Vassiliev Theory

by P. Vogel[1]

Abstract.

There exists a natural filtration on the module freely generated by knots (or links). This filtration is called the Vassiliev filtration and has many nice properties. In particular every quotient of this filtration is finite dimensional. A knot invariant which vanishes on some module of this filtration is called a Vassiliev invariant. Almost every knot invariant defined in algebraic term can be describe in term of Vassiliev invariants. Unfortunately the structure of all such invariants is completely unknown. The Kontsevich integral constructed in Christine Lescop's lecture [L] is, in some sense, the universal Vassiliev invariant. It takes values in a module $\mathcal{A}$ of 3-valent diagrams. So a good way to construct a knot invariant is to compose the Kontsevich integral with a linear homomorphism defined on $\mathcal{A}$.

Every Lie algebra equipped with a nonsingular bilinear symmetric invariant form produces a linear homomorphism on $\mathcal{A}$ and therefore a knot invariant. If the Lie algebra belongs to the A series, the induced knot invariant is the HOMFLY polynomial. If the Lie algebra belongs to the B-C-D series, one gets the Kauffman polynomial. The Kauffman bracket is obtained by the Lie algebra sl_2.

1. VASSILIEV INVARIANTS

1.1 Knots and links invariants

A link is a compact 1-dimensional smooth submanifold of $\mathbf{R}^3$. A connected link is called a knot. A link may be oriented or not. Every knot is the image of a embedding f from the circle S^1 into $\mathbf{R}^3$. For a link, the situation is similar but the embedding is defined on a disjoint union of finitely many copies of the circle.

1.2 Definitions. A link L is called banded if L is equipped with a vector field V from L to $\mathbf{R}^3$ such that $V(x)$ is transverse to L for every point $x \in L$.

A link L is called framed if it is oriented and banded.

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Two links L_0 and L_1 are isotopic if there exists an isotopy h_t of the ambiant space $\mathbf{R}^3$ such that h_0 is the identity and L_1 is the link $h_1(L_0)$. If the links are oriented we suppose also that L_1 has the same orientation as $h_1(L_0)$. If (L_0, V_0) and (L_1, V_1) are banded links, there are called isotopic if L_0 and L_1 are isotopic via an isotopy h_t in such a way that V_1 is homotopic to the vector field $h_1(V_0)$ by an homotopy which is allways transverse to L_1. So we have four isotopy relations corresponding to the four classes of links: non oriented, oriented, banded or framed.

An invariant of knots (or links) is a fonction from the set of knots (or links) to some module which is invariant under isotopy. It's also possible to define an invariant of oriented knots (or links), or an invariant of banded knots (or links) or an invariant of framed knots (or links).

Every link can be described by its projection on the plane if it is generic. Such a projection is called a diagram of a link. A diagram of a link is a finite graph D contained in the plane such that every vertex is of order 4. Moreover near every vertex x two edges arriving at x correspond to the over branch and the two other ones correspond to the under branch. The edges corresponding to the over branch are represented by a connected path.

Let L a link represented by a diagram D. If L is oriented, the orientation is represented by an orientation of D. If it is banded it is possible to choose the diagram D in such a way that the tranverse vector field is normal to the plan $\mathbf{R}^2$ with positive last coordonate.

Using this convention every diagram defines a banded link and every oriented diagram defines a framed link and these links are well defined up to isotopy.

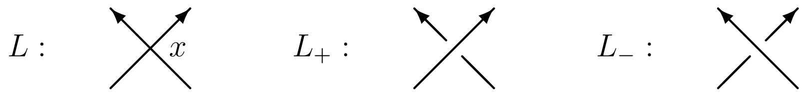
Suppose that L_0 and L_1 are two links related by a family L_t, $0 \leq t \leq 1$, of geometric objects. If every L_t is a link which depends smoothly on t (that is the union $L_{[0,1]}$ of all $L_t \times \{t\}$ is a submanifold of $\mathbf{R}^3 \times [0,1]$) the links L_0 and L_1 are isotopic. But it is possible to consider singular deformation when L_t becomes singular, for some particular values of t. The simplest example of such a singular deformation is when a branch of L crosses another one. When this crossing happens the link becomes a singular link in the following sense:

Definition. A singular link L is the image of an immersion f from a 1-dimensional compact manifold Γ to $\mathbf{R}^3$ such that f has only finitely many multiple points and every multiple point is double and transverse, together with local orientations in Γ near each singular point of f.

A singular link L is oriented if the source Γ of the immersion f is oriented and the local orientations are induced by the orientation of Γ. It is banded if L is equipped with a tranverse vertor field V such that, for every double point x of L, $V(x)$ is tranverse to the plan whiwh is tangent at x to the two branches of L containing x.

If D is a diagram of a link and P a subset of the set of vertices of D, one can associate to (D, P) a singular link L where the double points correspond to the points in P. With the same way as before, the diagram induces a well defined banded structure on L. If D is oriented, L is naturaly framed.

Let L be a singular link and x be a double point in L. One can modify L a little bit near x and obtain a new singular link L' with one double point less. But it is possible to do that in two different ways, and one gets two new links L_+ and L_-.



Since L is supposed to be oriented near x, there is no ambiguity between L_+ and L_-.

If L is banded, the two desingularized links L_+ and L_- are still banded.

1.3 Lemma. *Let I be an invariant of oriented knots. Then I extends uniquely to an invariant defined on the set of all singular oriented knots and satisfying the following property:*

If K is a singular otiented knot and K_+ and K_- are the two knots obtained by desingularization near a double point in K, one has:

$$I(K) = I(K_+) - I(K_-)$$

The extension of I may be defined in the following way:

Let K be a singular oriented knot. Denote by X the set of double points in K and by $\mathcal{F}$ the set of functions from X to $\{\pm 1\}$. If α is a function in $\mathcal{F}$ one can desingularize K near every every double point in K by using the positive or the negative move near a point x if $\alpha(x) = 1$ or -1. So for every $\alpha \in \mathcal{F}$ one gets a knot K_α. Then one sets:

$$I(K) = \sum_{\alpha \in \mathcal{F}} \varepsilon(\alpha) I(K_\alpha)$$

where $\varepsilon(\alpha)$ is the product of all numbers $\alpha(x)$, $x \in X$.

1.4 Definition. Let I be an invariant of knots. One said that I is a Vassiliev invariant of degree $\leq n$ if I vanishes on every oriented singular knot with at least $n + 1$ double points.

Remark. If I is an invariant of oriented links, or an invariant of knots (or links) or banded knots or links or framed knots or links, it is possible to extend I to the corresponding set of singular knots or links and one can define a Vassiliev invariant of knots (or links), or banded knots (or links) or framed knots (or links).

Example. Let L be a singular oriented link with only one double point x. One can modify L near x in three different ways:

These three links have no double point. The Conway polynomial ∇ is the only polynomial invariant of oriented links which is equal to 1 for the trivial knot and satisfies the following skein relation:

$$\nabla(L_+) - \nabla(L_-) = t\nabla(L_0)$$

For every oriented link L, $\nabla(L)$ is a polynomial in the ring $\mathbf{Z}[t]$.

1.5 Proposition. *The n^{th} coefficient of the polynomial ∇ is a Vassiliev invariant of degree n.*

Proof: The skein relation shows that $\nabla(L)$ is divisible by t^n if L is a singular link with at least n double points and the n^{th} coefficient a_n of the polynomial ∇ is an integral invariant of oriented links which vanishes on every singular links with at least $n+1$ double points. The result follows. □

If E is a module, denote by $\mathcal{I}(E)$ the set of invariants of knots with values in E. For every integer $n \geq 0$, denote by $V_n(E)$ the set of Vassiliev invariants of degree $\leq n$.

1.6 Proposition. *Let R be a ring. Then the R-modules $V_n(R)$ form an increasing family of finitely generated R-submodules of $\mathcal{I}(R)$:*

$$V_0(R) \subset V_1(R) \subset V_2(R) \subset \ldots \subset \mathcal{I}(R)$$

Moreover one has: $V_p(R)V_q(R) \subset V_{p+q}(R)$ for every $p, q \geq 0$.

Remark. This proposition is also true for the module of invariants of knots or banded knots or framed knots. For links it is also true but only for invariants of links with a fixed number of components.

In order to prove such a result one needs to study more precisely the set of singular links (or knots).

2. THE ALGEBRA OF KNOTS

Let $\mathcal{N}$ be the set of isopopy classes of oriented knots and $\mathbf{Z}[\mathcal{N}]$ be the free $\mathbf{Z}$-module generated by $\mathcal{N}$. The connected sum operation induces on $\mathcal{N}$ a structure of commutative monoid. With this structure $\mathbf{Z}[\mathcal{N}]$ becomes a commutative algebra. Moreover the map $K \mapsto K \otimes K$ induces a comultiplication Δ from $\mathbf{Z}[\mathcal{N}]$ to $\mathbf{Z}[\mathcal{N}] \otimes \mathbf{Z}[\mathcal{N}]$. With these structures $\mathbf{Z}[\mathcal{N}]$ is a commutative and cocommutative Hopf algebra (but without antipode map).

The inclusion $\mathcal{N} \subset \mathbf{Z}[\mathcal{N}]$ is obviously an invariant of oriented knots. Therefore it extends to singular knots and every singular link may be see as an element of $\mathbf{Z}[\mathcal{N}]$. So, using previous notations, one has the following, for every singular oriented knot K:

$$K = \sum_{\alpha \in \mathcal{F}} \varepsilon(\alpha) I(K_\alpha)$$

Let's denote by I_n the submodule of $\mathbf{Z}[\mathcal{N}]$ generated by singular knots with at least n double points. With this notation an invariant I of oriented knots is a Vassiliev invariant of degree $\leq n$ if it vanishes on I_{n+1} and the module $V_n(R)$ is isomorphic to the R-module $\mathrm{Hom}(\mathbf{Z}[\mathcal{N}]/I_{n+1}, R)$.

