WHAT IS A SIX FUNCTOR FORMALISM? (PART II)

REIN JANSSEN GROESBEEK

Abstract. We explain §III of [Sch22] on the definition of a 6 functor formalism. This is the third talk in the 6 functor formalism seminar at Jussieu.

Contents

Welcome back to the third talk of the seminar. Last time, Qixiang Wang introduced the ideas behind the 6 functor formalism and gave an introduction on ∞-categories.

Today we build on this work and give more details on the definition of the 6 functor formalism in the sense of [Sch22] and explain how the ∞-categorical tools introduced by Lurie are used in the definition. In particular, we recall the notion of commutative monoids in $\mathrm{Cat}_{(1,1)}$ and see how it extends to $\mathrm{Cat}_{(\infty,1)}$.

1. A three functor formalism

A six functor formalism is a special type of a three functor formalism.

Definition 1 (Three functor formalism)**.** A *three functor formalism* is a lax symmetric monoidal functor

$$D : \mathrm{Corr}(C, E)^{\otimes} \to \mathrm{Cat}_{(\infty,1)}\,.$$

$\Diamond$

Remark 1. Here as was explained by Qixiang last time, $\mathrm{Corr}(C, E)$ encodes the cartesian squares that can be formed by f^*, $f_!$ and $\otimes$, while the morphism $D : \mathrm{Corr}(C, E)^{\otimes} \to \mathrm{Cat}_{(\infty,1)}$ encodes the commutativity relations and higher coherences that all the diagrams involving f^*, $f_!$ and $\otimes$ have to satisfy.

Definition 2 (Six functor formalism, [Man22, p.306 Def A.5.7])**.** A *six functor formalism* is a three functor formalism such that for a morphism $f : X \to Y$ the morphism

$$f^* := D(Y \xleftarrow{f} X \xrightarrow{\mathrm{id}} X)$$

Date: 2023 Nov 16 Thu 14h00-16h00.

has a right adjoint f_* and the morphism

$$f_! := D(X \xleftarrow{\mathrm{id}} X \xrightarrow{f} Y)$$

has a right adjoint $f^!$. Moreover the tensor product

$$- \otimes - := \Delta^* \circ D(- \times -) : D(X) \times D(X) \xrightarrow{D(-\times -)} D(X \times X) \xrightarrow{\Delta^*} D(X)$$

has a right adjoint $\underline{\mathrm{Mor}}$. In other words, $D(X)$ is a closed symmetric monoidal ∞-category. ◊

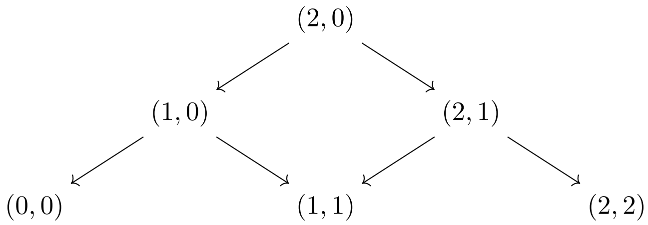
We will define the category of diagrams $\mathrm{Corr}(C, E)$ in two steps:
(1) first in Definition 4 as a simplicial set, which encodes f^* and $f_!$,
(2) then in section 3 as a commutative monoid, which encodes $\boxtimes := D(- \times -)$.

Definition 3 ($(\Delta^\bullet)^2_+$)**.** We define the cosimplicial category $C(\Delta^n) := (\Delta^\bullet)^2_+$ as the full subcategory of $(\Delta^\bullet)^{\mathrm{op}} \times \Delta^\bullet$, by defining as the objects

$$\mathrm{Obj}((\Delta^n)^2_+) := \{(i,j) \in \mathrm{Obj}((\Delta^n)^{\mathrm{op}} \times \Delta^n) = \{0, 1, \dots, n\}^2 \mid i \geq j\}.$$

◊

Remark 2. These objects can be visualised as correspondence diagrams or "roofs":



