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The Product Over All Primes is $4\pi^2$

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Abstract: We generalize the classical definition of zeta-regularization of an infinite product. The extension enjoys the same properties as the classical definition, and yields new infinite products. With this generalization we compute the product over all prime numbers answering a question of Ch. Soulé. The result is $4\pi^2$. This gives a new analytic proof, companion to Euler's classical proof, that the set of prime numbers is infinite.

1. Introduction

Christophe Soulé did ask several years ago (and the question is proposed in [SABK, p.101]) to give a meaning and find a value for the zeta-regularized product over all prime numbers, similar to the classical zeta-regularized product

$$\infty! = 1.2.3.\ldots = \sqrt{2\pi}.$$

Regularization of infinite products arises in geometry [RS], arithmetic geometry [SABK], theoretical physics [EORBZ], and, more recently, in analytic number theory [De]. One of the main applications is the computation of regularized determinants of infinite dimensional operators, as pseudo-differential operators on manifolds.

In a previous article [MG-PM1] we carry a computation "à la Euler" in order to show that

$$\prod_p p = 4\pi^2.$$

The computation in [MG-PM1] that we reproduce below is not rigorous. Its aesthetics "à la Euler" leaves no doubt about the correction of the end result. The aim of the present article is to make rigorous this computation. For this purpose, we generalize the classical definition of zeta-regularization of infinite products. The new definition extends the classical one and preserves its main properties.

Given an increasing sequence $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ one defines the regularized (or zeta-regularized) infinite product as

$$\prod_{n=1}^{+\infty} \lambda_n = \exp(-\zeta'_\lambda(0)),$$

where ζ_λ is the zeta function associated to the sequence (λ_n),

$$\zeta_\lambda(s) = \sum_{n=1}^{+\infty} \lambda_n^{-s}$$

(see [SABK], Chap. V, Definition 5, p. 97). Implicitly this assumes that the zeta function converges in a half plane and has an analytic extension up to 0. This definition can be extended to arbitrary sequences of complex numbers for which the procedure makes sense. Observe that the definition applies to a finite product, and gives the expected result.

In [MG-PM1] we took a "liberal" view on this definition, just assuming that we have some way of computing or making sense of $\zeta'_\lambda(0)$. This was necessary since, as observed in [SABK], the zeta function associated to the sequence of primes does not extend meromorphically to a neighborhood of 0 since the imaginary axes is a line of singularities. This fact was first proved by E. Landau and A. Walfisz [LW]. For this reason the regularization of this infinite product was believed to be impossible (see for example the introduction of [Il]).

The computation in [MG-PM1] runs as follows: Recall that (this is used for example in the definition of the Artin-Hasse exponential, see for example [Ko], Chap. IV)

$$\exp(X) = \prod_{n=1}^{+\infty} (1 - X^n)^{-\frac{\mu(n)}{n}},$$

where μ is Moebius function. This can be proved taking logarithmic derivatives and using The Moëbius inversion formula.

From this we get

$$e^{p^{-s}} = \prod_{n=1}^{+\infty} (1 - p^{-ns})^{-\frac{\mu(n)}{n}}.$$

We consider now the zeta function associated to the sequence of primes

$$\mathcal{P}(s) = \sum_p \frac{1}{p^s}.$$

So it follows that

$$\begin{aligned} e^{\mathcal{P}(s)} &= \prod_p e^{p^{-s}} \\ &= \prod_p \prod_{n=1}^{+\infty} (1 - p^{-ns})^{-\frac{\mu(n)}{n}} \end{aligned}$$

$$= \prod_{n=1}^{+\infty} \prod_{p} (1 - p^{-ns})^{-\frac{\mu(n)}{n}}$$

$$= \prod_{n=1}^{+\infty} \zeta(ns)^{\frac{\mu(n)}{n}},$$

where we have used in the last equality Euler's product for the Riemann zeta function (and following Euler's tradition we don't need to justify the product exchange, but it can be done here). This formula

$$e^{\mathcal{P}(s)} = \prod_{n=1}^{+\infty} \zeta(ns)^{\frac{\mu(n)}{n}}$$

can be found in [LW] and in [Da] (in this last reference there is a typo in the formula: n in the denominator of the exponent is missing).