2.1 Proposition. *The submodules I_n form a filtration of $\mathbf{Z}[\mathcal{N}]$ which is compatible with the Hopf algebra structure. The graded associated algebra $GA = \oplus I_n/I_{n+1}$ is a connected graded Hopf algebra.*

2.2 Proposition. *The Hopf algebra GA is finitely generated in each degree and $GA \otimes \mathbf{Q}$ is a polynomial algebra.*

Proof: We have to check the following conditions:

$$\mathbf{Z}[\mathcal{N}] = I_0 \supset I_1 \supset I_2 \supset \dots$$

$$\forall p, q \qquad I_p I_q \subset I_{p+q}$$

$$\forall n \qquad \Delta(I_n) \subset \bigoplus_{p+q=n} I_p \otimes I_q$$

The first property is obvious. The connected sum operation extends linearly to connected sum for singular links and the second property follows. The third one is a consequence of the following formula:

$$\Delta(K) = \sum K_\alpha \otimes K_{\alpha-1}$$

where K is any singular oriented knot, and α run in the set of all functions from the set X of double points of K to $\{0, 1\}$. If β is a function from X to $\{-1, 0, 1\}$, K_β denote the singular knot obtained from K by a positive (resp. negative) modification near every double point x where β is equal to 1 (resp. -1).

Then GA is a graded Hopf algebra which is connected, commutative and cocommutative. Therefore GA is rationaly a symmetric algebra over the module of primitive elements (i.e. elements x satisfying $\Delta(x) = 1 \otimes x + x \otimes 1$). If $\{x_i\}$ is a homogenious basis of the module of primitive elements, $GA \otimes \mathbf{Q}$ is the polynomial algebra $\mathbf{Q}[\{x_i\}]$.

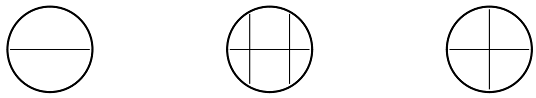
The last thing to do is to prove that I_0/I_n is finitely generated for every n. To do that, it is enough to show that I_n/I_{n+1} is finitely generated for every $n \geq 0$. This point will be proven in the next section.

Remark. If one consider the case of unoriented knots, the connected sum operation is no longer defined. We still have a Vassiliev filtration but the corresponding graded module is only a cocommutative coalgebra. The same hold for other classes of knots or links. Nevertheless, in the case of links (unoriented, oriented, banded or framed), one can use the disjoint union operation. In these cases the graded associated modules are Hopf algebras.

3. CHORD DIAGRAMS AND 3-VALENT DIAGRAMS

Definition. A *chord diagram* is a collection of disjoint pairs of points in the standard circle S^1. This structure is defined up to isotopy in the circle. Such a pair $\{a, b\}$ is called a chord. It is represented by the chord $[a, b]$ in the plan. In other words a chord diagram is a picture in the plan represented by the standard circle and finitely many chords with disjoint boundaries.

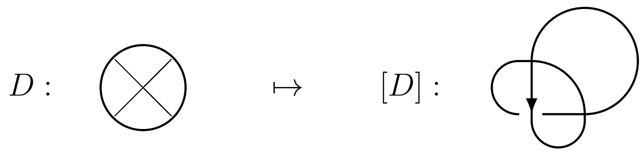
Notice that the set $\mathcal{D}^c_n$ of chord diagrams with exactly n chords is finite.



Let D be a chord diagram represented by n chords $c_i = [a_i, b_i]$, $i = 1, \ldots, n$ in the circle S^1. Then there exists a unique immersion f (unique up to regular homotopy) such that the image of f is a singular knots having exactly n double points: the points $f(a_i) = f(b_i)$, $i = 1, \ldots, n$.

If f and g are two such immersions, they are regularly homotopic. Then there exist p such immersions h_j such that $f = h_1$, $g = h_p$ and h_{j+1} is obtained from h_j by making a crossing change somewhere. Let K_j be the image of h_j. Then K_j and K_{j+1} are obtained from a singular knot K'_j with $n+1$ double points by the positive and negative moves near some double point of K'_j. Therefore the difference $K_{j+1} - K_j$ belongs to I_{n+1} and the two singular knots corresponding to f and g induce the same element in I_n/I_{n+1}.

Hence every chord diagram $D \in \mathcal{D}^c_n$ induces a well defined element $[D]$ in $I_n/I_{n+1} = GA_n$.



Since I_n is generated by singular knots with exactly n double points, this correspondence induces a surjective homomorphism from $\mathbf{Z}[\mathcal{D}^c_n]$ onto I_n/I_{n+1}. Therefore the Hopf algebra GA is finitely generated in each degree.

Remark. For unoriented knots the situation is different. In order to consider a singular knot as an element of $\mathbf{Z}[\mathcal{N}]$, one need to consider local orientation near each double point. So a chord diagram in this context is a manifold Γ diffeomorphic to a circle equipped with chords and local orientations near each endpoint of a chord. Moreover if one changes a local orientation in some place, the induced element in I_n/I_{n+1} is multiply by -1.

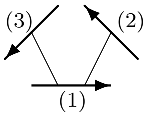
For banded knots there is another problem. If a chord diagram D is given, it is not possible to define naturally a banded structure on the singular knot constructed from D. The singular knot can be represented by a diagram Δ contained in the plan. This diagram is oriented and has n double points, p positive crossing and q negative crossing. If one positive crossing is replaced by a negative crossing, the difference of the two corresponding singular knots belongs to I_{n+1}. Therefore the class of the singular knot depends only on the class of $p - q$ mod 2.

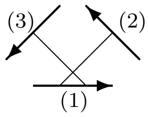
Actually there is two functions $D \mapsto [D]_0$ and $D \mapsto [D_1]$. The first one corresponds to the case $p - q$ even and the other one to the case $p - q$ odd. In the case of banded knots (or framed knots) one has a surjection from $\mathbf{Z}[\mathcal{D}_n^c] \oplus \mathbf{Z}[\mathcal{D}_n^c]$ onto I_n/I_{n+1}.

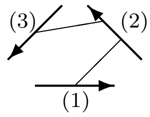
3.2 Proposition. *The morphism $D \mapsto [D]$ satisfy the following relations:*
— If D contains an isolated chord, then: $[D] = 0$.

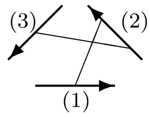
(1T) $D =$  $\Longrightarrow \quad [D] = 0$

— Let D_1, D_2, D_3 and D_4 be four chord diagrams which differ only in three parts of the circle and two chords and which have the following form near these chords:

D_1 : 

D_2 : 

D_3 : 

D_4 : 

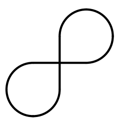
then one has the following:

(4T) $$[D_1] - [D_2] = [D_3] - [D_4]$$

Remark. The relation (1T) is called the 1T (one term) relation. It holds for oriented or unoriented knots but not for framed knots or banded knots. The relation (4T) is called the 4T (four terms) relation. It holds for every class of knots or links.

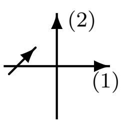
In the case of banded or framed knots, the 4T relation hlods for both functions $[?]_0$ and $[?]_1$.

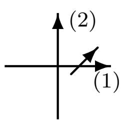
Proof: If D is the chord diagram with one chord, the corresponding singular knot K may be chosen to be represented by the following diagram:

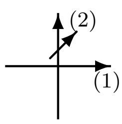
K : 

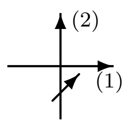
Therefore the knots K_+ and K_- are both trivial and $[K]$ is the zero element. In the case of banded or framed knots K_+ and K_- don't have the same banded structure or the same framing and the property 1T doesn't work.

Let K_1 be a singular knot corresponding to the diagram D_1. The diagrams D_2, D_3 and D_4 may be represented by singular knots K_2, K_3 and K_4 differing from K_1 only near parts (1), (2) and (3). Near this region the knots K_i look like the following:

K_1 : 

K_2 : 

K_3 : 

K_4 : 

In order to compute the class of each K_i in I_n, it is possible to modify K_i near the part (3). Denote by P the plan generated by parts (1) and (2) of the knot. This plan is cut into four pieces: the upper-right piece P_1, the upper-left piece P_2, the under-left piece P_3 and the under-right piece P_4.

If (3) doesn't cross parts (1) and (2), it crosses P in some P_i and defines a singular knot K'_i. With this conventions one has:

$$K_1 = K'_3 - K'_2 \qquad K_2 = K'_4 - K'_1 \qquad K_3 = K'_1 - K'_2 \qquad K_4 = K'_4 - K'_3$$

and the 4T relation follows. □

This notion of chord diagram is generalizable in the following way: Let Γ be a compact one-dimensional manifold. A n-chord diagram on Γ is a finite collection of n disjoint pairs of points (or chords) in the interior of Γ and a orientation of Γ near each point in this collection. The set of n-chord diagram on Γ will be denoted by $\mathcal{D}_n^c(\Gamma)$.

If Γ has no boundary, a n-chord diagram D on Γ induces a singular link $[D]$ which is oriented near each double point. Moreover $[D]$ is well defined modulo the submodule I_{n+1} generated by singular links with $n+1$ double points. If Γ has a boundary the same construction works but one has to consider embeddings of $(\Gamma, \partial\Gamma)$ in the pair $(B^3, \partial B^3)$ instead of links.

In all this cases the 4T relation holds.

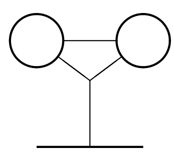
3.3 Definition. Let Γ be a curve (i.e. a compact one-dimensional manifold). A Γ-diagram is a triple (K, f, α) where:

— K is a finite graph and f is a homeomorphism from Γ to a sub-graph of K such that every point in $f(\partial\Gamma)$ is a univalent vertex of K and every other vertex of K has valency 3

— α is a function which associates to every 3-valent vertex x in K a cyclic ordering $\alpha(x)$ between the three edges of K starting from x.

Usually a Γ-diagram will be represented by a graph K immersed in the plane and containing Γ. The cyclic orderings are given by the orientation of the plane.

Such a diagram is also called 3-valent diagram in the litterature.



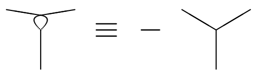
A chord diagram K is actually a particular S^1-diagram. The graph is the union of the circle and the chords. The vertices of K are the endpoints of the chords. If x is such a vertex, on has a natural cyclic ordering: first the chord, after the arc of the circle arriving at x and then the arc of the circle starting from x.

In the same way a n-cord diagram on a curve Γ is a particular Γ-diagram and we have a map from $\mathcal{D}_n^c(\Gamma)$ to the set of Γ-diagrams.

