

Total Convergence or General Divergence in Small Divisors

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Received: 19 September 2000 / Accepted: 27 February 2001

Dedicated to the memory of M. R. Herman

Abstract: We study generic holomorphic families of dynamical systems presenting problems of small divisors with fixed arithmetic. The characteristic features are delicate problems of convergence of formal power series due to Small Divisors. We prove the following dichotomy: We have convergence for all parameter values, or divergence everywhere except for an exceptional pluri-polar set of parameters. We illustrate this general principle in different problems of Small Divisors. As an application we obtain new richer families of non-linearizable examples in the Siegel problem when the Bruno condition is violated, generalizing and extending to higher dimension previous results of Yoccoz and the author.

Introduction

In this article we study generic (polynomial) holomorphic families of dynamical systems presenting problems of small divisors with fixed arithmetic. Generally speaking, the characteristic feature is the existence of a formal solution to a functional equation whose convergence is problematic due to the existence of small divisors. The principle our theorems illustrate is:

There is total convergence for all parameter values or general divergence except maybe for a very small exceptional set of parameter values.

The germinal idea can be traced back to Y. Ilyashenko where in [II] he studies divergence in problems of small divisors from divergence of the homological (or linearized) equation. Ilyashenko's paper contains a remarkable idea. We find there, for the first time in Small Divisors, the study of linear deformation of the system and the use of the polynomial dependence of the new formal linearizations. A similar idea, but not quite in the same problem, was used by H. Poincaré to show that linear deformations of completely

* Research supported by NSF

integrable hamiltonians are not generally completely integrable with analytic first integrals depending analytically on the parameter ([Poi] Vol I, Chap. V). It is the key preliminary step in his difficult proof of the non existence of non trivial local analytic first integrals in the three body problem (for some particular configurations of masses).

Linear deformations have been fruitfully used by J.-C. Yoccoz ([Yo] p. 58). He proves that in the Siegel problem the quadratic polynomial is the worst linearizable holomorphic germ. The only ingredient in this proof that is not in Ilyashenko’s one is the classical Douady–Hubbard straightening theorem for polynomial-like mappings. Yoccoz simplifies Ilyashenko’s argument replacing Nadirashvili’s lemma by the maximum principle. He loses in that way the strength of the original approach, in particular the potential theoretic aspects. Non-linear polynomial perturbations were used by the author in [PM1] to generalize Yoccoz result to higher degree polynomials.

In this article we clarify and strength the role played by potential theory in parameter space. A key point is the observation that Nadirashvili’s lemma can be improved by using Bernstein–Walsh lemma in approximation theory. In that way we make precise the thinnest notion for the exceptional set. In parameter space the exceptional set is pluri-polar (i.e. there is a pluri-sub-harmonic function identically $-\infty$ on this set) which is much stronger than the original measure 0 condition.

The techniques in this article are applicable to virtually any holomorphic problem in small divisors where the dependence on parameters of the coefficients of the divergent series are polynomial (as we will see this happens in most of the problems). We have selected a few illustrative ones guided mainly by our personal taste. We only consider here polynomial families. The same proof can be extended easily for more general holomorphic families (see Remark 4).

As far as the author knows, the first person who studied small divisors problems using ingredients from potential theory in parameter space is M. Herman (see [He1] and [He2]).

Linearization.

Theorem 1. *Let $n, m \geq 1$ and $d \geq 0$. For a multi-index $i = (i_1, \dots, i_m) \in \mathbb{N}^m$ with $0 \leq i_1 + \dots + i_m \leq d$, let φ_i be a germ of holomorphic map*

$$\varphi_i : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$$

of order larger or equal to 2 (i.e. $\varphi_i(z) = \mathcal{O}(z^2)$).

For $t = (t_1, \dots, t_m) \in \mathbb{C}^m$ we consider the holomorphic family of germs of holomorphic diffeomorphisms, $z \in \mathbb{C}^n$,

$$f_t(z) = Az + \sum_{\substack{i=(i_1, \dots, i_m) \\ 0 \leq i_1 + \dots + i_m \leq d}} t^i \varphi_i(z),$$

where $A \in GL_n(\mathbb{C})$ is a fixed linear map, $A = D_0 f$, with non-resonant eigenvalues.