Now, taking the logarithmic derivative we get

$$\mathcal{P}'(s) = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n} \frac{n\zeta'(ns)}{\zeta(ns)} = \sum_{n=1}^{+\infty} \mu(n) \frac{\zeta'(ns)}{\zeta(ns)},$$

thus

$$\mathcal{P}'(0) = \left(\sum_{n=1}^{+\infty} \mu(n) \right) \frac{\zeta'(0)}{\zeta(0)}.$$

Of course the infinite sum does not converge, but we recall that

$$\sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

thus

$$\mathcal{P}'(0) = \frac{1}{\zeta(0)} \frac{\zeta'(0)}{\zeta(0)} = -2 \log(2\pi).$$

We conclude that

$$\prod_{p} p = e^{-\mathcal{P}'(0)} = (2\pi)^2 = 4\pi^2.$$

After this "explanation" à la Euler, we developed in [MG-PM2] a generalization of classical regularization of infinite products that makes rigorous the previous computation. We present this theory.

2. Super-Regularization of Infinite Products

2.1. Definition. Consider a sequence $\lambda = (\lambda_n)_{n\geq 1}$ of complex numbers and define, as before, its associated zeta function

$$\zeta_\lambda(s) = \sum_{n=1}^{+\infty} \lambda_n^{-s}.$$

We assume that ζ_λ is absolutely convergent in the half plane $\mathrm{Re}\, s > s_0$.

Note that we have to make a choice of the branch of each λ_n^{-s}. The end result will not depend on the choice made for a finite number of these terms, but the analytic extension properties and the result may depend on infinitely many choices. Obviously when (λ_n) is a positive real sequence, which is the case on most of the applications (in particular when we consider the spectrum of an hermitian operator) we make the unique choice for which λ_n^{-s} is real for $s > 0$. For a directed sequence (λ_n) as considered in [Il], it is also natural to take a compatible branch for all the λ_n. Nevertheless we do not make any particular assumption on the choice of branches.

We consider now an extension of ζ_λ to two complex variables into a double Dirichlet series:

$$\zeta_\lambda(s,t) = \sum_{n,m=1}^{+\infty} c_{n,m} \lambda_n^{-s} \mu_m^{-t}.$$

We assume that this series is absolutely convergent in $U_0 = \{\mathrm{Re}\, s > s_0\} \times \{\mathrm{Re}\, t > t_0\}$, and defines in this domain $U_0 \subset \mathbf{C}^2$ a meromorphic function. We also assume that $t_0 < 0$ and that

$$\zeta_\lambda(s,0) = \zeta_\lambda(s).$$

Now the function

$$(s,t) \mapsto \frac{\partial \zeta_\lambda}{\partial s}(s,t)$$

is meromorphic in U_0. We assume that there exists $t_1 \geq t_0$ such that for each $t \in \mathbf{C}$ with $\mathrm{Re}\, t > t_1$ the meromorphic function of one complex variable

$$s \mapsto \frac{\partial \zeta_\lambda}{\partial s}(s,t)$$

that is meromorphic in $\mathrm{Re}\, s > s_0$, does extend meromorphically to a half plane $\{\mathrm{Re}\, s > s_1\}$, $s_1 < 0$, that is a neighborhood of $s = 0$. We denote by

$$s \mapsto \mathrm{ext}_s \frac{\partial \zeta_\lambda}{\partial s}(s,t)$$

this extension.