3.4 The module $\mathcal{A}(\Gamma)$. Let Γ be a compact one-dimensional manifold. Let K be a field of characteristic zero. The module $\mathcal{A}(\Gamma)$ is the K-module given by the following presentation:

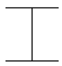
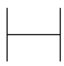
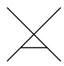
The generators are the Γ-diagrams. The relations are the following:

— If a Γ-diagram K is obtained from a Γ-diagram K' by changing a cyclic ordering in one place, then: $K \equiv -K'$.

(AS) 

— the IHX relation (also called Jacobi relation):

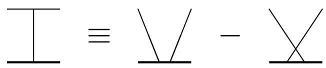
If three Γ-diagrams K, K' and K'' differ only near an edge in the following way:

K :  K' :  K'' : 

one has:

(IHX) $$K \equiv K' - K''$$

In the IHX relation, the edge is not necessary outside of Γ. If not the relation takes the following form:

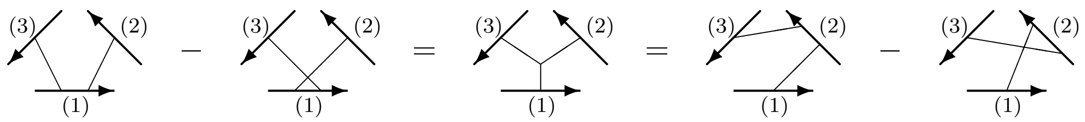
(STU) 

This relation is called the (STU) relation.

Remark. The module $\mathcal{A}(\Gamma)$ is a graded module. The degree of a Γ-diagram K is half the number of three-valent vertices of K.

3.5 Proposition. *Let Γ be a curve. Let $\mathcal{A}_c(\Gamma)$ be the submodule of $\mathcal{A}(\Gamma)$ spanned by Γ-diagrams K such that each component of K meets Γ. Then the natural map from $\mathcal{D}_n^c(\Gamma)$ to the set of Γ-diagrams induces an isomorphism from the quotient module $K[\mathcal{D}_n^c(\Gamma)]/(4T)$ to the degree n part $\mathcal{A}_c(\Gamma)_n$ of the module $\mathcal{A}_c(\Gamma)$.*

This result is proven by Bar-Natan [BN] in the case $\Gamma = S^1$. The general case can be done exactly in the same way. The fact that the 4T relation holds in $\mathcal{A}_c(\Gamma)$ is easy to check:



3.6 Algebraic properties of $\mathcal{A}$.

3.7 Proposition. *An inclusion i from a curve Γ to a curve Γ' induces a homomorphism of graded modules from $\mathcal{A}(\Gamma)$ to $\mathcal{A}(\Gamma')$.*

If Γ and Γ' are two curves the disjoint union operation induces a homomorphism from $\mathcal{A}(\Gamma)\otimes\mathcal{A}(\Gamma')$ to $\mathcal{A}(\Gamma\coprod\Gamma')$.

If f is a continous map from a curve Γ to a curve Γ' sending boundary to boundary, f induces a well define homomorphism f^* from $\mathcal{A}(\Gamma')$ to $\mathcal{A}(\Gamma)$. Moreover if f and g are two homotopic maps from $(\Gamma,\partial\Gamma)$ to $(\Gamma',\partial\Gamma')$, the homomorphisms f^* and g^* are equal.

Sketch of proof: The first homomorphism send a Γ-diagram K to the union $K\cup\Gamma'$. It is easy to see that AS and IHX relations are satisfied and this homomorphism is well defined.

The second homomorphism send $K\otimes K'$ to the disjoint union $K\coprod K'$.

The last homomorphism is more complicated to define. First suppose that f is smooth and has only finitely many critical values corresponding to distinct critical points. Let K be a Γ'-diagram. Let $\{x_i\}$ be the set of vertices of K contained in Γ' and H be the closure in K of the complement $K-\Gamma'$. Suppose also that every x_i is a regular value of f. The space H is a finite graph and the set of univalent vertices of H is exactly the set $\{x_i\}$. The diagram K is the union (over $\{x_i\}$) of Γ' and H. Moreover, for each vertex x_i, the cyclic ordering near x_i induces an orientation ω_i of a neighborough of x_i in Γ. Let F be the set of functions s from $\{x_i\}$ to Γ such that $s(x_i)\in f^{-1}(x_i)$ for every x_i. This set F is finite because f is a finite covering over a neighborough of $\{x_i\}$.

For every $s\in F$ the union of Γ and H, where each $x_i\in H$ is identify with the point $s(x_i)\in\Gamma$ is a finite graph K_s. Since f is étale near every point $s(x_i)$, the local orientations ω_i induces local orientations ω'_i near each point $s(x_i)$. The graph K_s equipped with these local orientations induces a Γ-diagram still denoted by K_s. Then one defines $f^*(K)$ as the sum in $\mathcal{A}(\Gamma)$ of all K_s.

Suppose now that f is any continous map from $(\Gamma,\partial\Gamma)$ to $(\Gamma',\partial\Gamma')$. This fonction is homotopic to a smooth function f' such that every point x_i is a regular value of f'. Then we set: $f^*(K)=f'^*(K)$. If f is homotopic to another function f'' satisfying the same property, f' is smoothly homotopic to f'' by a homotopy h_t such that every h_t is smooth and only finitely many h_t has some x_i as critical value. One mays also suppose that each of these critical functions h_t has only one x_i as critical value corresponding to only one non degenerate critical point. The element $h_t^*(K)$ is well defined for every non critical function h_t. It is easy to see that this function is locally constant in t and the AS relation show that each jump of this function is trivial. Therefore this function is constant and $f'^*(K)$ is equal to $f''^*(K)$. Consequently $f'^*(K)$ depends only on K and the homotopy class of f.

In order to prove that f^* is compatible with AS and IHX relation it is enough to consider the case where f is smooth and has no critical value where the geometry of the diagrams is modified by the relation. In this case f^* send an AS relation to a sum of AS relation and an IHX relation to a sum of IHX relations. □

Corollary. *For every curve Γ, $\mathcal{A}_c(\Gamma)$ is a graded cocmmutative coalgebra.*

Proof: Let f be the first projection from $\Gamma \times \{0,1\}$ to Γ. The homomorphism f^* sends $\mathcal{A}_c(\Gamma)$ to $\mathcal{A}_c(\Gamma \times \{0,1\})$. Consider the module $\mathcal{A}_c(\Gamma \times \{0,1\})$ quotiented by the submodule M spanned by all diagrams K for which some component meets $\Gamma \times \{0\}$ and $\Gamma \times \{1\}$. We have a natural map:

$$\mathcal{A}_c(\Gamma \times \{0\}) \otimes \mathcal{A}_c(\Gamma \times \{1\}) \longrightarrow \mathcal{A}_c(\Gamma \times \{0,1\})/M$$

wich is obvioulу an isomorphism. Then the homomorphism f^* composed with the quotient map induces a homomorphism Δ from $\mathcal{A}_c(\Gamma)$ to $\mathcal{A}_c(\Gamma) \otimes \mathcal{A}_c(\Gamma)$. It is not difficult to see that $\mathcal{A}_c(\Gamma)$ equipped with this homomorphism Δ is a graded cocommutative coalgebra. □

3.8 Proposition. *The graded modules $\mathcal{A}_c([0,1])$ and $\mathcal{A}_c(S^1)$ are isomorphic graded commutative and cocommutative Hopf algebras.*

Proof: Let f be an embedding from $[0,1]$ to the circle S^1 which is compatible with the orientations. This injection induces a homomorphism φ from $\mathcal{A}_c([0,1])$ to $\mathcal{A}_c(S^1)$. This homomorphism depends only on the isotopy class of f and then it is well defined. It is easy to see that φ is an epimorphism of coalgebras. The fact that φ is an isomorphism is proven in [BN1]. The algebra structure on $\mathcal{A}_c([0,1])$ comes from the embedding from $[0,1] \coprod [0,1]$ to $[0,1]$ which is an increasing bijection from the first copy of $[0,1]$ to $[0,1/3]$ and an increasing bijection from the second copy of $[0,1]$ to $[2/3,1]$.

The fact that the product is commutative can be see in the circle. If u and v are represented by $[0,1]$-diagrams K and K', uv is represented by the diagram L obtained by placing K' after K on the interval. But in the circle, one can move K' around the circle and uv is equal to vu in $\mathcal{A}_c(S^1)$. Since the inclusion map from $\mathcal{A}_c([0,1])$ to $\mathcal{A}_c(S^1)$ is bijective, one has: $uv = vu$. □

4. THE CATEGORY OF DIAGRAMS

A good way to understand knots or link is to cut such a link by horizontal planes. So one gets one-dimensional submanifolfs of $\mathbf{R}^2 \times [0,1]$ with boundary in $\mathbf{R}^2 \times \{0,1\}$. These objects are called tangles. The tangles form a category $\mathcal{T}$. An object in the category $\mathcal{T}$ is a finite subset X in the plane $\mathbf{R}^2$. A morphism from X to Y is an isotopy class of tangle in $\mathbf{R}^2 \times [0,1]$ with boundary $X \times \{0\} \cup Y \times \{1\}$.

One may also consider the category of oriented tangles, banded tangles or framed tangles. In view on the Kontsevich integral it is also convenient to consider categories of parenthetized tangles (or non associative tangles). In this category the morphisms are the same but the objects are more general. They are points written in a non-associative way in $\mathbf{R} \subset \mathbf{R}^2$. See Christine Lescop's lecture [L].

The same consideration holds for diagrams.

Definition. Let Γ be a curve (i.e. a compact one-dimensional manifold). Let X be a finite set. A (Γ, X)-diagram is a triple (K, f, g, α) where:

— K is a finite graph and g is an injection from X to the set of univalent vertices of K

— f is a homeomorphism from Γ to a sub-graph of K such that the set of univalent vertices of K is the disjoint union of $g(X)$ and $f(\partial\Gamma)$

— every vertex in K is univalent or three-valent

— α is a function which associates to every 3-valent vertex x in K a cyclic ordering $\alpha(x)$ between the three edges of K starting from x.

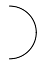
A $(\Gamma, \emptyset)$-diagram is nothing else but a Γ-diagram. Usually a (Γ, X)-diagram will be represented by a graph immersed in the plan and containing Γ and X, and the cyclic orderings are induced by the orientation of the plan.

The AS and IHX relations make sense for (Γ, X)-diagrams. So one can define the quotient:

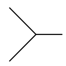
The graded module $\mathcal{A}(\Gamma, X)$. The module $\mathcal{A}(\Gamma, X)$ is the module freely generated by all (Γ, X)-diagram and quotiented by the AS and IHX relations. The degree of an element of this module which is represented by a diagram K is half the number of 3-valent vertices of K. This degree is half an integer. Twice this degree is congruent to the order of X modulo 2.