Then all maps $f_t, t \in \mathbb{C}^m$ are formally linearizable, i.e. there exists a unique formal map h_t with $h_t(0) = 0$ and $D_0 h_t = I$ such that the formal equation

$$h_t \circ f_t = A \circ h_t$$

is satisfied.

We have the following dichotomy:

- 1) *The holomorphic family $(f_t)_{t \in \mathbf{C}^m}$ is holomorphically linearizable, that is for all $t \in \mathbf{C}^m$, h_t defines a germ of holomorphic diffeomorphism. Moreover, the radius $R(h_t)$ of convergence of the linearization h_t is bounded from below on compact sets and, more precisely, for some $C_0 > 0$, and any $t \in \mathbf{C}^m$,*

$$R(h_t) \geq \frac{C_0}{1 + \|t\|^d}.$$

- 2) *Except for an exceptional pluri-polar set $E \subset \mathbf{C}^m$ of values of t , f_t not holomorphically linearizable.*

We remind that $E \in \mathbf{C}^m$ is a pluri-polar set if for each $z \in E$ there is a neighborhood U of z and a pluri-sub-harmonic function u such that $E \cap U \subset u^{-1}(-\infty)$. This implies that the set has Lebesgue measure 0 and is small in a strong sense. For example, there are C^∞ smooth arcs in \mathbf{C}^m which are not pluri-polar. We refer to [Kl] for basic notions of pluri-potential theory. In dimension 1 ($m = 1$) this means that the set is polar in the usual sense of potential theory, that is, it has 0 logarithmic capacity. Such a set has not only measure 0 but even Hausdorff dimension 0 (see [Ra] or [Tsu]).

Remarks. 1) We recall that the eigenvalues $(\lambda_1, \lambda_2, \dots, \lambda_n)$ of A , counted with multiplicity, are non-resonant if

$$\lambda_i - \lambda_1^{i_1} \dots \lambda_n^{i_n} \neq 0$$

for all $(i_1, \dots, i_n) \in \mathbf{N}^n$ with $i_1 + \dots + i_n \geq 2$, and $i = 1, \dots, n$. We state later a theorem for holomorphic germs with resonant linear parts. (proved in [PM3]).

- 2) The linear part A is in the Poincaré domain if

$$\min(\max_i |\lambda_i|, \max_i |\lambda_i^{-1}|) < 1.$$

In that case it is well known that we are always in Case (1). Otherwise the linear part of A belongs to the Siegel domain.

- 3) In general the exceptional set $E \subset \mathbf{C}^m$ is not empty. For example if $\varphi_0 = 0$, then $0 \in E$ when we are in the second case.
 4) With the same type of proof, one can prove the same result (but with a weaker estimate on the radius of convergence in (1)) for holomorphic non-polynomial families of the form

$$f_t(z) = Az + \sum_{\substack{i=(i_1, \dots, i_m) \\ |i|=i_1+\dots+i_m \geq 0}} t^i \varphi_i(z),$$

where the holomorphic germs $(\varphi_i(z))$ have increasing orders such that $\varphi_i(z) = \mathcal{O}(z^{\varepsilon_0 |i|})$ for some $\varepsilon_0 > 0$.

Some illustrative corollaries follow now. For $n = 1$ and the special case of entire functions we have directly from Theorem 1:

Corollary 1. *Let $(f_t)_{t \in \mathbf{C}^m}$ be a finite dimensional holomorphic family of entire functions as above with*

$$f'_t(0) = \lambda,$$

where $\lambda = e^{2\pi i \alpha}$ with $\alpha \in \mathbf{R} - \mathbf{Q}$.

Then the family is linearizable or, except for an exceptional polar set $E \in \mathbf{C}$ of values of t , all f_t are non-linearizable.

Assuming that the family contains a non-linearizable structurally stable polynomial (for example a quadratic polynomial) we can break the dichotomy. This just follows from the observation that in a neighborhood of this polynomial all elements of the family are quasi-conformally conjugated (by the Douady–Hubbard straightening theorem), thus they are linearizable or not simultaneously.

Corollary 2. *Let $(f_t)_{t \in \mathbb{C}^m}$ be a finite dimensional holomorphic family of entire functions as above with*

$$f'_t(0) = \lambda,$$

where $\lambda = e^{2\pi i \alpha}$ with $\alpha \in \mathbf{R} - \mathbf{Q}$. We assume that for some value t_0 f_{t_0} is a structurally stable polynomial in the space of polynomials with fixed point at 0 and multiplier λ .

