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Mathematical Pulsation at the root of invention

" quand ils arrivent à la vérité, c'est en heurtant de ce côté et d'autre qu'ils y sont tombés "
Évariste Galois

René Guitart

Summary. Our purpose is to explain in an historical perspective how the true fundamental skill for mathematical invention is the strength of Mathematical Pulsation.

Invention of the objective rigour as a subjective act

The ideology of the *Didactic Transposition* [Chevallard (1991)] (see also [Verret (1975)]) is based on the claim that the mathematics (M) of the mathematicians *are not* the mathematics of the teachers (T), which *are not* those of the pupils (P). Guy Brousseau insists on this point, and maintains [Brousseau (2001), note 5] that the *Mathematical Pulsation* is not a natural state or a law, but an ideal and a useful fiction for the didactic relation. I assert exactly the opposite : the mathematical pulsation is proposed as a law of the natural state of practise of mathematics, roughly expressed by M = T = P [Guitart (1999), p. 286]. Let call that the *pulsation* or *pulsative nexus* MTP. By its radical separation between the M, T and P positions, the Didactical Transposition is disastrous with respect to any good mathematical instruction : doing mathematics, either as an M, as a T or as a P, requires at the very beginning a strong commitment in the full pulsation MTP. An adequat comparison by Jean-Yves Degos [Degos (2002)] is with some Glenn Gould's ideas on music [Gould (1988)] : Gould would like to unite the composer, the performer, the auditor. Paradoxically, in order to do mathematics we have to be simultaneously a mathematician, a teacher and a pupil ; we have to construct, to explain and to learn what we are doing. Certainly the good mental state for entrance at mathematics viewed as a craft is to admit that paradox and to experience it effectively. But this basic observation (*the pulsative nexus* MTP) is only the first step in the idea of the Mathematical Pulsation.

Now a second step is the *closed/open bearing* (CO). Let us look at mathematicians as craftmen working with some mental objects. They have some specific dexterities, and the point is to work out the sheer specific skill they use. If we could to know this skill, it will be the top priority to teach it to teachers. This artisanal question is even more basic than the very important psychological question of motivations and pulsions, and in teaching of mathematics we have to take care of both aspects. As a solution for this problem of the specific skill in mathematics, I introduced the notion of the *Mathematical Pulsation* [Guitart (1999]. Two interesting analysis of this work are [Bareil (2000)] and [Legrand (2000/a) and Legrand (2000/b)]. But do not mistake this *Mathematical Pulsation* with something like a "Mathematical Pulsion". The later is rather the psychological cause of the former. In fact, in concrete terms, the *Mathematical Pulsation* is nothing else but the thing that everyone does when doing mathematics, even the most elementary ones. It is a very special gesture in understanding ("geste de pensée"), well known by each mathematician. The mind have to go to and fro between to antinomical postures : to have the situation under control, to leave the door open. To master and to fix (a clear unique meaning) or to neglect and to change (toward other possible meanings). Because of the similarity of the pulsation of inspiration and expiration in breath with the pulsation of closing and opening phases in mathematical thinking, at the end of [Guitart (2003/a)] I suggested to consider the famous book "Zen in the Art of Archery" [Herrigel (1997)] as a true treatise in didactic of mathematics : just you have to replace everywhere the words "archery" by "mathematical proof". So you get a first approximation of the idea of the so called Mathematical Pulsation [Guitart (1999)] as a kind of paradoxal mental breath inside every mathematical act. In order to do mathematics you have both to be rigorous *and* to be zen with rigour. Dancing with the rigour, surfing on rigour [Guitart (1999), P. 53]. In order to do mathematics you have to pick up this knack, exactly as when going cycling. The point [Guitart (1999), p. 17] is that free fall forward is what we have to risk in order to get lateral stability. In the area of music, Glenn Gould said that the secret to play piano is partially in the manner you succeed to be separated from the instrument. He also said that he has to be at some distance from himself *and* to take part in the process. He was looking for a piano with keys initially firm under vertical sollicitation but with some lateral looseness when fully pressed [Schneider (1988), p. 107, 109, 74].