Remarks. 1. Rothstein's theorem (see [Siu]) shows that the function of two complex variables

$$(s,t) \mapsto \mathrm{ext}_s \frac{\partial \zeta_\lambda}{\partial s}(s,t)$$

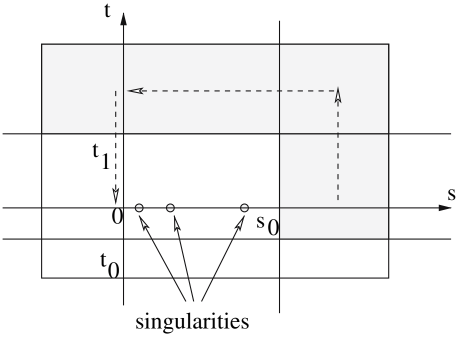

Fig. 1.

is meromorphic, but we won't need this fact.

2. In the definition of [SABK] of the classical regularized product, it is assumed that the zeta function extends meromorphically to the whole complex plane. Not so much is necessary since only the derivative at 0 matters, thus it is natural to request a meromorphic extension to 0. But this simple assumption would not prove to be a good definition. We may have isolated singularities of the extension with non-trivial monodromy. We don't know if actually there are Dirichlet series where this can happen. It is unlikely that this happens for Dirichlet series of arithmetic origin. These in general are meromorphic on half planes. Clearly, just assuming meromorphic extension to a half plane containing 0 avoids the monodromy problem and gives a good definition. It is this approach that we generalize.

Now we assume that the function

$$t \mapsto \operatorname{ext}_s \frac{\partial \zeta_\lambda}{\partial s}(0, t)$$

defined in the half plane $\operatorname{Re} t > t_1$ has a meromorphic extension to a neighborhood of $\{t = 0\}$ not identically infinite. Denote this extension by

$$t \mapsto \operatorname{ext}_t \left(\lim_{s \to 0} \operatorname{ext}_s \frac{\partial \zeta_\lambda}{\partial s}(s, t) \right).$$

We assume, for the scope of this article, that this extension has no pole at $t = 0$, but one can figure out a reasonable extension also in that situation.

This procedure that circumvents the eventual singularities in $\{s = 0\}$ is illustrated by Fig. 1. If it can be carried over, we define:

Definition 1 (Super-regularized product). *The super-regularized product of the sequence $\lambda = (\lambda_n)_{n \geq 1}$ is by definition, provided that the limits and meromorphic extensions exist,*

$$\prod_n \lambda_n = \exp\left(- \lim_{t \to 0} \left(\operatorname{ext}_t \left(\lim_{s \to 0} \operatorname{ext}_s \frac{\partial \zeta_\lambda}{\partial s}(s, t) \right) \right) \right).$$

We write in short

$$\prod_n \lambda_n = \exp\left(- \lim_{t \to 0} \frac{\partial \zeta_\lambda}{\partial s}(0, t) \right).$$

Zeta-regularization being a procedure cherished by physicists, the terminology "super-regularization" seems appropriate. The idea of avoiding singularities by "going through the complex" is not new in Theoretical Physics, nor in Pure Mathematics, but the procedure of increasing the complex dimension appears to be new in the theory of regularization of infinite products and resummation of divergent series. Nevertheless some classical procedures as Abel's summation can be viewed as going from complex dimension 0 *to complex dimension* 1 (*one can consult [Bo] and [Ha] for classical procedures, and [Ra] for modern extensions*). *The introduction of supplementary variables has proved to be useful in other classical problems in Number Theory* (*see [Re] for several examples described by a specialist of several complex variables, in particular the discussion of Eisenstein's trick in Sect.* I.3.4).

Several justifications of this definition are in order. First, we prove that the end result is independent of the two complex variable extension and the whole procedure.

Proposition 2. *Let* $(s,t) \mapsto \hat{\zeta}_\lambda(s,t)$ *be another two complex variable extension of* $s \mapsto \zeta_\lambda(s)$ *satisfying the same properties as* $(s,t) \mapsto \zeta_\lambda(s,t)$*, so that the procedure described above to compute the super-regularized product can be carried over. Then*

$$\lim_{t\to 0} \frac{\partial \hat{\zeta}_\lambda}{\partial s}(0,t) = \lim_{t\to 0} \frac{\partial \zeta_\lambda}{\partial s}(0,t),$$

thus the super-regularization is independent of the choice of the complex extension.