Definition 4 (Two functor formalism)**.** As a simplicial set, $\mathrm{Corr}(C, E)$ is defined as the cartesian diagrams where the vertical arrows are in E, or more precisely as follows:

$$\mathrm{Corr}(C,E)_n := \left\{ f \in \mathrm{Mor}_{\mathrm{Cat}_{(1,1)}}((\Delta^n)^2_+, C) \;\middle|\; \begin{array}{c} f(i,j) \\ \downarrow \\ f(i,j+1) \end{array} \in E, \quad \begin{array}{ccc} f(i,j) & \longleftarrow & f(i+1,j) \\ \downarrow & \lrcorner & \downarrow \\ f(i,j+1) & \longleftarrow & f(i+1,j+1) \end{array} \right\}$$

◊

Proposition 1 ([LZ17, p.83 Lem 6.1.2])**.** *The simplicial set* $\mathrm{Corr}(C, E)$ *is in* $\mathrm{Cat}_{(\infty,1)}$, *i.e. all inner horns can be filled.*

2. Commutative monoids

The simplicial set structure on $\mathrm{Corr}(C, E)$ encodes the cartesian squares involving f^* and $f_!$, hence the name "two functor formalism". To also encode diagrams involving $\otimes$, which is a functor of a different arity, we instead put a commutative monoid structure on $\mathrm{Corr}(C, E)$.

Recall: In the case of $C \in \mathrm{Cat}_{(1,1)}$ we define

Definition 5 ([nLa23])**.** A *commutative monoid in* C is a unit object $1_C \in C$ and a morphism

$$- \otimes - : C \times C \to C$$

such that we have:

(1) A unitality constraint $u_X : 1_C \otimes X \xrightarrow{\sim} X$ functorially in X
(2) A commutativity constaint $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ functorially in X, Y
(3) An associativity constraint $a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ functorially in X, Y, Z.
(4) For all W, X, Y, Z the pentagon diagram commutes

$$\begin{array}{ccccc} W \otimes (X \otimes (Y \otimes Z)) & \xrightarrow{a_{W,X,Y\otimes Z}} & (W \otimes X) \otimes (Y \otimes Z) & \xrightarrow{a_{W\otimes X,Y,Z}} & ((W \otimes X) \otimes Y) \otimes Z \\ \downarrow{\scriptstyle \mathrm{id}_W \otimes a_{X,Y,Z}} & & & & \uparrow{\scriptstyle a_{W,X,Y}\otimes \mathrm{id}_Z} \\ W \otimes ((X \otimes Y) \otimes Z) & & \xrightarrow{\qquad a_{W,X\otimes Y,Z} \qquad} & & (W \otimes (X \otimes Y)) \otimes Z \end{array}$$

(5) for all X, Y, Z the hexagon diagram commutes

$$\begin{array}{ccccc} (X \otimes Y) \otimes Z & \xrightarrow{a_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{c_{X,Y\otimes Z}} & (Y \otimes Z) \otimes X \\ \downarrow{\scriptstyle c_{X,Y}\otimes \mathrm{id}_Z} & & & & \downarrow{\scriptstyle a_{Y,Z,X}} \\ (Y \otimes X) \otimes Z) & \xrightarrow{a_{Y,X,Z}} & Y \otimes (X \otimes Z) & \xrightarrow{\mathrm{id}_Y \otimes c_{X,Z}} & Y \otimes (Z \otimes X) \end{array}$$

$\Diamond$

Problem 1. *This language does not work in the* $\mathrm{Cat}_{(\infty,1)}$ *setting as it is unknown how to enumerate all coherence diagrams (analogues of the pentagon axiom and hexagon axiom).*

Solution 1. Therefore we take a different point of view, where we add for all n the n-ary operations $\bigotimes_n$ all at once, instead of just the binary operation $- \otimes -$. This was introduced by Lurie in [Lur17, §2].