The third step is the idea of the *pulsation of the rigour* itself (RP). To enforce our second step, we have to be precise on objective rigour, and on the link between rigour and intimate dexterity. The idea of *Pulsation* cannot be well understood without the admission of three thesis on *dexterity*, in connection with analytical rigour, usefulness and talent. The first thesis is a very particular picture of the mathematical rigour. The mathematical rigour is not only the strict formal rigour, but it is also, at a higher level in the area of reason, the feeling we get when an intuition stop dead on a wording : a paradoxal surfing on rigour takes place there. The absolute obsessive fixation on formal analytical rigor only increases the dexterity. The true rigour is the pulsative surfing on rigour. The second thesis says that in order to be able to do mathematics you do not have to ask yourself of what earthly use it is to you. The usefulness of mathematics increases the pulsion but not dexterity; the true usefullness happens from the pulsation between reality and aesthetic sense. The last thesis say that dexterity is not talent or genius. Everyone can do mathematics like everyone can ride a cycle. The main injonction here is just : do it ! And then we have just to know what it is as an act. Paradoxically, in order to do mathematic we have to be rigorous and not to control if we are rigorous. We have to keep in mind a very strong requirement of accuracy, an absolute one in fact, but it is also necessary in exactly the same time to leave aside questions on analytical rigor, usefulness and talent, and we do not have to control rigorously our practise with rigour.

Of course for training and invention in mathematics we have to think seriously about three things : an *intimate motivation* for mathematical activity, the power to start effectively and *to go on* with mathematical computations and reasonings ("s'y mettre"), a true *dexterity with the paradoxality* of the mathematical rigour. But these three factors of pulsion and desire, moving off and risk taking, pulsation, are closely interlinked. From this link we can understand the will for searching in mathematical problems. In this perspective the search for tools to develop the mathematical imagination [Sinclair (2003)] could be more accurate. By experiment we learn that the closed/open standpoint of the pulsation allows the risk (the risk for the reason in the harsh choice between "to known" and "to do") to be fruitful or not, and the deepest mathematical pulsion will come back from the productive experiment with pulsation in the breath between intuition and *rigorous invention of true*. The Mathematic is a game which is identical to its own invention, and the best pulsion for mathematics is the wish to invent the rigour. In this sense the rigour is the object of mathematics [Guitart (2000/a), p. 87]. And for this *game of rigour*, the invention of mathematical supervising by mathematical logic and theory of proof, or by the description of fundations, were just some possibilities *a posteriori*, when the die is cast. The mathematicians at work proceed differently, by trial and error ([Hadamard (1954)] and [Rostand (1960)]), "they do not deduce, they combine, they compose" (Galois). It is the reason why the mathematicians discover necessities but in a contingent way (as said by Jean Cavaillès). The Mathematical Pulsation is the strength there, before the proof, at the point of contingency of invention.

To see versus To say

A main aspect at macro-level of the pulsation is the alternative between "to see" and "to say" (SS). As Felix Klein claimed, "the charm of geometry is to see what you think". Closed to this observation is our following claim : in mathematical practise you always have to see what you say, and to say what you see [Guitart (2000/b)]. See also [Barbin (2001)]. This is a deep pulsation driving the mathematical mind toward a greater evidence (in the sense of Descartes). Of course this works under the assumption that all evidences are logically equivalent. So in fact a real mathematician knows that *he does not have to choose* between the evidence for the eye and the evidence for the ear, between logic and geometry.

Famous is the case of the Pythagorean Proposition [Loomis (1972)]. Among the great collection of proofs of the Pythagorean Proposition we have not to choose the more geometrical (chinese) or the more logic (greek) or algebraic (modern) ; the point is rather the appreciation of the mathematical fact of this going to and fro.

In [Guitart (2003/b)] I constructed a calculus of "assimilations" as a common tool for analyzing pictures or figures, and for analyzing speeches or discourses. We can considere this theory as a mathematical thought about the fact of the pulsation between "to see" and "to say", and a mathematical achievement of this pulsation, an *abreviation* of this pulsation. By this calculus the two areas communicate and exchange intuitions.

Abreviations and bifurcations in algebra, differential calculus, probability.

In order to get to the heart of the matter, we go on to show more how our explanations above on MTP, CO and RP and SS works in practice, in the true mathematical life. At first, the pulsation works *inside* each mathematical frame, under the angle of bifurcation and abreviation (AB) [Guitart (2000/a), p. 161].

So in the frame of elementary algebra [Guitart (1999)], a basic pulsation is to be able to take the risk to write down *and* to erase letters. In fact all the calculus support this idea : when you compute you have to sollicit appearance *and* disappearance of letters. When in an addition process you get x+0, you have the possibility to erase this 0 and to let x ; but you can also replace 0 by y-y. A second pulsation here is with the use of equality, because very often you have to change your feeling on the meaning of "=" : sometimes it means an absolute identity *and sometimes* it means only a possibility of substitution in a given context. A third pulsation is the possibility to reverse a = b in b = a. And a fourth pulsation is the one invented by François Viète, between knowns and unknowns: to denote by letters unknown *and* known quantities is the crucial decision finishing the creation of algebra.

