.3 Corollary:** *For $n>1$, $H^2(H(n))$ is a 1-dimensional module generated by the central extension $[\widehat{H}(n),\widehat{H}(n)]\longrightarrow H(n)$.*

**Proof:** Let $L$ be the algebra $[\widehat{H}(n),\widehat{H}(n)]$ and $L_0$ be the quotient $\widehat{H}(n)/L$. By looking in low degree the spectral sequence of the cohomology of the extension:

$$0\longrightarrow L\longrightarrow \widehat{H}(n)\longrightarrow L_0\longrightarrow 0$$

we get the following:

$$n \text{ even} \quad\Longrightarrow\quad H^1(L)\simeq H^2(L)\simeq 0$$

$$n=1 \quad\Longrightarrow\quad d_2: H^1(L)\xrightarrow{\simeq} H^2(L_0)$$

$$n>2,\ n \text{ odd} \quad\Longrightarrow\quad \mathrm{H}^1(L)\simeq 0 \ \text{ and } d_3: H^2(L)\longrightarrow H^3(L_0) \text{ is injective.}$$

So for $n>1$, $H^1(L)$ is trivial.

Let $Z$ be the center of $L$. The spectral sequence of the central extension:

$$0\longrightarrow Z\longrightarrow L\longrightarrow H(n)\longrightarrow 0$$

implies that $H^1(H(n))$ is trivial and the morphism $d_2$ is an isomorphism from $H^1(Z)$ to $H^2(H(n))$. □

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