Definition 6. Let $\mathrm{Fin}^{\mathrm{part}}$ (in [Sch22, p.19]) or Fin_* (in [Lur17, p.165 Not 2.0.0.2]) be the category of finite sets with partially defined maps. $\Diamond$

Definition 7 ([Lur17, p.293 Def 2.4.2.1], [Sch22, p.19 Def 3.3])**.** Let $\mathcal{C} \in \mathrm{Cat}_{(\infty,1)}$ be an ∞-category. Then a (straightened) monoid in $\mathcal{C}$ is a functor $M : \mathcal{N}(\mathrm{Fin}^{\mathrm{part}}) \to \mathcal{C}$ satisfying the following commutativity and associativity property:
For every finite set I, the map induced by the universal property of the product

$$M(I) \to \prod_{1 \le i \le n} M(\{i\})$$

is an isomorphism in $\mathcal{C}$. $\Diamond$

Problem 2. *This definition uses the "straightened" point of view, and it is harder to construct monoidal functors to* ∞*-categories than it is to construct fibrations of* ∞*-categories.*

Solution 2.

$$\underline{\mathrm{Mor}}_{\mathrm{Cat}_{(\infty,1)}}(C, \mathrm{Cat}_{(\infty,1)}) \underset{\mathrm{St}}{\overset{\mathrm{Un}}{\rightleftarrows}} \{\varphi \in \underline{\mathrm{Mor}}_{\mathrm{Cat}_{(\infty,1)}}(D, C) \mid \varphi \text{ coCartesian}, D \in \mathrm{Cat}_{(\infty,1)}\}.$$

Therefore we apply Lurie's unstraightening functor to obtain the unstraightened point of view, expressing the condition of a monoid in terms of fibrations.

Definition 8 ([Lur17, p.123 Prop 2.4.2.8], [Sch22, p.20 Def 3.5])**.** A *coCartesian fibration* is a map $F : D \to C$ such that

(1) F is an inner fibration: any inner horn can be lifted
(2) any morphism of C admits a locally F-coCartesian lift
(3) The composition of locally F-coCartesian lifts is locally F-coCartesian.

A map $g : Y \to Y'$ in D is a locally F-coCartesian lift of $f : X \to X'$ if it is initial among all morphism with source Y that lift f. ♢

Definition 9 ([Lur17, p.169 Def 2.0.0.7], [Sch22, p, 21 Def 3.7])**.** A (unstraightened) symmetric monoidal ∞-category is a coCartesian fibration

$$C^{\otimes} \to N(\mathrm{Fin}^{\mathrm{part}})$$

such that for all finite sets $I \in \mathrm{Fin}^{\mathrm{part}}$

$$C_I^{\otimes} \to \prod_{i \in I} C$$

is an equivalence. ♢

2.1. **Morphisms of monoids.** To conclude the study of the category of symmetric monoidal ∞-categories, we have to define what are the morphisms between them.

Recall the situation in the classical $(1,1)$-categorical world. Here, a functor $F : (C, \otimes, 1_C) \to (D, \otimes, 1_D)$ between monoidal categories has to preserve the monoidal structure, so we have the following coherence maps

$$\phi_{X,Y} : FX \otimes FY \to F(X \otimes Y)$$
$$\phi : 1_D \to F1_C$$

which make the following diagrams commute [Wik22]

$$\begin{array}{ccc} (FX \otimes FY) \otimes FZ & \xrightarrow{a_{FX,FY,FZ}} & FA \otimes (FB \otimes FZ) \\ \big\downarrow {\scriptstyle \phi_{X,Y} \otimes \mathrm{id}_{FZ}} & & \big\downarrow {\scriptstyle \mathrm{id}_{FX} \otimes \phi_{Y,Z}} \\ F(X \otimes Y) \otimes FZ & & FX \otimes F(Y \otimes Z) \\ \big\downarrow {\scriptstyle \phi_{X \otimes Y, Z}} & & \big\downarrow {\scriptstyle \phi_{X, Y \otimes Z}} \\ F((X \otimes Y) \otimes Z) & \xrightarrow{Fa_{X,Y,Z}} & F(X \otimes (Y \otimes Z)) \end{array} \quad \text{and} \quad \begin{array}{ccc} FX \otimes 1_D & \xrightarrow{\mathrm{id}_{FA} \otimes \phi} & FX \otimes F1_C \\ \big\downarrow {\scriptstyle u_{FA}} & & \big\downarrow {\scriptstyle \phi_{X,1_C}} \\ FX & \xleftarrow[Fu_X]{} & F(X \otimes 1_C) \end{array} \tag{1}$$