In the invention of differential calculus and differential geometry we can observe a basic pulsation between the idea of a "touchante" to a curve and the idea of the "derivative" of a function [Guitart (1999), p.98-105]. On the one hand we have a curve C and a straight line outside S0 (given by y = px +h0) approaching parallel to itself until in a position S (given by y = px+h, with h such that there is a unique solution u for px+h = f(x)) it touchs C at a point P (of coordinates (u, f(u)) ; on the other hand we consider a point P on C (of coordinates (u, f(u))) and the tangente as the limit position S of a chord PM when M approach to P (given by y-f(u) = f'(u)(x-u)). Then f'(u) = p, v = f(u), h = v-pu, and the pulsation is reversing of order from (p, u) to (u, f'(u)). We can today consider that the involutive Legendrian contact transformation L(u,v,p) = (-p,-(v-pu), -u) will provide a mathematical setting for this pulsation. This transformation L exhibite the ambiguity between the two versions of a contact element : a straight line with a point on it, ot a point with a line through it ; it is the Galois group of the situation.

Another example is a probabilistic pulsation [Mazliak, (2002)], which takes place in the alternative between on the one hand the data of the sample space of events followed by the description of random variables as numerical function on this space, and on the other hand the direct consideration at first of random variables and there laws : in practice we have to do both, to forget one aspect for the other, and conversely. Historically also this point was essential in the invention of the probability theory.

The changing of frame and polytranslation, the arrow without a target.

An essential observation is that, obviously, when you are doing mathematics in a given setting, in fact you have also to move outside the scope of this setting, to move into a new framework. From algebra to geometry, from complex analysis to minimal surfaces [Douady A & Douady R. (1994)], and so on. Grasping this feature at the level of didactic, Régine Douady introduced her notion of *changing of frame* ("Changement de cadre") (CF) [Douady R. (1984)]. This CF is one aspect of the pulsation, at a macro-level.

Another thing is in a famous letter of André Weil to his sister Simone [Weil (1940)] where he explains how his work is engaged within a process of translation and construction of a trilingual text, written in "theory of algebraic functions of one variable", in "theory of fields of numbers", and in "theory of fields of algebraic functions with finite fields of constants". Such a "polytranslation" is a little different from a standard changing of frame, because here the frames in consideration are not known and fixed a priori before the change, but they are more explicitly partially invented and constructed *by* the act of changing itself. The pulsation at macro-level could be even possible from a given well known discipline toward just the open area outside of this discipline, without an established target. Retrospectively the target will appear as an implicite constituent of the arrow. In this type of practice we could speak of *Open Changing of Frame* (OCF).

The reflexivity, the axiomatic attitude, the indirect way

A well known observation finally is that mathematics are *reflexive*, i.e. that they are able to say in a mathematical way something on what they can do and on what they cannot do. They are plenty of examples, as in the theory of equations (Galois theory), in logic (theorems of Gödel). The axiomatic attitude (AA) is a natural consequence of this reflexivity and of the basic fact that mathematics always work in a roundabout way, *indirectly*. It is absolutely necessary that axioms are assumed as true hypotheses, but simultaneously we know that the choice of hypotheses is always contingent. By axiomatizing we keep at a respectful distance from an ultimate decision. And moreover we can study mathematically the organization of choices of axioms. More systematically nowdays, the theory of categories take as objects of mathematical investigations today the mathematical gestures of yesterday [Guitart (2005)].

An interesting historical question is when the roundabout way in mathematics appeared explicitly, among means rather than among obstacles. At least in the XVIIth century, in the hands of Fermat (calculus of adequations) and Leibniz ("useful fictions" e.g. imaginary and infinitesimal elements), a significant gesture was invented : at first, *to write down what you want* (entities and relations) and then to see what this imply and produce mathematically. Here, following the words of Leibniz you have not to be careful with the existence of points and indivisible and infinitely small elements [Leibniz (1866), p. 592] because "les vérités mathématiques portent avec elles leurs contrôles et leurs confirmations" [Leibniz (1966), p. 80-1] (cited in [Clero J.-P.-Le Rest E. (1980), p. 177]). To be comparated with the Analysis of Ancients: here, in the case of Fermat and Leibniz, we do not start from a wished result in a known field, but we start with a wish of a new field. The new infinitesimal calculus is constructed inside the cartesian pulsation between algebra and geometry, a kind of OCF, supported on the cartesian pulsation between logic and calculus.

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