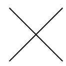
4.1 The category of diagrams $\mathcal{D}$. The objects of $\mathcal{D}$ are the finite sets $[n] = \{1, \ldots, n\}$, $n \geq 0$. A morphism from an object X and an object Y is an element of the module $\mathcal{A}(\emptyset, X \coprod Y)$. The composite of a morphism from X to Y and a morphism from Y to Z is obtained by taking the union of diagrams over Y.

4.2 Proposition. *The category $\mathcal{D}$ is a monoidal linear category. As a monoidal category it is generated by one object (the oject $[1]$) and four morphisms:*

— *the morphism d_1 from $[2]$ to $[0]$ represented by* 

— *the morphism d_2 from $[0]$ to $[2]$ represented by* 

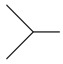
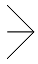
— *the morphism d_3 from $[2]$ to $[1]$ represented by* 

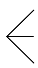
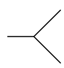
— *the morphism d_4 from $[2]$ to $[2]$ represented by* 

Proof: The monoidal structure is given by the disjoint union. The tensor product of two objects $[p]$ and $[q]$ is the object $[p+q]$, and the first p points in $[p+q]$ correspond in the standard way to the points in $[p]$ and the last q points in $[p+q]$ to the points in $[q]$. The tensor product of two morphisms u and v represented by two diagrams K and K' is the morphism represented by $K \coprod K'$.

It is easy to see that the category $\mathcal{D}$ is a monoidal category and the monoidal structure is strictly associative. Moreover for every objects X and Y the set $\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathcal{A}(\emptyset, X \coprod Y)$ is a module and the composition and the tensor product are both bilinear.

The object $[p]$ is the tensor product of p copies of $[1]$. A morphism between $[p]$ and $[q]$ is a linear combination of morphisms corresponding to diagrams. Consider now such a diagram K. This diagram is a finite graph containing $[p]\coprod[q]$. There exist a subdivision K' of K and a function f from K to $[0,1]\times\mathbf{R}$, which is affine in every edge of K' and send every point $i\in[p]$ to $(0,i)$ and every point $j\in[q]$ to $(1,j)$. Let X be the set of double points of f and Y be the image under f of the set of vertices of K'. If f is chosen to be generic enough, the image of $X\cup Y$ under the first projection consists of distinct points. Then one can cut $[0,1]\times\mathbf{R}$ into pieces $U_i=[a_i,a_{i+1}]\times\mathbf{R}$, with: $0=a_0<a_1<\ldots<a_n=1$, and $X\cup Y$ meets U_i in at most one point. Then the morphism induced by K is the composite of n elementary morphisms corresponding to the diagram $K_i=K\cap U_i$. By construction each such morphism is on the form $\mathrm{Id}\otimes u\otimes\mathrm{Id}$ where u is the morphism d_4 or a morphism corresponding to a connected graph H with at most one 3-valent vertex. If H has no 3-valent vertex u is 1 or d_1 or d_2. If H has one 3-valent vertex, we have four possibilities for H, but each of these may be express as composite of morphisms d_1, d_2 and d_3. Precisely the four possibilities are:

 $= d_3$  $= d_1\circ(1\otimes d_3)$

 $=(1\otimes d_3\otimes 1)\circ(1\otimes 1\otimes d_2)\circ d_2$  $=(d_3\otimes 1)\circ(1\otimes d_2)$

Therefore the four morphisms d_i generate the full category. □

4.3 Remark. Actually it is possible to describe the monoidal category $\mathcal{D}$ by generators and relations. The generators are the object [1] and morphisms d_1, d_2, d_3 and d_4. The relations are the following:

— $d_3\circ d_4=-d_3$
— $d_3\circ(d_3\otimes 1)\circ\Big(1\otimes 1\otimes 1+(d_4\otimes 1)\circ(1\otimes d_4)+(1\otimes d_4)\circ(d_4\otimes 1)\Big)=0$
— $d_1\circ d_4=d_1$
— $d_1\circ(d_3\otimes 1)=d_1\circ(1\otimes d_3)$
— $d_4\circ d_4=1\otimes 1$
— $1=(d_1\otimes 1)\circ(1\otimes d_2)=(1\otimes d_1)\circ(d_2\otimes 1)$
— $(d_4\otimes 1)\circ(1\otimes d_4)\circ(d_4\otimes 1)=(1\otimes d_4)\circ(d_4\otimes 1)\circ(1\otimes d_4)$
— $(1\otimes d_3)\circ(d_4\otimes 1)\circ(1\otimes d_4)=d_4\circ(d_3\otimes 1)$
— $(1\otimes d_1)\circ(d_4\otimes 1)=(d_1\otimes 1)\circ(1\otimes d_4)$

4.4 The category of diagrams $\mathcal{D}(E_0,E_1)$. Consider two sets E_0 and E_1. Consider a Γ-diagram K where Γ is a curve. Suppose that only some components of Γ are oriented. Suppose that each oriented component of Γ is "colored" by an element of E_1 and each unoriented component by an element of E_0. If one cuts such a diagram into pieces one gets diagrams with univalent vertices. This univalent vertices are of different type:

– points outside of Γ

– points in a unoriented component of Γ. Such a point is colored by an element of E_0.

– points in an oriented component of Γ. Such a point is colored by an element of E_1 and the orientation of Γ defines a sign on it.

The points of the first type are called standard points. The points of the second type are called unoriented point. They are colored by E_0. The points in the last type are called oriented points. They are colored by E_1 and equipped with a sign.

So we are able to define a category corresponding to this situation.

An object of the category $\mathcal{D}(E_0, E_1)$ is a 5-tuple $(X, X_0, X_1, \alpha, \beta)$, where X, X_0 and X_1 are disjoint finite sets, α is a function from $X_0 \coprod X_1$ to $E_0 \coprod E_1$ sending X_0 into E_0 and X_1 into E_1, and β is a function from E_1 to $\{\pm 1\}$.

A morphism from $(X, X_0, X_1, \alpha, \beta)$ to $(X', X'_0, X'_1, \alpha', \beta')$ is a triple (Γ, f, u), where Γ is a compact partially oriented one-dimensional manifold with boundary $\partial\Gamma = X_0 \coprod X_1 \coprod X'_0 \coprod X'_1$, f is a function from the set of components of Γ to E and u is an element of $\mathcal{A}(\Gamma, X \coprod X')$, such that:

– the oriented components of Γ are sent by f into E_1

– the unoriented components of Γ are sent by f into E_0

– the restriction of f on the boundary of Γ is the function $\alpha \coprod \alpha'$

– the sign induced by the partial orientation of Γ on its boundary is the function $\beta' \coprod -\beta$.

It is easy to see that all these data define a category. The composition is given by gluing. We have also a monoidal structure obtained by disjoint union.

Remark. There is also a completed version of this category, where a morphism is a triple (Γ, f, u) satisfying the same condition as above except that u lies now in the completion $\mathcal{A}(\Gamma, X \coprod X')\widehat{\ }$ of $\mathcal{A}(\Gamma, X \coprod X')$ (completion with respect to the degree).

The most important cases are the unoriented category $\mathcal{D}(un) = \mathcal{D}([1], \emptyset)$ and the oriented categy $\mathcal{D}(or) = \mathcal{D}(\emptyset), [1])$ and also their completed versions $\mathcal{D}(un)\widehat{\ }$ and $\mathcal{D}(or)\widehat{\ }$.

4.5 Proposition. *The Kontsevich integral induces a monoidal functor $\widetilde{Z}$ from the category of non associative framed tangle to the category $\mathcal{D}(or)\widehat{\ }$.*

Proof: An object in the category $\mathcal{D}(or)\widehat{\ }$ is a triple (X, X_+, X_-) where X is the set of standard points and X_+ (resp. X_-) is the set of oriented points with positive (resp. negative) sign. A morphism between an object (X, X_+, X_-) and an object (Y, Y_+, Y_-) is a pair (Γ, u) where Γ is a compact oriented one-dimensional manifold with boundary $(Y_+ \cup X_- \cup Y_- \cup X_+)$ with positive sign for Y_+ and X_- and negative sign for Y_- and X_+ and u lies in the module $\mathcal{A}(\Gamma, X \cup Y)\widehat{\ }$.

Consider now a non associative framed tangle T. This tangle may be considered as a compact oriented one-dimensional manifold Γ contained in $[0,1] \times \mathbf{R}^2$. The Kontsevich integral $Z(T)$ is an element of $\mathcal{A}(\Gamma, \emptyset)\widehat{\ }$ with constant term 1. Then one sets $\widetilde{Z}(T) = (\Gamma, u)$. The properties of the Kontsevich integral shows that $\widetilde{Z}$ is a functor of monoidal categories. □

Remark. For the standard Kontsevich integral, one needs to have $\mathbf{C}$ as coefficient ring. But it is possible to consider other functors with coefficients in some smaller field. Actually it is possible to apply a gauge transformation to the Kontsevich integral in order to obtain an associator with rational coefficients. Then the rational coefficients are available for this new functor.

4.6 Theorem. *Let Θ be the chord diagram with only one chord. Let $\mathcal{A}_c(S^1)\widehat{\ }$ be the completion of $\mathcal{A}_c(S^1)$ with respect to the degree. Then the Kontsevich integral Z induces an isomorphism from the completion of $\mathbf{Q}[\mathcal{N}]$ with respect to the Vassiliev filtration to $\mathcal{A}_c(S^1)\widehat{\ }/(\Theta)$.*

Let $\mathcal{N}'$ be the monoid of framed knots. Then the Kontsevich integral Z induces an isomorphism from the completion of $\mathbf{Q}[\mathcal{N}']$ with respect to the Vassiliev filtration to the product of two copies of $\mathcal{A}_c(S^1)\widehat{\ }$.

Proof: The Kontsevich integral sends a framed knot K to an element $Z(K)$ in the completion $\mathcal{A}_c(S^1)\widehat{\ }$ of $\mathcal{A}_c(S^1)$. The element $Z(K)$ is an infinite series:

$$Z(K) = \sum_0^\infty Z_n(K)$$

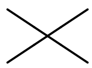
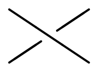
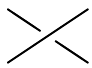
where $Z_n(K)$ belongs to $\mathcal{A}_c(S^1)_n$. Moreover the constant term $Z_0(K)$ is equal to 1 for every knot K. The completion of $\mathcal{A}_c(S^1)$ is the product of all modules $\mathcal{A}_c(S^1)_n$.