Depending on how strictly we want to preserve the monoidal structure, we have three possibilities [Wik22]:

- (Lax monoidal functors): The coherence maps $\phi_{X,Y}, \phi$ satisfy no additional properties, they are just maps.
- (Strong monoidal functors): The coherence maps $\phi_{X,Y}, \phi$ are invertible.
- (Strict monoidal functors): The coherence maps $\phi_{X,Y}, \phi$ are identity maps. This definition violates the "principle of equivalence" [1] so its usage is uncommon.

To translate these notions to the $(\infty, 1)$-categorical setting, we again pass to the unstraightened point of view. By doing so, we can automatically generate the infinite families of commuting diagrams, which are the ∞-categorical counterparts of the two in Equation 1.

Definition 10 ([Sch22, p.8 Def 3.8], lax symmetric monoidal functor)**.** Let $(C, \otimes)$ and $(D, \otimes)$ be (unstraightened) symmetric monoidal ∞-categories. Then a functor $F^{\otimes} : C^{\otimes} \to D^{\otimes}$ over $N(\mathrm{Fin}_*)$ is

[1] https://ncatlab.org/nlab/show/principle+of+equivalence

(1) *lax symmetric monoidal* if $F^{\otimes}$ preserves locally coCartesian lifts of the morphisms of the form $I \dashrightarrow \{i\}$.
(2) *strong symmetric monoidal* if $F^{\otimes}$ preserves all locally coCartesian lifts of all morphisms $I \dashrightarrow J$.

$\diamondsuit$

Remark 3.
- In [Man22, p.305] a 6 functor formalism was defined to be a map of ∞-operads ([Lur17, p. 180 Def 2.1.2.7]) instead of a lax symmetric monoidal functor. But these two definitions are the same by [Lur17, p. 180 Rmk 2.1.2.9].
- Similarly, the definition of a strong symmetric monoidal functor in Definition 10 is the same as a "symmetric monoidal functor" in [Lur17, p.185 Def 2.1.3.7].

3. Monoidal structure on $\mathrm{Corr}(C, E)$

We will now apply the definitions of the previous section and define a monoidal structure on $\mathrm{Corr}(C, E)$. We first choose the following monoidal structure on C^{op} ([Lur09, p.26 1.2.1]):

Definition 11. Let

$$(C^{\mathrm{op}})^{\amalg} \to N(\mathrm{Fin}_*)$$
$$(f : \{Y_j\}_{1 \le j \le n} \to \{X_i\}_{1 \le i \le m}) \mapsto (\alpha : \langle m \rangle \to \langle n \rangle)$$

be the coCartesian symmetric monoidal ∞-category with monoidal structure on C^{op} induced by the coproduct $\amalg$. $\diamondsuit$

Proposition 2 ([LZ17, p. 84 Prop 6.1.3])**.** *The fibration*

$$\mathrm{Corr}((C^{\mathrm{op}})^{\amalg,\mathrm{op}}, E) \to N(\mathrm{Fin}_*)$$

$$\begin{array}{ccc} \{Y_j\}_{1 \le j \le n} & \longrightarrow & \{X_i\}_{1 \le i \le m} \\ \big\downarrow & & \\ \{Z_j\}_{1 \le j \le n} & & \end{array} \mapsto (\alpha : \langle m \rangle \to \langle n \rangle)$$

defines a coCartesian symmetric monoidal ∞-category with underlying ∞-category $\mathrm{Corr}(C, E)$.

Remark 4 ($(C^{\mathrm{op}})^{\amalg,\mathrm{op}}$ versus $C^{\times}$). Why do we consider the complicated ∞-category $(C^{\mathrm{op}})^{\amalg,\mathrm{op}}$ instead of the simpler $C^{\times}$?