In particular, the super-regularization coincides with the classical zeta-regularization when this last one is well defined.

Proof. We assume that $\hat{\zeta}(s,t)$ is absolutely convergent in the domain $\hat{U}_0 = \{\mathrm{Re}\, s > \hat{s}_0\} \times \{\mathrm{Re}\, t > \hat{t}_0\}$, with $\hat{t}_0 < 0$, and that in the half plane $\{\mathrm{Re}\, t > \hat{t}_1\}$ the meromorphic function

$$s \mapsto \frac{\partial \hat{\zeta}_\lambda}{\partial s}(s,t)$$

extends meromorphically to a half plane $\{\mathrm{Re}\, s > s_1\}$, $s_1 < 0$, neighborhood of $\{s = 0\}$.

Now, since $t_0 < 0$ and $\hat{t}_0 < 0$, the difference $\hat{\zeta}_\lambda - \zeta_\lambda$ is meromorphic in $V_0 = \{\mathrm{Re}\, s > \max(s_0, \hat{s}_0)\} \times \{\mathrm{Re}\, t > \max(t_0, \hat{t}_0)\}$ and vanishes in $\{\mathrm{Re}\, s > \max(s_0, \hat{s}_0)\} \times \{t = 0\}$, thus there exists a meromorphic function g defined in V_0 such that

$$\hat{\zeta}_\lambda(s,t) - \zeta_\lambda(s,t) = t\, g(s,t).$$

Taking partial derivatives,

$$\frac{\partial \hat{\zeta}_\lambda}{\partial s}(s,t) - \frac{\partial \zeta_\lambda}{\partial s}(s,t) = t\, \frac{\partial g}{\partial s}(s,t).$$

Using this equation, we have that for $\mathrm{Re}\, t > \max(\hat{t}_1, t_1)$,

$$s \mapsto \frac{\partial g}{\partial s}(s,t)$$

extends meromorphically to a neighborhood of $s = 0$, and

$$\mathrm{ext}_s \frac{\partial \hat{\zeta}_\lambda}{\partial s}(s,t) - \mathrm{ext}_s \frac{\partial \zeta_\lambda}{\partial s}(s,t) = t\, \mathrm{ext}_s \frac{\partial g}{\partial s}(s,t).$$

Again using this equation and the properties of $\hat{\zeta}_\lambda$ and ζ_λ we get that

$$t \mapsto \mathrm{ext}_s \frac{\partial g}{\partial s}(0, t),$$

which is well defined for $\mathrm{Re}\, t > \max(\hat{t}_1, t_1)$, has a meromorphic extension to $t = 0$, and

$$\mathrm{ext}_t \left(\lim_{s \to 0} \mathrm{ext}_s \frac{\partial \hat{\zeta}_\lambda}{\partial s}(s, t) \right) - \mathrm{ext}_t \left(\lim_{s \to 0} \mathrm{ext}_s \frac{\partial \zeta_\lambda}{\partial s}(s, t) \right) = t.\, \mathrm{ext}_t \left(\lim_{s \to 0} \mathrm{ext}_s \frac{\partial g}{\partial s}(s, t) \right),$$

and making $t \to 0$ in this equation we get the result.

The super-regularization does extend the classical regularization because we can always consider the trivial extension

$$\zeta_\lambda(s, t) = \zeta_\lambda(s).$$

□

2.2. Properties. The main properties of the classical regularization are preserved.

Proposition 3. *If $\lambda = (\lambda_n)$ is a finite sequence, then the super-regularization of the product coincides with the classical finite product. More generally, if $\lambda = (\lambda_n)$ is an infinite sequence for which the infinite product can be regularized then the super-regularization of the product coincides with the classical regularized product.*