Let Z' be a functor obtained by a gauge transformation of the Kontsevich integral which has rational coefficients.

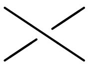
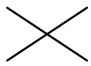
Denote by $\mathcal{T}$ the category of non associative framed tangles. If we replace the morphisms of this category by the sets of formal linear combinations of framed tangles, we get a new category $\mathbf{Q}[\mathcal{T}]$. By linearity Z' induces a functor still denoted by Z' from the category $\mathbf{Q}[\mathcal{T}]$ to the oriented category $\mathcal{D}(or)\widehat{\ }$.

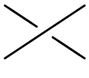
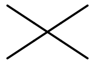
A framed singular knot K is represented by an oriented diagram Δ. Let D be the chord diagram associated to K. The diagram Δ has n double points, p positive crossings and q negative crossings. Let a be the class of $p-q$ mod 2. Then in $\mathbf{Q}[\mathcal{N}']$ one has: $K = [D]_a$. The knot K may be see as an endomorphism of the emptyset in the category $\mathbf{Q}[\mathcal{T}]$. Cutting D into pieces produces a decomposition of K as a composite (in $\mathbf{Q}[\mathcal{T}]$) of elementary morphisms K_i. Some of these K_i are standard tangles, but n of these morphisms have the form $\mathrm{Id}\otimes T_i\otimes\mathrm{Id}$, where T_i is a morphism between two points and two points (with some sign) represented by a singular tangle with one double point. In this case the singular tangle represents the difference between a positive and a negative half-twist.

If K_i is standard, $Z'(K_i)$ has constant term 1. If K_i has a double point, $Z'(K_i)$ is on the form $\mathrm{Id}\otimes Z'(T_i)\otimes\mathrm{Id}$ and we have:

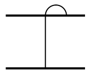
$Z'(T_i) = Z'\Big($  $\Big) = \varepsilon Z'\Big($  $\Big) - \varepsilon Z'\Big($  $\Big)$

where ε depends on the local orientations near the double points in T_i. On the other hand the images under Z' of the half twists are given by the following exponentials:

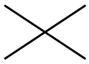
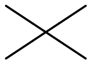
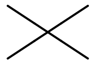
$Z'\Big($  $\Big) =$  $.\ \exp(\frac{u}{2})$

$Z'\Big($  $\Big) =$  $.\ \exp(\frac{-u}{2})$

where u is the following diagram:

$u = -$  $=$ 

Then we have:

$Z'\Big($  $\Big) = \varepsilon$  $\Big(\exp(\frac{u}{2}) - \exp(\frac{-u}{2})\Big) = \varepsilon$  $(u + \cdots)$

Therefore the degree one term of $Z'(K_i)$ is exactly the chord diagram corresponding to K_i. Hence the Kontsevich integral $Z'(K) = Z(K)$ is trivial in degree $< n$ and its degree n term is represented by the chord diagram $[D]_a$ associate to the diagram Δ.

Let I'_n be the n^{th} filtration of $\mathbf{Q}[\mathcal{N}']$ in the Vassiliev filtration. The degree n term of the Kontsevich integral Z induces a homomorphism Z_n from the quotient I'_n/I'_{n+1} to $\mathcal{A}_c(S^1)_n$ and, for every chord diagram $D \in \mathcal{D}^c_n$ and every $a \in \{0,1\}$, $Z_n([D]_a)$ is the class of D in the module $\mathcal{A}_c(S^1)_n$.

Consider the ring $R = \mathbf{Q}[\theta]/(\theta^2 - 1)$. Every framed knot K has a self linking number $a = \lambda(K,K)$. If K is represented by a diagram Δ this number a is the difference between the number of positive crossings of Δ and the number of negative crossings. Consider the following homomorphism φ from $\mathbf{Q}[\mathcal{N}']$ to $R \otimes \mathcal{A}_c(S^1)\widehat{\ }$: $K \mapsto \varphi(K) = \theta^{\lambda(K,K)} \otimes Z(K)$.

This homomorphism induces a function $\varphi_n : I'_n/I'_{n+1} \to R \otimes \mathcal{A}_c(S^1)_n$. We have the following diagram:

$$\mathbf{Q}[\mathcal{D}^c_n]/(4T) \oplus \mathbf{Q}[\mathcal{D}^c_n]/(4T) \xrightarrow{\psi} I'_n/I'_{n+1} \xrightarrow{\varphi_n} R \otimes \mathcal{A}_c(S^1)_n$$

where ψ is the map $D \oplus D' \mapsto [D]_0 + [D]_1$.

We have shown that ψ is surjective. On the other hand the composite $\varphi_n \circ \psi$ is the map: $D \oplus D' \mapsto f(D) + \theta f(D')$ where f is the isomorphism from $\mathbf{Q}[\mathcal{D}^c_n]/(4T)$ to $\mathcal{A}_c(S^1)_n$. Hence ψ and φ_n are isomorphisms.

In the case of oriented knots, we have only one function $D \mapsto [D]$ which satisfies also the 1T relation. On the other hand the Kontsevich integral is well defined, but only in the quotient $\mathcal{A}_c(S^1)\widehat{\ }/(\Theta)$. Since the natural map f from $\mathbf{Q}[\mathcal{D}^c_n]/(4T)$ to $\mathcal{A}_c(S^1)_n$ is an isomorphism, it induces an isomorphism f_1 from $\mathbf{Q}[\mathcal{D}^c_n]/(4T,1T)$ to $\mathcal{A}^r_c(S^1)_n = \big(\mathcal{A}_c(S^1)/(\Theta)\big)_n$. So we have the following diagram:

$$\mathbf{Q}[\mathcal{D}^c_n]/(4T,1T) \xrightarrow{\psi_1} I_n/I_{n+1} \xrightarrow{Z_n} \mathcal{A}^r_c(S^1)_n$$

where ψ_1 is the map $D \mapsto [D]$. Since ψ_1 is surjective and the composite $Z_n \circ \psi_1 = f_1$ bijective, ψ_1 and Z_n are isomorphisms. □

5. WEIGHT FUNCTIONS

The Kontsevich integral of a link or a tangle lies in a completion of a module of diagrams $\mathcal{A}(\Gamma, X)$. In order to have an invariant of links or tangles it's enough to construct linear homomorphisms from a module of diagram to some module.

A *weight function* on (Γ, X) is a homomorphism from the module $\mathcal{A}(\Gamma, X)$ to some module E. Such a function associates to every (Γ, X)-diagram K an element $\varphi(K) \in E$ in such a way that φ satisfy AS and IHX relations.

The standard way to construct a weight function into a ring or a module is obtained with the help of a Lie algebra or a Lie superalgebra.

Definitions. Let k be a charasteristic zero field. A Lie algebra over k is a pair $L = (L, [\ ,\ ])$ where L is a k-vector space and $[\ ,\ ]$ a bilinear homomorphism from $L \otimes L$ to L (called the bracket or the Lie bracket) such that:

— the Lie bracket is antisymmetric: $[x, y] = -[y, x]$ for every x, y in L

— the Lie bracket satisfies the Jacobi identity: $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for every x, y, z in L.

Let L be a Lie algebra. A L representation (or a L-module) is a vector space E together with a bilinear map $x \otimes e \mapsto xe$ from $L \otimes E$ to E such that;

— for every x, y in L and every e in E one has: $x(ye) - y(xe) = [x, y]e$

Let Mod_L be the category of finite dimensional L-modules. The objects of this category are the finite dimensional L-modules and the morphisms are the linear maps compatible with the L-action. The category Mod_L is a k-linear category, but it is also a monoidal category. If E and E' are L-modules the vector space $E \otimes E'$ is also a L-module by the action: $x(e \otimes e') = xe \otimes e' + e \otimes xe'$. The dual E^* of a L-module is also a L-module by the rule: $(xf)(e) = -f(xe)$.

The simplest example of L-module is the module L itself with the action: $x(y) = [x, y]$. This module is called the adjoint representation.

In order to obtain weight function we need to consider Lie algebras equipped with a bilinear form.

Let E be a finite dimensional k-vector space equipped with a non singular bilinear form $b : E \otimes E \to k$. With such a form there is a well defined associated element $\omega = \sum e_i \otimes e'_i$ in $E \otimes E$ satisfying:

$$\forall e \in E \qquad e = \sum b(e, e_i)e'_i = \sum e_i b(e'_i, e)$$

This element is called the Casimir element of E. If the form b is symmetric the Casimir element is symmetric too.

If E isn't equipped with a bilinear form, one still have a Casimir element $\omega = \sum e_i \otimes e'_i \in E \otimes E^*$. It satisfies the following:

$$\forall e \in E \qquad e = \sum e_i\ e'i(e)$$

Definition. A quadratic Lie algebra is a triple $L = (L, [\ ,\ ], <\ ,\ >)$ where $(L, [\ ,\ ]$ is a finite dimensional Lie algebra over k and $<\ ,\ >$ is a non singular symmetric bilinear form on L satisfying the following:

$$\forall x, y, z \in L \qquad < [x, y], z >=< x, [y, z] >$$

This condition is equivalent to said that b is a L-linear map from $L \otimes L$ to k, where L is considered as the adjoint representation and k the trivial module (the module k equipped with the trivial action).

Supermodules and Lie superalgebras. A supermodule is a $\mathbf{Z}/2$-graded module $E = E_0 \oplus E_1$. The degree of an element in E_i is $i \in \mathbf{Z}/2$. If E and E' are supermodules their tensor product is also a supermodule: $(E \otimes E')_0 = E_0 \otimes E'_0 \oplus E_1 \otimes E'_1$ and $(E \otimes E')_1 = E_0 \otimes E'_1 \oplus E_1 \otimes E'_0$. In the classical case one has a symmetry T from $E \otimes E'$ to $E' \otimes E$ sending $e \otimes e'$ to $e' \otimes e$. In the super case T is replaced by the supersymmetry T sending $e \otimes e'$ to $\varepsilon e' \otimes e$ where ε is the sign $(-1)^{pq}$, with $p = \partial^\circ e$ and $q = \partial^\circ e'$. In this context, a supersymmetric bilinear form on a super module E is a symmetric bilinear form on E_0 together with an antisymmetric bilinear form on E_1.