(1) First, as was remarked by Yifeng Liu via e-mail, $(C^{\mathrm{op}})^{\amalg,\mathrm{op}}$ lives over $N(\mathrm{Fin}_*)^{\mathrm{op}}$ and $C^{\times}$ lives over $N(\mathrm{Fin}_*)$ so the two are not interchangeable.
(2) Another reason is that the definition of commutative monoids is not self-dual. Hence for example $(C^{\mathrm{op}})^{\amalg,\mathrm{op}}$ is not in an obvious way a monoid, nor is $(C^{\times})^{\mathrm{op}}$.

This means that if we were to choose as correspondence category $\mathrm{Corr}(C^{\times}, E)$, we would have an inclusion of the horizontal arrows

$$C^{\times,\mathrm{op}} \subset \mathrm{Corr}(C^{\times}, E)$$

but where $C^{\times,\mathrm{op}}$ does not have an obvious monoidal structure!

Hence to remedy this, we want to "conjugate" by $(-)^{\mathrm{op}}$ and consider $(C^{\mathrm{op}})^{\amalg,\mathrm{op}}$ instead. Then indeed we have the inclusion of the horizontal arrows

$$(C^{\mathrm{op}})^{\amalg} \subset \mathrm{Corr}((C^{\mathrm{op}})^{\amalg,\mathrm{op}}, E) \tag{2}$$

where by construction $(C^{\mathrm{op}})^{\amalg}$ has a monoidal structure given by the coproduct $\amalg$. Recall that an inclusion of monoidal categories like in Equation 2 is needed to apply [Lur17, p.302 Thm 2.4.3.18]. See also the discussion in [Man22, p. 306 Def A.5.6(a)].

Now that $\mathrm{Corr}(C, E)$ is a coCartesian (unstraightened) symmetric monoidal ∞-category, Definition 1 makes sense.

4. Three examples

To illustrate how Definition 1 encodes the different commuting diagram relations such as projection formula, we give three examples of commuting diagrams.

Example 1 ([Sch22, p.24])**.** Let D be a three functor formalism. Let $f : X \to Y$ in E, $A \in D(X)$ and $B \in D(Y)$. Then we have in Figure 1 the projection formula in terms of correspondences.

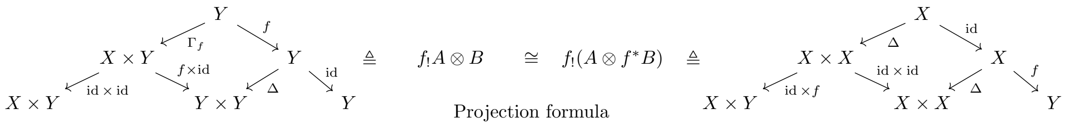


Figure 1. Projection formula.

Example 2. Let D be a three functor formalism. Let

$$\begin{array}{ccccc} X' & \xrightarrow{\overline{g}} & X & \xrightarrow{\overline{t}} & Y \\ \downarrow{\scriptstyle \overline{f}} & \lrcorner & \downarrow{\scriptstyle f} & \lrcorner & \downarrow{\scriptstyle h} \\ S' & \xrightarrow{g} & S & \xrightarrow{t} & T \end{array}$$

be a composition of fibred squares. We illustrate in Figure 2 several alternative descriptions of the functor $\overline{f}_!(\overline{t} \circ \overline{g})^*$ and its associated correspondence diagrams.

Example 3. Let D be a three functor formalism. Let

$$\begin{array}{ccc} X' & \xrightarrow{\overline{g}} & X \\ \downarrow{\scriptstyle \overline{f}} & \lrcorner & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a cartesian square. Then for $M \in D(X)$ and $N \in D(S)$, we can consider the multiple ways we can write

$$T := g^* f_!(M \otimes f^* N) \in D(S').$$

This is illustrated in Figure 3.