Proposition 4. *The super-regularized product is finitely associative. That is, if we partition the sequence λ into N parts*

$$\lambda = \lambda^{(1)} \cup \lambda^{(2)} \cup \ldots \cup \lambda^{(N)}$$

with $\lambda^{(j)} = (\lambda_n^{(j)})_{n \geq 1}$, and we assume that the super-regularized product exists for each sequence $\lambda^{(j)}$, then the super-regularized product exists for the sequence λ, and we have the formula (where we assume all infinite products to be super-regularized products)

$$\prod_{n=1}^{+\infty} \lambda_n = \left(\prod_{n=1}^{+\infty} \lambda_n^{(1)} \right) \cdot \left(\prod_{n=1}^{+\infty} \lambda_n^{(2)} \right) \cdots \left(\prod_{n=1}^{+\infty} \lambda_n^{(N)} \right).$$

Proof. Just observe that

$$\zeta_\lambda = \sum_{j=1}^{N} \zeta_{\lambda^{(j)}}.$$

□

Proposition 5. *Let $a \in \mathbf{C}^*$. We assume that the super-regularized product of the sequence λ is well defined. Then the super-regularized product of the sequence $\lambda^a = (\lambda_n^a)_{n \geq 0}$ exists and we have*

$$\prod_{n=1}^{+\infty} \lambda_n^a = \left(\prod_{n=1}^{+\infty} \lambda_n \right)^a.$$

Proof. We have

$$\zeta_{\lambda^a}(s) = \zeta_\lambda(as).$$

□

The next property is more involved. It relies on the analytic property that we can give a meaning to $\zeta_\lambda(0)$. We do strengthen the assumptions requesting not only that

$$s \mapsto \frac{\partial \zeta_\lambda}{\partial s}(s, t),$$

extends meromorphically to a half plane $\{\operatorname{Re} s > s_1\}$ with $s_1 < 0$, but also that

$$s \mapsto \zeta_\lambda(s, t),$$

has this property. Then we assume that for $s = 0$, we also have a meromorphic extension of

$$t \mapsto \zeta(0, t),$$

to $t = 0$ and that $t = 0$ is not a pole. In practice the analytic extension of ζ and not just $\partial\zeta/\partial s$ always happens and it almost follows from the first assumption. One has to be careful when only using the first assumption because of the possible different branches of the extension of ζ obtained by integration due to the fact that

$$s \mapsto \frac{\partial \zeta_\lambda}{\partial s}(s, t)$$

may have poles which introduce a monodromy around them for the primitive. One can overcome these difficulties by assuming that the branch of the extension chosen depends continuously on the parameter t. More precisely, at $t \in \mathbf{C}$, $\operatorname{Re} t > t_1$, we consider the meromorphic extension

$$\zeta_\lambda(s, t) = \int_{\gamma_t} \frac{\partial \zeta_\lambda}{\partial s}(s, t)\, ds,$$

where we integrate over a path $\gamma_t \subset \mathbf{C} \times \{t\}$ from a base point $(t, \tilde{s}_0)$, where $\operatorname{Re} \tilde{s}_0 > s_0$, and we assume that the path γ_t depends continuously on t and avoids the poles of the meromorphic extension of

$$s \mapsto \frac{\partial \zeta_\lambda}{\partial s}(s, t)$$

(note that the integral does not really depend on γ_t but only on the homotopy class of paths out of the poles). The existence of such a continuous choice of paths is problematic when we have poles that escape to infinity in finite time. We assume in what follows that we have such a well behaved extension.

Theorem 6. *Let $a \in \mathbf{C}^*$. If the super-regularized product of the sequence $\lambda = (\lambda_n)_{n\geq 1}$ exists with the stronger assumption mentioned before, then the super-regularized product of the sequence $a\lambda = (a\lambda_n)_{n\geq 0}$ exists and we have*

$$\prod_{n=1}^{+\infty}(a\lambda_n) = a^{\zeta_\lambda(0)}\left(\prod_{n=1}^{+\infty}\lambda_n\right),$$

where $\zeta_\lambda(0)$ has to be interpreted as

$$\lim_{t\to 0}\lim_{\substack{s\to 0\\ \mathrm{Re}\, t>t_1}}\zeta_\lambda(s,t),$$

which is well defined by successive analytic extensions and does not depend on the two complex variable extension.