As in the classical case there is a Casimir element associated to every supermodule equipped (or not) with a non singular bilinear form.

Definition. A Lie superalgebra is a pair $(L, [\ ,\ ])$ where L is a k-supermodule and $[\ ,\ ]$ is a morphism from $L \otimes L$ to L which is antisymmetric (in the super sense) and satisfies the super Jacobi identity:

$$\forall x \in L_p, y \in L_q, z \in L_r \qquad [[x, y], z] + (-1)^{p(q+r)}[[y, z], x] + (-1)^{r(p+q)}[[z, x], y] = 0$$

A L-supermodule is a supermodule E equipped with an action $L \otimes E \to E$ such that:

$$\forall x \in L_p, \forall y \in L_q, \quad \forall e \in E \qquad x(ye) - (-1)^{pq} y(xe) = [x, y]e$$

The category of L-supermodules is still a monoidal k-linear category.

A quadratic Lie superalgebra is a triple $(L, [\ ,\ ], <\ ,\ >)$ where $(L, [\ ,\ ])$ is a finite dimensional Lie superalgebra and $<\ ,\ >$ is a non singular supersymmetric bilinear form en L which is invariant (i.e. the map from $L \otimes L$ to k is L-linear, where L is the adjoint representation and k the trivial L-module).

Let L be a quadratic Lie (super)algebra. Let Γ be an oriented curve. A *L-coloring* of Γ is a map f which associates to each component of Γ a finite dimensional L-(super)module. With such a coloring each point x in $\partial\Gamma$ has an associated module $E_0(x)$. On the other hand each point x in $\partial\Gamma$ is equipped with a sign $\varepsilon(x)$. If Γ start from x, the sign is negative. It is positive in the other case. We'll said that the color of x is the dual module $E(x) = E_0(x)$ if $\varepsilon(x) = 1$ and the module $E(x) = E_0(x)^*$ if $\varepsilon(x) = -1$.

Suppose now that Γ is only partially oriented. Then we'll define a L-coloring of Γ as a map f which associates to each oriented component of Γ a finite dimensional L-module and to each unoriented component a finite dimensional L-module E equipped with a non singular (super)symmetric invariant bilinear form: $E \otimes E \to k$. Let x be a point in $\partial\Gamma$. If x lies in the oriented part of Γ, its sign $\varepsilon(x)$ and its color $E(x)$ are defined. In the other case x has only a color: the module $E(x)$ which is the coloring of the component of Γ containing x.

5.1 Proposition. *Let L be a quadratic Lie (super)algebra. Let Γ be an partially oriented curve and $X = \{x_1, \ldots, x_n\}$ be a finite set. Let $\{y_1, \ldots, y_p\}$ be the set of points in $\partial\Gamma$. Let f be a L-coloring of Γ and $E_1, \ldots, E_p$ be the corresponding colors of $y_1, \ldots, y_p$.*

Then these data induce a well defined homomorphism $\Phi(L, f)$ from $\mathcal{A}(\Gamma, X)$ to the module $L^{\otimes n} \otimes E_1 \otimes \cdots \otimes E_p$.

Proof: Let K be a (Γ, X)-diagram. Let K' be a subdivision of K such that K' has only standard edges (i.e. no circle). Let A be the set of vertices of K' and A_3 (resp. A_2, A_1) be the set of 3-valent (resp. 2-valent, univalent) vertices. Let B be the set of edges in K' and $\widetilde{B}$ be the set of edges in K' equipped with an orientation. The orientation changing is an involution $\alpha \mapsto -\alpha$ in $\widetilde{B}$. The end-point mapping is a map ∂_+ from $\widetilde{B}$ to A and the starting point mapping is a map ∂_- from $\widetilde{B}$ to A. For every $\alpha \in B$ one has: $\partial_+(-\alpha) = \partial_-(\alpha)$.

Let α be an oriented edge in $\widetilde{B}$. If α is not contained in Γ, we set: $E(\alpha) = L$. If α is contained in a oriented component Γ_0 of Γ, there is two possibilities: if the orientation of α is compatible with the orientation of Γ_0 we define $E(\alpha)$ as the color of Γ_0. If the orientations don't agree we define $E(\alpha)$ as the dual of the color of Γ_0. If α is contained in an unoriented component Γ_0, $E(\alpha)$ is defined as the color of Γ_0.

Let a be an (unoriented) edge in K'. Let α and $-\alpha$ be the corresponding oriented edges. If a is not contained in Γ denote by $\omega(a)$ the Casimir element $\Omega \in L \otimes L = E(\alpha) \otimes E(-\alpha)$. Suppose that a is contained in an oriented component Γ_0 which is colored by E. Up to taking another choice for α we may as well suppose that the orientations of α and Γ_0 are compatible. In this case we set $\omega(a)$ to be the Casimir element in $E \otimes E^*$. Suppose a is contained in a unoriented component Γ colored by E. On E we have a non singular (super)symmetric invariant bilinear form and we define $\omega(a)$ as the Casimir element associated to it.

So for every $a \in B$, the element $\omega(a)$ is an element in $E(\alpha) \otimes E(-\alpha)$. Notice that $\omega(a)$ doesn't depend on the choice of α, because a Casimir element is (super)symmetric.

Take a numbering $\{\alpha_0, \ldots \alpha_{2q-1}\}$ of $\widetilde{B}$ such that $\alpha_{2j+1} = -\alpha_{2j}$ for every $j < q$, and denote by a_j the unoriented edge corresponding to α_{2j} (and α_{2j+1}). Let $\mathcal{E}$ be the following module: $\mathcal{E} = E(\alpha_0) \otimes E(\alpha_1) \cdots \otimes E(\alpha_{2q-1})$. The tensor product $\Omega = \omega(a_1) \otimes \cdots \otimes \omega(a_{q-1})$ belongs to the module $\mathcal{E}$.

Consider another numbering $\{\beta_0, \ldots, \beta_{2q-1}\}$ of $\widetilde{B}$ satifying the following properties:

– for every 3-valent vertex x in K the three oriented edges arriving to x are $\beta_j, \beta_{j+1}, \beta_{j+2}$ for some j (in the right cyclic ordering)

– for every 2-valent vertex x in K' the two oriented edges arriving to x are β_j, β_{j+1} for some j

– the oriented edges arriving to some vertex in X or in $\partial\Gamma$ appair in the ordering corresponding to the given ordering of $X \cup \partial\Gamma = \{x_1, \dots, x_n, y_1, \dots, y_p\}$.

This new numbering is obtained by a permutation σ of the set $\widetilde{B}$. The group of permutations of $\widetilde{B}$ acts on $\mathcal{E}$. In the super case this action permutes a pure tensor and multiply it by a sign. In particular a transposition acts as the supertransposition: $x \otimes y \mapsto (-1)^{ij} y \otimes x$ where x and y are of degree i and j.

Denote by Ω' the image of Ω under σ.

Let x be a 2-valent vertex of K' and α_j and α_{j+1} the oriented edges arriving to x. In any case one has a canonical bilinear form b_x defined on $E(\alpha_j) \otimes E(\alpha_{j+1})$. This form is the given form on $L \otimes L$, the evaluation map from some $E^* \otimes E$ to k or a given form on $E \otimes E$ where E is a color corresponding to a unoriented component of Γ.

Let x be a 3-valent vertex of K'. This vertex is a vertex of K. Let α_j, α_{j+1} and α_{j+2} be the oriented edges arriving to x. If x is not contained in Γ we have a trilinear form b_x on $E(\alpha_j) \otimes E(\alpha_{j+1}) \otimes E(\alpha_{j+2}) = L^{\otimes 3}$: the form $u \otimes v \otimes w \mapsto < u, [v, w] > = < [u, v], w >$. If x belongs to the curve Γ, the module $E(\alpha_j) \otimes E(\alpha_{j+1}) \otimes E(\alpha_{j+2})$ is, up to a (unique) cyclic permutation a module on the form $E^* \otimes L \otimes E$ and the form b_x is defined (up to this permutation) as the form: $e^* \otimes u \otimes e \mapsto e^*(ue)$.

Now we are able to define the element $\Phi(L, f)(K)$ as the image of Ω under the tensor product of all forms b_x. The fact that $\Phi(L, f)(K)$ doesn't depend on any choice follows from the construction. The morphism $\Phi(L, f)$ is compatible with the AS relation because the form b_x is completely antisymmetric (if x is not contained in Γ). The compatibility of the IHX relation comes from the Jacobi identity and the STU relation from the algebraic property of an action of L on a L-module. □

Let L be a quadratic Lie (super)algebra. The category Mod_L is a monoidal category. Denote by L the adjoint representation.

The Casimir element may be see as a homomorphim from $k = L^{\otimes 0}$ to $L \otimes L = L^{\otimes 2}$. The form <?,?> is a homomorphism from $L^{\otimes 2}$ to $L^{\otimes 0}$ and the bracket is a homomorphism from $L^{\otimes 2}$ to $L = L^{\otimes 1}$. The symmetry (or the supersymmetry) T is a homomorphism from $L^{\otimes 2}$ to itself.

5.2 Theorem. *Let L be a quadratic Lie (super)algebra. Then there exists a unique functor Φ_L of monoidal categories from the category of diagrams $\mathcal{D}$ to the category Mod_L sending* [1] *to the adjoint representation L and morphisms $d_1 - d_4$ to the invariant form, the Casimir element, the Lie bracket and the symmetry T respectivily.*

Proof: The morphisms d_i are defined in 4.2. The unicity of such a functor comes from the fact that $\mathcal{D}$ is generated by [1] and the d_i's morphisms. In order to construct Φ_L it is enough to prove that Φ_L is compatible with all the relations satisfied by the d_i's.

The first relation means that the bracket is antisymmetric. The second one which correspond to the IHX relation is send by Φ_L to the Jacobi relation. The other relations are easy to check.

A direct way to construct Φ_L is the following: Set $\Phi_L([n]) = L^{\otimes n}$. Let u be a morphism from an object $[p]$ to an object $[q]$. This morphism belongs to the module $\mathcal{A}(\emptyset, [p] \coprod [q])$. Then $\Phi(L, -)(u)$ is an element of the module $L^{\otimes p} \otimes L^{\otimes q}$. But L is canonically isomorphic to its dual (as a L-module). Then $\Phi(L,-)(u)$ may be see as an element of the module $L^{*\otimes p} \otimes L^{\otimes q}$ or an element $\Phi_L(u)$ of $\mathrm{Hom}(L^{\otimes p}, L^{\otimes q})$.