Then if α is not a Bruno number almost all entire functions f_t , except maybe for an exceptional polar set $E \subset \mathbf{C}$ of values of t , are not linearizable.

When $\alpha \in \mathbf{R} - \mathbf{Q}$ is not a Bruno number, no examples were known of non-linearizable entire functions not quasi-conformally conjugated to polynomials in a neighborhood of 0. This was due to the shortcomings of Yoccoz’ maximum principle approach [Yo].

A particular case of this corollary is the theorem proved in [PM1] about polynomial germs. The author showed, generalizing Yoccoz’ result for the quadratic polynomial, that if α is not a Bruno number, in the space

$$\mathcal{P}_{\lambda,d} = \{P(z) = \lambda z + a_2 z^2 + \dots + a_d z^d; (a_2, \dots, a_d) \in \mathbf{C}^{d-1}\}$$

the polynomials that are of degree d and structurally stable (this is an open dense set of pluri-polar complement) are not linearizable.

It is worth mentioning that the question to decide if the exceptional set $E_{\lambda,d}$ is trivial (i.e. reduced to 0) for the polynomial family $\mathcal{P}_{\lambda,d}$ when $\alpha \in \mathbf{R} - \mathbf{Q}$ is not a Bruno number, is still open, even for the cubic family:

$$P_b(z) = \lambda z + b z^2 + z^3.$$

Contrary to extended belief, the author will not be surprised that $E_{\lambda,d}$ is not trivial for appropriate values of λ and d . For Liouville numbers α with extremely good rational approximations, by an argument of Cremer (see [Cr] and [PM1]), $E_{\lambda,d}$ is known to be reduced to 0.

To illustrate the strength of the precedent theorem, we present the following variations.

Corollary 3. *Let $\alpha \in \mathbf{R} - \mathbf{Q}$ be not Bruno.*

- 1) *Let $f(z) = e^{2\pi i \alpha} z + \mathcal{O}(z^2)$ be non-linearizable. Any polynomial family $(f_t)_{t \in \mathbf{C}}$ as above containing f has all of its members f_t non-linearizable except for an exceptional polar set of parameters t .*
- 2) *For an arbitrary holomorphic germ $\varphi(z) = \mathcal{O}(z^2)$ and for almost all values $t \in \mathbf{C}$ except a polar set E , we have that*

$$f_t(z) = e^{2\pi i \alpha} z + z^2 + t \varphi(z)$$

is not linearizable.

- 3) *Let*

$$f(z) = e^{2\pi i \alpha} z + \sum_{n \geq 2} f_n z^n$$

be an arbitrary entire function. Keeping all coefficients fixed except f_2 , there is a polar set E such that if $f_2 \in \mathbf{C} - E$, then f is not linearizable.

Also, we have the same type of results for rational functions (in the proof we use Remark 4).

Corollary 4. *Let*

$$\mathcal{R}_{\lambda,d} = \{f \in \mathbf{C}(z); f(0) = 0; f'(0) = \lambda; \deg f = d\}.$$

When $\alpha \in \mathbf{R} - \mathbf{Q}$ is not a Bruno number, except for an exceptional set, all rational functions in $\mathcal{R}_{\lambda,d}$ are not linearizable.

The corollaries presented here are by no means restricted to dimension 1. There follows just one example of new result.

Corollary 5. *We consider the space $\mathcal{P}_{A,d}$ of polynomial germs of holomorphic diffeomorphisms with non-resonant linear part $A \in GL_n(\mathbf{C})$ of total degree d . The existence of one non-linearizable example forces all the others, except maybe a pluri-polar exceptional set, to be non-linearizable. This happens for instance when one eigenvalue of A does not satisfy Bruno’s condition.*

We can prove also a version of Theorem 1 for resonant linear parts A which has an independent interest (for example when applied to symplectic holomorphic mappings). When the linear part is resonant, the linearization is not uniquely determined. Nevertheless, given a polynomial family (f_t) as in Theorem 1 whose elements are all formally linearizable, there always exist a canonical family of linearizations (h_t) whose coefficients depend polynomially on t (see [PM3]). The complete treatment of this situation requires some algebraic preliminaries. We do not develop them in this article. We refer to [PM3] for a complete treatment. We are content to prove here the following:

Theorem 2. *We consider a family $(f_t)_{t \in \mathbf{C}^m}$ as in Theorem 1 but we allow $A \in GL_n(\mathbf{C})$ to be resonant. We are also given a family of formal linearizations $(h_t)_{t \in \mathbf{C}^m}$ whose coefficients depend polynomially on t . We assume that the monomial of order l has a coefficient a polynomial of degree bounded above by $C_0 + C_1 l$ for some $C_0, C_1 > 0$.*

We have the following dichotomy:

- 1) *The family $(f_t)_{t \in \mathbf{C}^m}$ is holomorphically linearizable by the family $(h_t)_{t \in \mathbf{C}^m}$.*
- 2) *For all $t \in \mathbf{C}^m$ except for an exceptional set E of Γ -capacity 0, h_t is diverging.*

One can also prove a statement similar to Theorem 2 when (f_t) is not formally linearizable but the family (h_t) conjugates the family to a formal normal form ([PM3]). A particular relevant case is the one of a symplectic holomorphic diffeomorphism with an elliptic fixed point. The formal conjugacy to Birkhoff’s normal form is then in general diverging (see [Si-Mo], Sect. 30). The formal normal form situation is also relevant when A is not invertible.

Central manifolds. In situations where the dynamics is not linearizable, one can still have invariant manifolds through the fixed point (see for example [Pos], and [St]). Usually one has a formal equation whose coefficients depend polynomially on the coefficients of f_t thus on t . In these situations the following theorem applies.

Theorem 3. *Under the same assumptions as in Theorem 1, we assume the existence of a formal invariant submanifold through 0 with equation*

$$F_t(z) = 0$$

with $F_t : \mathbb{C}^n \rightarrow \mathbb{C}^p$ a formal mapping whose coefficients depend polynomially on $t \in \mathbb{C}^m$. More precisely, the coefficient of the monomial of valuation l is a polynomial on t of degree less than $C_0 + C_1 l$ where $C_0, C_1 > 0$ are constants.

We have the dichotomy:

- 1) F_t converges and defines an invariant submanifold for all $t \in \mathbb{C}^m$.
- 2) Except for a pluri-polar exceptional set of parameter values $t \in \mathbb{C}^m$, F_t diverges.

We have the same theorem for holomorphic vector fields. To be more specific, consider the situation treated by L. Stolovitch [Sto], for $1 \leq j \leq n$,

$$\dot{z}_j = \lambda_j z_j + \sum_{i=1}^d t^i f_{j,i}(z),$$

where $f_{j,i} = \mathcal{O}(2)$. We assume that the linear part (which does not depend on t) is in the Siegel domain, that is 0 belongs to the convex hull of $\{\lambda_1, \dots, \lambda_n\}$. We assume that the linear part is resonant, and the resonances, $n_1, \dots, n_2 \geq 1$ and any $1 \leq j \leq n$,

$$\sum_{i=1}^n n_i \lambda_i - \lambda_j \neq 0$$

are generated by a finite number of resonances, $1 \leq j \leq l, r_j = (r_1, \dots, r_n) \neq 0, r_j \in \mathbb{N}^n$,

$$(r_j, \lambda) = 0.$$

Then there exists a formal change of variables $w = h_t(z)$ with $h_t(0) = 0$ and $D_0 h_t = I$ which transforms the system into

$$\dot{w}_i = \lambda_i w_i + g_{i,t}(w)$$

with $g_{i,t}(w) = \sum_{j=1}^l g_{i,j,t} y^{r_j}$, and if $\|r_j\| = 1$ then $g_{i,j,t}(0) = 0$. As constructed in [Sto], the coefficients of the formal normalization do depend polynomially on t .

Theorem 4. *With the previous assumption, we have the following dichotomy,*

- 1) For all value of $t \in \mathbb{C}^m$ the formal normalization h_t converges, thus the sub-manifold $\{w^{r_1} = 0, \dots, w^{r_n} = 0\}$ is invariant.
- 2) Except for an exceptional pluri-polar set of values of t , the normalization mappings h_t diverge.

According to [Sto], and assuming that the higher dimensional resonant Bruno condition on $(\lambda_1, \dots, \lambda_n)$ holds, we are always in Case (1).