Note that indeed the right-hand side depends on the choice of the branch of the power of a. But also the left-hand side depends on the choice of the branches of the $(a\lambda_n)^{-s}$ in order to compute the zeta function. The claim of the theorem is that there is such a choice of $a^{\zeta_\lambda(0)}$, compatible with the implicit choices in order to compute the left-hand side, so that the formula holds.

Lemma 7. *Denoting by $(s,t)\mapsto\zeta_\lambda(s,t)$ the extension defined, we have that the limit*

$$\lim_{t\to 0}\zeta_\lambda(0,t)$$

is independent of the two complex variable extension $(s,t)\mapsto\zeta_\lambda(s,t)$ chosen.

Proof. For another two complex variable extension $(s,t)\mapsto\hat{\zeta}(s,t)$, we have as before, once the meromorphic extensions are performed,

$$\hat{\zeta}_\lambda(0,t)-\zeta_\lambda(0,t)=tg(0,t),$$

thus

$$\lim_{t\to 0}\hat{\zeta}_\lambda(0,t)=\lim_{t\to 0}\zeta_\lambda(0,t)$$

as claimed. □

Proof of Theorem 6. We prove first that the super-regularization of the infinite product of the sequence $a\lambda$ exists. We can define the two variable complex extension of the zeta function associated to the sequence $a\lambda$ as

$$\zeta_{a\lambda}(s,t)=a^{-s}\zeta_\lambda(s,t).$$

Then in the region $\{\mathrm{Re}\, s>s_0\}\times\{\mathrm{Re}\, t>t_1\}$ we have

$$\frac{\partial\zeta_{a\lambda}}{\partial s}(s,t)=\frac{\partial\zeta_\lambda}{\partial s}(s,t)-(\log a)a^{-s}\zeta_\lambda(s,t).$$

We have that for $\mathrm{Re}\, t>t_1$ fixed,

$$s\mapsto\frac{\partial\zeta_{a\lambda}}{\partial s}(s,t)$$

extends meromorphically to $s = 0$, and

$$\frac{\partial \zeta_{a\lambda}}{\partial s}(0, t) = \frac{\partial \zeta_\lambda}{\partial s}(0, t) - (\log a)\zeta_\lambda(0, t).$$

Now we have a meromorphic extension of this last equation to a neighborhood of $t = 0$. This proves the existence of the super-regularized product associated to the sequence $a\lambda$. Moreover,

$$\lim_{t\to 0} \frac{\partial \zeta_{a\lambda}}{\partial s}(0, t) = \lim_{t\to 0} \frac{\partial \zeta_\lambda}{\partial s}(0, t) - (\log a) \lim_{t\to 0} \zeta_\lambda(0, t),$$

which gives the formula stated. □

3. Application

Theorem 8. *The super-regularized product over all prime numbers is* $4\pi^2$,

$$1 \times 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times \cdots = \prod_p p = 4\pi^2.$$

Remark. Many reasons indicate that 1 should be considered also as a prime number. But this, obviously, does not matter here.

Proof. Consider the zeta function associated to the sequence of prime numbers

$$\mathcal{P}(s) = \sum_p p^{-s}.$$

We have seen in the introduction that

$$\mathcal{P}(s) = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n} \log \zeta(ns),$$

where ζ is Riemann zeta function. We define the extension to two complex variables

$$\mathcal{P}(s, t) = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^{1+t}} \log \zeta(ns).$$

We have

$$\mathcal{P}(s, 0) = \mathcal{P}(s),$$

and $\mathcal{P}(s, t)$ is a Dirichlet series of s and t (expand the logarithm). For $\{\mathrm{Re}\, s > 0\}$, $(\log \zeta(ns))_{n\geq 1}$ decays geometrically to 0. Thus $\mathcal{P}(s, t)$ is a meromorphic in $\{\mathrm{Re}\, s > 0\} \times \mathbf{C}$. Now we can compute

$$\frac{\partial \mathcal{P}}{\partial s}(s, t) = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^t} \frac{\zeta'(ns)}{\zeta(ns)},$$