References

[Lur09] Jacob Lurie. *Higher topos theory.* Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: 10.1515/9781400830558. URL: https://doi.org/10.1515/9781400830558 (cit. on p. 5).

[Lur17] Jacob Lurie. *Higher Algebra.* 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf (visited on 10/30/2023) (cit. on pp. 3–6).

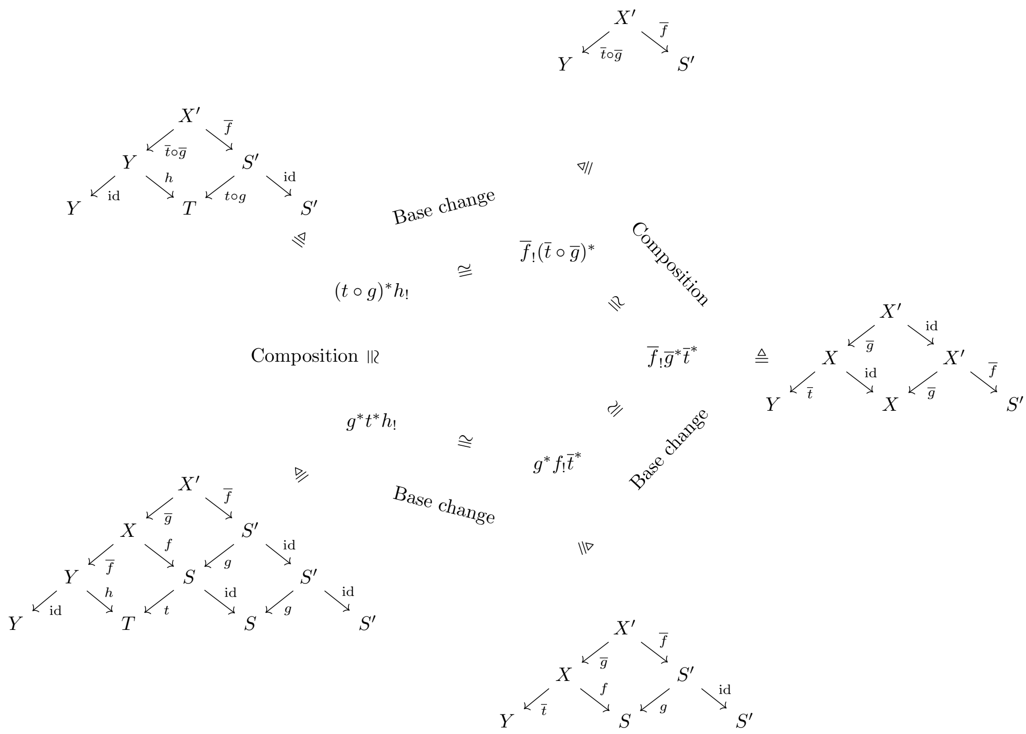


FIGURE 2. Five different ways to write $\overline{f}_!(\overline{t} \circ \overline{g})^*$.