So Φ_L is well defined. The fact Φ_L satisfy the desired properties is a formal consequence of the construction. □

Remark. This theorem gives a very efficient way to compute the image under $\Phi(L,-)$ of a (Γ, X)-diagtram when Γ is empty. Just take a bijection between X and some $[n]$, and the diagram becomes a morphism from $[0]$ to $[n]$ and can be decomposed in a composite of the morphisms d_i's. Then the theorem give the desired computation.

This theorem may be generalized in the case of colored curves.

5.3 Theorem. *Let L be a quadratic Lie (super)algebra. Let E_0 and E_1 be two sets. Let f_0 and f_1 be two coloring functions sending every $x \in E_1$ to a finite dimensional L-module $f_1(x)$ and every $x \in E_0$ to a finite dimensional L-module $f_0(x)$ equipped with a non singular bilinear (super)symmetric invariant form. Then there exist a unique monoidal functor $\Phi(L, E_0, E_1, f_0, f_1)$ from the category $\mathcal{D}(E_0, E_1)$ to the category Mod_L satisfying the following properties:*

— An object of $\mathcal{D}(E_0, E_1)$ reduced to a standard point is sent to L. An oriented point with color $e \in E_1$ and sign ε is sent to the module $f_1(e)$ if $\varepsilon = 1$ or to its dual in the other case. An unoriented point with color $e \in E_0$ is sent to the module $f_0(e)$

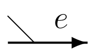
— The functor send morphisms d_1, d_2, d_3 and d_4 to the bilinear form $< \, , \, >$, the Casimir element, the bradket and the (super)symmetry

— If X is an object with two points x and y and K is a diagram composed with two edges joining x to y and y to x, then the image of this morphim under the functor is the (super)symmetry

— If X is an object with only two points x and y colored by E_0 or E_1 and u is a morphism from X to $\emptyset$ represented by a diagram K with only one edge, the image of u is the evaluation map if K is oriented and the given bilinear form in K if not

— If X is an object with only two points x and y colored by E_0 or E_1 and u is a morphism from $\emptyset$ to X represented by a diagram K with only one edge, the image of u is the corresponding Casimir element

— If u is the morphism corresponding to the following diagram, its image is the action map:



$$\Longrightarrow \qquad (x \otimes u \mapsto xu)$$

where x is in L and u in the color module associated to e (or its dual if the curve is oriented in the other way).

Sketch of proof: It is easy to see that the objects and morphisms described in the theorem generate the monoidal category $\mathcal{D}(E_0, E_1)$ and the functor is unique.

The construction of the functor is exactly the same as above. The functor is defined on the objects. On morphisms the definition uses the functions $\Phi(L, f)$ constructed in proposition 5.1.

6. INVARIANTS OF LINKS

Let L be a quadratic Lie (super)algebra and E be a finite dimensional L-module. Let K be a framed link. The Kontsevich integral $Z(L)$ of K may be see as a morphism from $\emptyset$ to itself in the oriented category $\widehat{\mathcal{D}(or)}$. This integral has an expansion $Z(K) = \sum Z_n(K)$ and each $Z_n(K)$ is an morphism of degree n in the category $\mathcal{D}(or)$. Let Φ be the functor $\Phi(L, \emptyset, [1], -, E)$, where E is the coloring $1 \mapsto E$, we have a series of numbers: $a_n(K) = \Phi(Z_n(K))$.

In order to force the series $\sum a_n(K)$ to be convergent, we'll consider a formal modification of the functor Φ.

Let X and Y be two objects in the category $\mathcal{D}(E_0, E_1)$ and u be a morphism from X to Y. Suppose that X and Y have p and q elements and that u is represented by a diagram with n 3-valent vertices. set: $\partial^\circ u = (n + q - p)/2$. This new degree is an integer and, with this degree, the category $\mathcal{D}(E_0, E_1)$ becomes a graded monoidal category.

Let $\Phi = \Phi(L, E_0, E_1, f_0, f_1)$ be a functor from $\mathcal{D}(E_0, E_1)$ to Mod_L corresponding to some colorings. Let $\mathbf{F}$ be the field of Laurent series: $\mathbf{F} = k[[t]][t^{-1}]$. Then we have a new graded functor $\widetilde{\Phi}$ from $\widehat{\mathcal{D}(E_0, E_1)}$ to $K \otimes \mathrm{Mod}_L$ defined by: $\widetilde{\Phi}(\sum u_n) = \sum \Phi(u_n) t^{\partial^\circ u_n}$.

With these conventions, we have, for every framed link K: $\widetilde{\Phi}(Z(K)) = \sum a_n(K) t^n \in k[[t]]$, and $\widetilde{\Phi} \circ Z$ is an invariant or framed links.

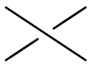
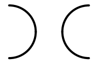
If we want to have an invariant of banded links, we have to consider other data. If K is a banded link, $Z(K)$ is a morphism from $\emptyset$ to itself in the unoriented category $\widehat{\mathcal{D}(un)}$. In order to have a weight function in this case, we have to take a L-module E equipped with a non singular (super)symmetric bilinear invariant form. The same construction as before applied to the functor $\Phi = \Phi(L, [1], \emptyset, E, -)$ gives rise to an invariant $\widetilde{\Phi} \circ Z$ of banded links.

The Kauffman bracket. Consider the simplest Lie algebra: $L = sl_2$ of all 2×2 matrices with zero trace. This Lie algebra is equipped with a form: $\alpha \otimes \beta \mapsto <\alpha, \beta> = tr(\alpha\beta)$. With this form, L becomes a quadratic Lie algebra. Consider the standard representation E of dimension 2. An isomorphism $\bigwedge^2 E \simeq k$ induces an antisymmetric bilinear invariant form b from $E \otimes E$ to k. Consider now the following super L-module E': the degree 0 part of E' is trivial and the degree 1 part of E' is the module E. Then the (super)dimension of E' is -2 and the form b induces a supersymmetric form b' on E'. Let Φ be the functor $\Phi(L, [1], \emptyset, E', -)$. The construction above gives rise to a functor $\widetilde{\Phi}$ and an invariant of banded links.

6.1 Theorem. *Let $K \mapsto <K>$ be the invariant of banded links induced by the Lie algebra sl_2 equipped with the standard representation consider as a supermodule of superdimension -2. Set: $A = -\exp(t/4)$. Then this invariant satisfies the following properties:*

— *for every banded link K, $<K>$ belongs to $k[[t]]$*

— $< >$ *is multiplicative with respect to the disjoint union operation*
— $< \emptyset >= 1$ *and the invariant of the trivial banded knot is* $-A^2 - A^{-2}$
— *the invariant* $< >$ *satisfy the following skein relation:*

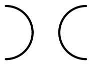
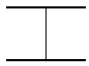
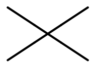
$<$  $>= A <$  $> +A^{-1} <$  $>$

Proof. Let $\{e_1, e_2\}$ be a basis of the standard representation E. The Casimir element $\Omega \in L \otimes L$ is the following:

$$\Omega = \sum_{ij} e_{ij} \otimes e_{ji} - \frac{1}{2}\mathrm{Id} \otimes \mathrm{Id}$$

where e_{ij} is the elementary matrix with 1 in the (i, j) place. Consider the supermodule E'. We have the following decomposition: $E' \otimes E' \simeq k \oplus L$, where k is the trivial L-module.

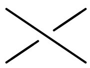
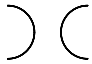
We have particular endomorphisms of $E' \otimes E'$: the identity, and the images under $\widetilde{\Phi}$ of the following diagrams:

$h =$  $u = -$  $T =$ 

An easy computation gives the following:

$$\mathrm{Id} = (1, 1) \qquad \widetilde{\Phi}(h) = (-2, 0) \qquad \widetilde{\Phi}(u) = (-3t/2, t/2) \qquad \widetilde{\Phi}(T) = (1, -1)$$

Let K_+, K_0 and K_∞ be the following tangles:

$K_+ =$  $K_0 =$  $K_\infty =$ 

For every tangle K denote its invariant $\widetilde{\Phi}(Z(K))$ by $< K >$. With this notation we have the following:

$$< K_+ >= \widetilde{\Phi}(T \circ \exp(u/2)) = (1, -1)(\exp(-3t/4, t/4) = (\exp(-3t/4), -\exp(t/4))$$

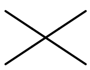
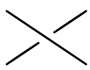
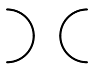
$$< K_0 >= \widetilde{\Phi}(\mathrm{Id}) = (1, 1)$$

$$< K_\infty >= \widetilde{\Phi}(K_\infty) = (a, 0)$$

for some $a \in k[[t]]$. Therefore there exists an element $b \in k[[t]]$ such that:

$$< K_+ >= -\exp(t/4) < K_0 > +b < K_\infty >$$

Consider a singular banded link L with only one double point x. We can modify L near x in oder to get three banded links:

$L :$  $L_+ :$  $L_0 :$ $L_\infty :$ 

In the category of non associative tangles we can express this links on the following form:

$$L_+ = u \circ ((\mathrm{Id} \otimes K_+) \otimes \mathrm{Id}) \circ v$$

$$L_0 = u \circ ((\mathrm{Id} \otimes K_0) \otimes \mathrm{Id}) \circ v$$

$$L_\infty = u \circ ((\mathrm{Id} \otimes K_\infty) \otimes \mathrm{Id}) \circ v$$

and we have:

$$< L_+ > = < u > \circ(\mathrm{Id}\otimes < K_+ > \otimes\mathrm{Id})\circ < v >$$

$$= -\exp(t/4) < u > \circ(\mathrm{Id}\otimes < K_0 > \otimes\mathrm{Id})\circ < v >$$

$$+b < u > \circ(\mathrm{Id}\otimes < K_\infty > \otimes\mathrm{Id})\circ < v > = -\exp(t/4) < L_0 > +b < L_\infty >$$

Applying this method to the negative crossing (with $-u$ instead of u), we get also the following:

$$< L_- > = -\exp(-t/4) < L_0 > +b' < L_\infty >$$

for some $b' \in k[[t]]$. By applying a 90° retation of the picture we get:

$$< L_+ > = -\exp(-t/4) < L_\infty > +b' < L_0 >$$

and b is equal to $-\exp(-t/4)$. So the invariant $K \mapsto < K >$ satisfies the desired skein relation. The other relations are easy to check. □

Remark. This invariant is actually the Kauffman bracket. It can be computed using the skein relation. It is easy to prove by induction that $< K >$ is an element of $\mathbf{Z}[A, A^{-1}]$.