Singularities of holomorphic vector fields. We consider a polynomial family of germs of holomorphic vector fields as before. But we assume here that the linear part is non-resonant, that is, for any $n_1, \dots, n_2 \geq 1$ and any $1 \leq j \leq n$,

$$\lambda_j - \sum_{i=1}^n n_i \lambda_i \neq 0.$$

Theorem 5. *Under the above hypothesis, we have the dichotomy*

- 1) *The family of holomorphic vector fields is linearizable for all t .*
- 2) *Except for an exceptional pluri-polar set of values of t , the holomorphic vector fields are non-linearizable.*

In the case $n = 2$ one has a complete correspondence of the problem of linearization of holomorphic vector fields as above and the problem of linearization of germs of holomorphic diffeomorphisms of $(\mathbf{C}, 0)$. This was established in [PM-Yo] where, as corollary, Yoccoz and the author proved that the Bruno condition is optimal for the problem of linearization.

Centralizers. We discuss here the situation of one complex variable. The analysis generalizes similarly to higher dimension.

The study of centralizers of holomorphic germs generalizes the problem of linearization. We refer to [PM2] for proofs and references. In the group of holomorphic diffeomorphisms $G = (\text{Diff}(\mathbf{C}, 0), \circ)$, composed by holomorphic germs f with $f(0) = 0$ and $f'(0) \neq 0$, and with group operation given by the composition \circ , we consider the centralizer of f ,

$$\text{Cent}(f) = \{g \in \text{Diff}(\mathbf{C}, 0); g \circ f = f \circ g\}.$$

It is a subgroup of G that can be interpreted as the group of symmetries of f (i.e. those changes of variables conjugating f to itself). We have the following cases:

- 1) For germs with attracting or repelling fixed point at 0, i.e. $f'(0) = e^{2\pi i\alpha}$ with $\alpha \notin \mathbf{R}$, the centralizer is a complex flow of dimension 1.
- 2) For germs with indifferent rational fixed point at 0, i.e. $f'(0) = e^{2\pi i\alpha}$ with $\alpha \in \mathbf{Q}$, the centralizer is generated by root (for composition) of the germ (then it is discrete), or it is a one dimensional complex flow.

These cases are well understood. We discuss the last case in what follows.

- 3) For germs with an indifferent irrational fixed point at 0, $f'(0) = e^{2\pi i\alpha}$ with $\alpha \in \mathbf{R} - \mathbf{Q}$, the centralizer can be a one-dimensional real flow (the linearizable case), discrete or uncountable.

The existence of examples where the last possibility holds was only proved recently in [PM2].

In this case the centralizer is abelian and isomorphic to a subgroup of the circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ by the rotation number morphism,

$$\rho : \begin{matrix} G & \longrightarrow & \mathbf{T} \\ f & \longmapsto & \log f'(0). \end{matrix}$$

We denote

$$G(f) = \rho(\text{Cent}(f)).$$

Note that $\mathbf{Z}\alpha \subset G(f)$. The holomorphic germ f is linearizable if and only if the centralizer is full $G(f) = \mathbf{T}$, otherwise it is an F_σ and dense set of \mathbf{T} with 0 measure (and indeed 0 capacity). Moreover, all elements $g \in \text{Cent}(f)$ are non-linearizable.

Thus how small $G(f)$ is can be thought of as a measure of how far f is from being linearizable. Thus the study of centralizers (apart from the motivation coming from the theory of foliations, see [PM2]) is motivated as a finer study of linearization. The

question of determining if $\beta \in G(f)$ is intimately connected with the common rational approximations of α and β , as the following theorem of J. Moser shows ([Mo]). Let f be non-linearizable. If there exists $\gamma, \tau > 0$ such that for any $p \geq 1, q \in \mathbf{Z}$,

$$\min(|q\alpha - p_1|, |q\beta - p_2|) \geq \frac{\gamma}{q^\tau},$$

then $\beta \notin G(f)$. The necessity of an arithmetic condition in Moser’s theorem is proved in [PM2].

Using the techniques developed in this article, we prove:

Theorem 6. *Let f_t be a family of holomorphic germs as in Theorem 1, with fixed linear part $f'(0) = e^{2\pi i\alpha}$, $\alpha \in \mathbf{R} - \mathbf{Q}$. For any $\beta \in \mathbf{T}$, we have the following dichotomy:*

- 1) For all $t \in \mathbf{C}$, $\beta \in G(f_t)$.
- 2) Except for an exceptional polar set $E \subset \mathbf{C}$, $\beta \notin G(f_t)$.