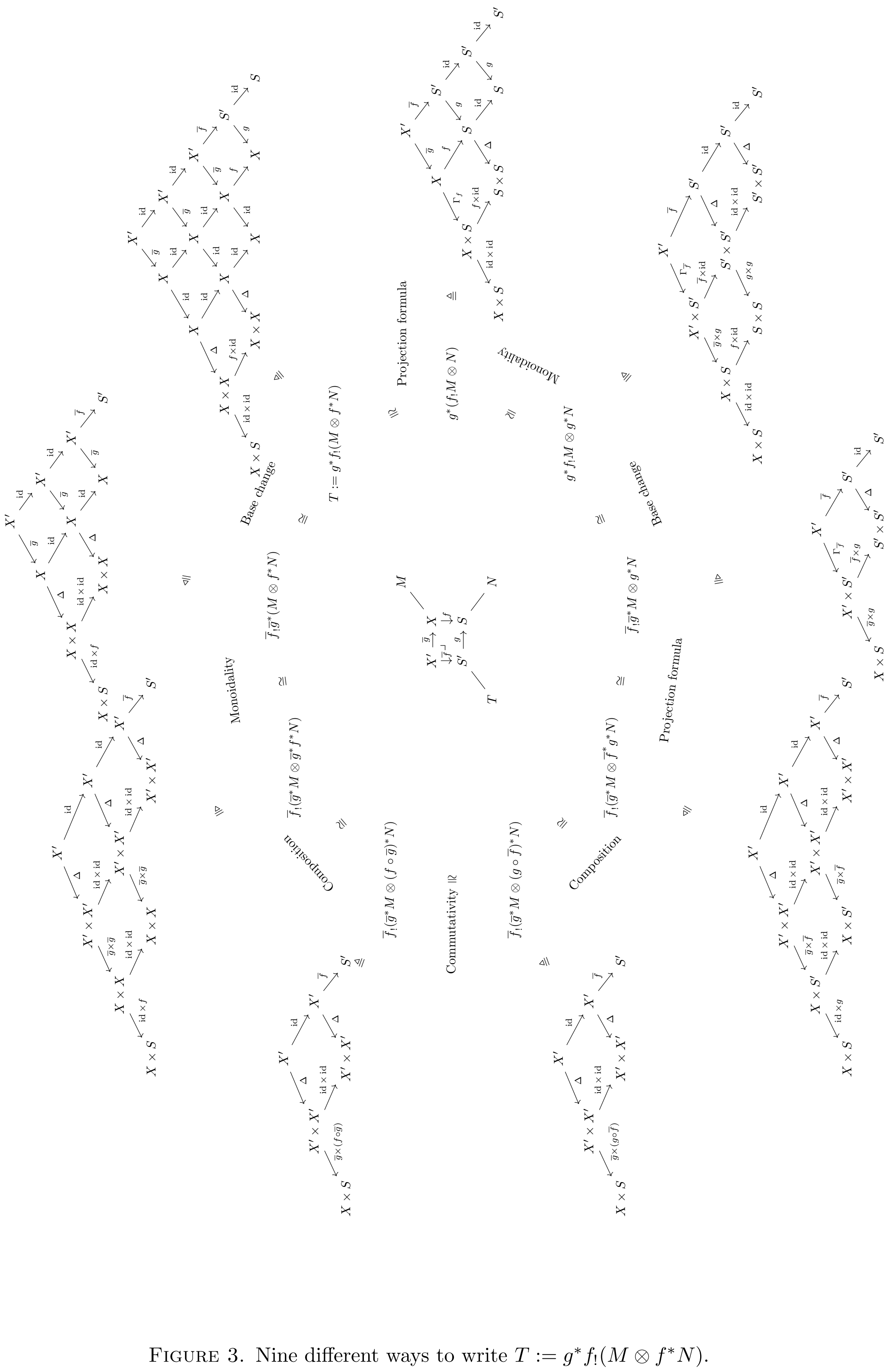


FIGURE 3. Nine different ways to write $T := g^* f_!(M \otimes f^*N)$.

[LZ17] Yifeng Liu and Weizhe Zheng. *Enhanced six operations and base change theorem for higher Artin stacks*. 2017. arXiv: 1211.5948 [math.AG] (cit. on pp. 2, 5).

[Man22] Lucas Mann. *A p-Adic 6-Functor Formalism in Rigid-Analytic Geometry*. 2022. arXiv: 2206.02022 [math.AG] (cit. on pp. 1, 5, 6).

[nLa23] nLab authors. *symmetric monoidal category*. https://ncatlab.org/nlab/show/symmetric+monoidal+category. Revision 53. Nov. 2023 (cit. on p. 3).

[Sch22] Peter Scholze. *Six-Functor Formalisms*. 2022. URL: https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf (visited on 10/30/2023) (cit. on pp. 1, 3, 4, 6).

[Wik22] Wikipedia contributors. *Monoidal functor — Wikipedia, The Free Encyclopedia*. https://en.wikipedia.org/w/index.php?title=Monoidal_functor&oldid=1110520596. [Online; accessed 15-November-2023]. 2022 (cit. on p. 4).