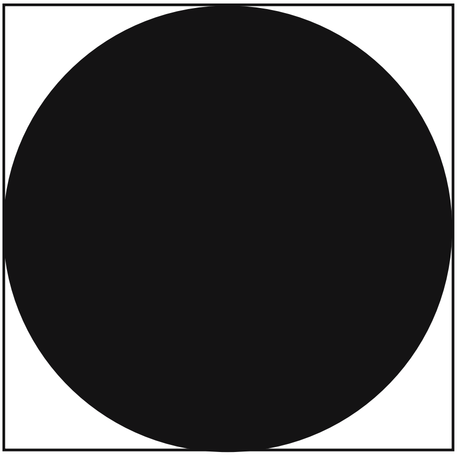

Fig. 2.

which for $\operatorname{Re} t > t_1 = 1$ fixed has a meromorphic extension to a neighborhood of $s = 0$. We compute the value at $s = 0$,

$$\frac{\partial \mathcal{P}}{\partial s}(0,t) = \left(\sum_{n=1}^{+\infty} \frac{\mu(n)}{n^t}\right) \frac{\zeta'(0)}{\zeta(0)} = \frac{1}{\zeta(t)} \log(2\pi).$$

Now we have a meromorphic extension to $t = 0$, which gives the result

$$\lim_{t\to 0} \frac{\partial \mathcal{P}}{\partial s}(0,t) = \frac{1}{\zeta(0)} \log(2\pi) = -2\log(2\pi).$$

So

$$\prod_p p = 4\pi^2.$$

□

Corollary 9. *The set of prime numbers is infinite.*

Proof. If the set was finite then the super-regularized product would be the usual product by Proposition 3. But $2 \times 3 \times 5 \times 7 > 4\pi^2$ because $\pi < 4$, as we show inscribing a circle inside a square. We can also proceed using the slightly less elementary argument that π^2 is not an integer. □

Therefore the following picture gives a geometric proof that the set of prime numbers is infinite.

Remark. 1. It is interesting to note the relation between the non-integer value of π^2 and the infinitude of prime numbers. An unexpected relation between two main topics of "The Elements" of Euclid. The first known proof of the infinitude of prime numbers is given by Euclid in Book IX, Prop. 20 of his "Elements" ([Euc], Vol. 2, p. 412). In book XII the "method of exhaustion" discovered by Eudoxus is presented ([Euc], Vol.3, p. 365). Historically this is the first algorithm that provides arbitrarily accurate approximations of the number π.

2. The first analytic proof of the existence of infinitely many prime numbers was given by L. Euler by finding Euler's prime factorization of the Riemann zeta function ([Eul], Chap. XV, p. 271), and showing that the series

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots=\sum_p \frac{1}{p}=\infty.$$

is infinite ([Eul], Chap. XV, Example 1, p. 277) where this formula appears. Note that Euler's formula also contains $p = 1$ as a prime number. We don't know of any other analytic proof other than Euler's and the one given above.

It is also interesting to observe that in the "Introductio in Analysin Infinitorum" ([Eul]) we can find this basic result about prime numbers together with many formulas involving π, and the value of π to many decimal places (that apparently was due to De Lagny). The notation π was consolidated after the publication of the "Introductio" (but apparently was first used by W. Jones in 1706). Euler's approach is equivalent to the fact that the Riemann zeta function has a pole at $s = 1$. Observe the noteworthy difference with our approach where only the value at $s = 0$ matters. Euler's idea was exploited by P.G.L. Dirichlet in order to prove his theorem on the infinitude of primes in arithmetic progressions.

For other non-analytic proofs of the infinitude of primes see the first "proof from the book" in [AZ].

3. One may ask if the formula

$$\infty! = \sqrt{2\pi}$$

does prove the infinitude of positive integers (!?). We leave to the reader this metaphysical question.

4. We observe that the value $4\pi^2$ obtained for the super-regularized product over all prime numbers coincides with the regularized determinant of the Laplacian on the circle.

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