Further applications. One can use the argument presented here, to improve on the original result of Ilyashenko in [II], and replace one parameter by several, and measure 0 set by pluri-polar set. The same remark applies to all results derived from Ilyashenko’s argument, as those one can find in [He2] and [Yo].

A complete treatment for the problem of linearization of resonant holomorphic germs is given in [PM3]. These techniques also apply to other small divisors problems. Behind the technique used here, there is an abstract elementary theorem on holomorphic extension of Rothstein type for a certain type of power series(see [PM4]).

1. Proof of Theorem 1

1.1. Nadirashvili and Bernstein lemmata. We first present the potential theory tools in dimension 1 which is probably more familiar to the readers, and is enough to prove the theorems for one dimensional parameter families. For the definition of Green function, polar sets and other notions in potential theory we refer the reader to [Ra] for example (for a more encyclopedic treatment see [Tsu]).

Y. Ilyashenko in his article [II] makes use of the following lemma attributed to N. S. Nadirashvili ([Na]).

Lemma (Nadirashvili). *Let $E \subset \mathbf{C}$ be a compact set with positive measure in the disk \mathbf{D}_R of center 0 and radius $R > 0$. Let P be a polynomial of degree n such that for some $M > 0$,*

$$\|P\|_{C^0(E)} \leq M^n.$$

Then there exists a constant C only depending on the measure of E and $R > 0$ such that

$$\|P\|_{\mathbf{D}_R} \leq C^n M^n.$$

The key idea of this lemma is that it is enough to control a polynomial on a set of E positive measure to get a bound in any bounded domain. Note that this idea is very different in nature than the maximum principle.

We improve on [II] observing that the measure of E is not the relevant quantity. Nadirashvili’s lemma is a direct corollary of the classical Bernstein (or Bernstein–Walsh) lemma in approximation theory and classical potential theory (see [Ra], p. 156) and the fact that a set of positive measure is non-polar:

Lemma (Bernstein). *Let $E \subset \mathbf{C}$ be a non-polar compact set (i.e. $\text{cap}(E) > 0$). Let Ω be the connected component of $\overline{\mathbf{C}} - E$ containing ∞ . Then for any polynomial P of degree n , we have for $t \in \mathbf{C}$,*

$$|P(t)| \leq e^{ng_{\Omega}(t, \infty)} \|P\|_{C^0(K)},$$

where g_{Ω} denotes the Green function of Ω .

We recall that the non-polarity of E implies the existence of a Green function $g_{\Omega}(z, \infty) = g_{\Omega}(z)$ such that $g_{\Omega}(\infty) = \infty$, g_{Ω} is harmonic in Ω , for $z \rightarrow \infty$,

$$g_{\Omega}(z) - \log |z|$$

is bounded, and when $z \rightarrow z_0$, z_0 regular point of E , $g_{\Omega}(z) \rightarrow 0$. These properties determine g_{Ω} uniquely.

The proof of this lemma is quite simple.

Proof. We can assume the polynomial monic. Then

$$u(t) = \frac{1}{n} \log P(t) - \frac{1}{n} \log \|P\|_{C^0(K)} - g_{\Omega}(t, \infty)$$

is sub-harmonic, is negative near ∞ (because $g_{\Omega}(t, \infty) = \log |t| + \text{cap}(E) + o(1)$), and $\limsup u(t) \leq 0$ when $t \rightarrow K$. The maximum principle concludes the proof. \square

For future reference we recall here that a countable union of polar sets is polar.

1.2. Pluri-potential theory. There is a relatively recent extension to \mathbf{C}^m of the classical potential theory on \mathbf{C} . We refer to [Kli], Chap. 5 for proofs.

We consider the set \mathcal{L} of pluri-subharmonic functions u defined in \mathbf{C}^m and of minimal growth, i.e. $u(z) - \log \|z\|$ is bounded above when $\|z\| \rightarrow \infty$. Given a subset $E \subset \mathbf{C}^m$, we define

$$V_E(z) = \sup\{u(z); u \in \mathcal{L}, u|_E \leq 0\}.$$

The upper semi-continuous regularization V_E^* of V_E is called the pluri-sub-harmonic Green function of E . This function V_E^* is either pluri-sub-harmonic or identically $+\infty$. We are in the former case when E is not pluri-polar, then V_E^* has logarithmic growth at ∞ , that is $V_E^*(z) - \log \|z\|$ is bounded above when $z \rightarrow \infty$.