The Kauffman polynomial. Consider now a n-dimensional vector space E equipped with a non singular symmetric form b. Let $L = o(E)$ be the Lie algebra of antisymmetric endomorphisms of E. The trace of the product induces a form $< , >$ on L and L is a quadratic Lie algebra. The module E is a L-module and functors Φ and $\widetilde{\Phi}$ are defined. So we get an invariant of banded links.

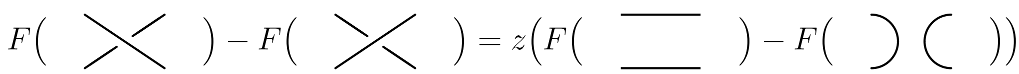
6.2 Theorem. *Let $K \mapsto F(K)$ be the invariant of banded links induced by the quadratic Lie algebra $o(E)$ equipped with the standard representation E. Set: $\alpha = \exp((n-1)t/4)$ and $z = 2sh(t/4)$. Then this invariant satisfies the following properties:*

— *for every banded link K, $F(K)$ belongs to $k[[t]]$*
— *F is multiplicative with respect to the disjoint union operation*
— *$F(\emptyset) = 1$ and the invariant of the trivial banded knot δ is:*

$$F(\delta) = 1 + \frac{\alpha - \alpha^{-1}}{z}$$

— *if K' is obtained from a banded link K by a positive twist, one has: $F(K') = \alpha F(K)$*

— *the invariant F satisfies the following skein relation:*



Proof: The module $\bigwedge^2 E$ is isomorphic to L by the rule:

$$e \wedge e' \mapsto \Big(u \mapsto b(u,x)y - b(y,u)x\Big)$$

The module S^2E contains the trivial module generated by the Casimir of E as a direct summand. So we get the following decomposition:

$$E \otimes E \simeq k \oplus L \oplus F$$

The situation is similar as above except that we have three modules instead of two. With the same notations, we have:

$$\mathrm{Id} = (1,1,1) \qquad \widetilde{\Phi}(h) = (n,0,0) \qquad \widetilde{\Phi}(u) = ((1-n)t/2, -t/2, t/2)$$
$$\widetilde{\Phi}(T) = (1,-1,1)$$

If we extend F to all non associative banded tangle, we have, with the same notation as above (and K_- as the inverse of K_+):

$$F(K_+) = (\exp((1-n)t/4), -\exp(-t/4), \exp(t/4))$$
$$F(K_-) = (\exp((n-1)t/4), -\exp(t/4), \exp(-t/4))$$
$$F(K_0) = (1,1,1)$$
$$F(K_\infty) = (a,0,0)$$

for some $a \in k[[t]]$. Then there exists an element $b \in k[[t]]$ such that:

$$F(K_+) - F(K_-) = 2\mathrm{sh}(t/4)F(K_0) - bF(K_\infty)$$

and the same argument as before shows that F verifies the following skein relation for every banded link L:

$$F(L_+) - F(L_-) = 2\mathrm{sh}(t/4)F(L_0) - bF(L_\infty)$$

By applying a 90° retation of the picture we get:

$$F(L_+) - F(L_-) = bF(L_0) - 2\mathrm{sh}(t/4)F(L_\infty)$$

and therefore:

$$b = 2\mathrm{sh}(t/4) \qquad \Longrightarrow \qquad a = 1 + \frac{\mathrm{sh}((n-1)t/4)}{\mathrm{sh}(t/4)}$$

On the other hand we have:

$$F(K_\infty K_\infty) = (a^2, 0, 0) = F(\delta)F(K_\infty)$$

where δ is the trivial banded knot. Then we have: $F(\delta) = a$. Let L be a link and L' and L'' be the link obtained from L by applying a positive or negative twist. By construction there exists a series α in $[[t]]$, with constant term 1, such that:

$$F(L') = \alpha F(L) \qquad\qquad F(L'') = \alpha^{-1}F(L)$$

and the skein relation shows the following:

$$\alpha - \alpha^{-1} = 2\mathrm{sh}(t/4)(F(\delta) - 1)$$

which implies:

$$\alpha = \exp((n-1)t/4)$$

□

Remark. This invariant is called the Kauffman polynomial. For every banded link L, $F(L)$ belongs the the algebra $\mathbf{Z}[z, z^{-1}, \alpha, \alpha^{-1}]$.

The HOMFLY polynomial. Consider now the Lie algebra $L = sl_n$ of $n \times n$-matrices with zero trace. This Lie algebra is quadratic by taking the trace of the product as bilinear form. The standard representation E is n-dimensional. The module E is a L-module and functors Φ and $\widetilde{\Phi}$ are defined on the category of non associative framed tangles. So we get an invariant of framed links.

6.3 Theorem. *Let $K \mapsto P(K)$ be the invariant of framed links induced by the quadratic Lie algebra $sl_n = sl(E)$ equipped with the standard representation E. Set: $\alpha = \exp(t/(2n))$, $\beta = \exp(nt/2)$ and $z = \exp(t/2) - \exp(-t/2)$. Then this invariant satisfies the following properties:*

— *for every framed link K, $P(K)$ belongs to $k[[t]]$*
— *P is multiplicative with respect to the disjoint union*
— *$P(\emptyset) = 1$ and the invariant of the trivial banded knot δ is:*

$$F(\delta) = \frac{\beta - \beta^{-1}}{z}$$

— *if K' is obtained from a banded link K by a positive twist, one has:*

$$P(K') = \beta\alpha^{-1}P(K)$$

— *If K_+, K_- and K_0 are obtained from a singular framed link by the three standard modifications, on has:*

$$\alpha P(K_+) - \alpha^{-1}P(K_-) = zP(K_0)$$

Proof: Let $\{e_i\}$ be a basis of the standard representation E. The Casimir element $\Omega \in L \otimes L$ is the following:

$$\Omega = \sum_{ij} e_{ij} \otimes e_{ji} - \frac{1}{n} \mathrm{Id} \otimes \mathrm{Id}$$

where e_{ij} is the elementary matrix with 1 in the (i, j) place.

With the same notations as above, we have three endomorphisms of $E \otimes E$: The identity, the symmetry T, and the endomorphism u. An easy computation shows the following:

$$u = tT - \frac{t}{n} \mathrm{Td}$$

We have the decomposition:

$$E \otimes E \simeq S^2 E \oplus \bigwedge{}^2 E$$

With respect to this decomposition, we have:

$$\mathrm{Id} = (1, 1) \qquad T = (1, -1) \qquad u = (t - t/n, -t - t/n)$$

If K_+ and K_- are the framed tangle corresponding to positive and negative half twists, we have:

$$P(K_+) = T \circ \exp(u/2) = (\exp(\frac{t}{2} - \frac{t}{2n}), -\exp(-\frac{t}{2} - \frac{t}{2n}))$$

and that implies:

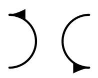
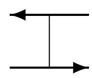
$$\alpha P(K_+) - \alpha^{-1} P(K_-) = z P(K_0)$$

With the same argument as above, this formula becomes true for every framed link, and the skein relation is proven.

On the othe hand, we have the following decomposition:

$$E^* \otimes E \simeq k \oplus L$$

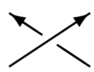
We have also three endomorphisms: the identity and the image under $\widetilde{\Phi}$ of the following diagrams:

h :  u : 

In this decomposition, we have:

$$\mathrm{Id} = (1, 1) \qquad \widetilde{\Phi}(h) = (n, 0) \qquad \widetilde{\Phi}(u) = (nt - t/n, -t/n)$$

Let K_+ be the following tangle:



we have:

$$F(K_+) = T \circ \exp(u/2) = T \circ (\exp(\frac{nt}{2} - \frac{t}{2n}), \exp(-\frac{t}{2n}))$$

Denote by K_- the tangle K_+, but with a negative crossing. We have:

$$\alpha F(K_+) - \alpha^{-1} F(K_-) = (\exp(nt/2) - \exp(-nt/2)) T \circ (1, 0)$$

On the other hand the skein relation is the following:

$$\alpha F(K_+) - \alpha^{-1} F(K_-) = (\exp(t/2) - \exp(-t/)) F(K_0)$$

and: $F(K_0) = F(\delta) T \circ (1, 0)$. Thus we get the desired formula for $F(\delta)$.

Let L be a framed link and L' and L'' be the link obtained from L by applying a positive or negative twist. By construction there exists a series γ in $[[t]]$, with constant term 1, such that:

$$P(L') = \gamma P(L) \qquad\qquad F(L'') = \gamma^{-1} P(L)$$

If one applies the skein relation to L_+ one gets:

$$\alpha\gamma - \alpha^{-1}\gamma^{-1} = zP(\delta) = \beta - \beta^{-1}$$

Since the constant term of γ the only possibility is:

$$\alpha\gamma = \beta \qquad \Longrightarrow \qquad \gamma = \beta\alpha^{-1}$$

and that finishes the proof. □

Remark. This polynomial invariant can be computed using the skein relation. It belongs to the subring of $k[[t]]$ generated by α, α^{-1}, β, β^{-1}, z and $(\beta - \beta^{-1})/z$. But the three variables α, β and z are algebraically independant in the following sense: there is no polynomial Q such that $Q(\alpha, \beta, z)$ vanishes in $k[[t]]$ for every value of n.

Then the polynomial invariant P belongs to a ring contained in the polynomial algebra $\mathbf{Z}[\alpha, \alpha^{-1}, \beta, \beta^{-1}, z, z^{-1}]$. If we want to have an invariant of oriented link, it's enough to set: $\beta = \alpha$. This polynomial is the HOMFLY polynomial. It satisfies all properties of the theorem, except that $\alpha = \beta$ and z are formal variables.

Remark. If one consider a k-supermodule E with non zero superdimension n as a module over the Lie superalgebra $L = sl(E)$, one gets exacty the same polynomial invariant as before in theorem 6.3. If one takes a k-supermodule of superdimension n, equipped with a non singular symmetric form and the Lie superalgebra $L = osp(E)$, ones gets the same polynomial invariant as before in theorem 6.2. Roughly speaking, the A series give the HOMFLY polynomial and the B-C-D series give the Kauffman polynomial.

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