As in one dimension we immediately prove

Lemma. *If E is not pluri-polar and P is a polynomial of degree d , then we have for $z \in \mathbf{C}^m$,*

$$|P(z)| \leq \|P\|_{C^0(E)} e^{dV_E(z)}.$$

We also have that a countable union of pluri-polar sets is pluri-polar.

1.3. *Proof of Theorem 1.* We start with the following elementary lemma.

Lemma A. *The coefficient vectors $h_i(t)$ of the formal linearization*

$$h_t(z) = z + \sum_{\substack{i=(i_1, \dots, i_n) \\ i_1 + \dots + i_n \geq 2}} h_i(t)z^i$$

have coordinates that are polynomials in the parameter $t = (t_1, \dots, t_m)$ of degree less than $d(i_1 + \dots + i_n)$.

Proof. We can assume that A is in upper triangular Jordan normal form. We solve the functional equation

$$A \circ h_t = h_t \circ f_t$$

identifying coordinates and developing in homogeneous vector monomials. By induction on $|i| = i_1 + \dots + i_n$ we do determine successively the vectors $h_i(t)$ that depend on coefficients of f_t and lower order $h_j(t)$'s, $|j| < |i|$. By induction, the linear equations determining $h_i(t)$ do have the form

$$(A - M_i)h_i(t) = \sum_{|j| < |i|} c_j(t)h_j(t),$$

where the matrix M_i is upper triangular, only depends on A and i (but not on the parameter t), has diagonal coefficients products of eigenvalues of A (thus $A - M_i$ is invertible) and in the left-hand side the coefficients $c_j(t)$ are polynomials in t of total degree at most $(|i| - |j|)d$. To see this, note that $c_j(t)$ is obtained collecting the coefficient of z^i in the expansion of

$$(f_{t,1}(z))^{j_1} \dots (f_{t,n}(z))^{j_n} = \prod f_{t,k}$$

with

$$|j| + |\{k; k \geq 2\}| \leq |\{k; k = 1\}| + 2|\{k; k \geq 2\}| \leq \sum k = |i|$$

and

$$\deg_t c_j(t) \leq d|\{k; k \geq 2\}| \leq d(|i| - |j|)$$

By induction the result follows. \square

Remark. In the case of a more general family as the ones in remark 4 after the theorem, we get, by the same proof, that the degree of $c_j(t)$ is bounded by $\varepsilon_0^{-1}(|i| - |j|)$.

Proof of Theorem 1. Let

$$E = \{t \in \mathbf{C}; f_t \text{ is linearizable}\}.$$

We want to show that E is pluri-polar or the whole complex plane. We have

$$E = \bigcup_{j \geq 1} E_j,$$

where E_j the set of parameters t such that h_t has radius of convergence larger or equal to $1/j$ and h_t is holomorphic and bounded by 1 in the ball of center 0 and radius $1/j$. It is clear that any convergent h_t belongs to some E_j since $h_t(0) = 0$. The sets E_j are

clearly closed. If E is not pluri-polar, we have that for some $j_0 \geq 1$, E_{j_0} is not pluri-polar. Thus, by Cauchy, there exists $\rho_0 > 0$ such that for all $t \in E_{j_0}$,

$$\varphi(t) = \sup_{|i| \rightarrow +\infty} \|h_i(t)\| \rho_0^{-|i|} < +\infty.$$

The function φ is lower semicontinuous (as a supremum of continuous functions), and

$$E_{j_0} = \bigcup_p L_p$$

where $L_p = \{z \in E_{j_0}; \varphi(t) \leq p\}$ is closed. Again some L_{p_0} is not pluri-polar. Finally, we find a non-pluri-polar closed set $C = L_{p_0}$ for which there exists $\rho_1 > 0$ such that for any $t \in C$ and all $i \in \mathbb{N}^n$,

$$\|h_i(t)\|_{C^0(C)} \leq \rho_1^{|i|}.$$

Using Bernstein’s lemma and Lemma A we get for any $t \in C^m$

$$|h_i(t)| \leq e^{d|i|V_C(t)} \rho_1^{|i|}.$$

Thus h_t has non-zero radius of convergence and f_t is linearizable for any $t \in C^m$. The radius of convergence can be estimated by the precise form of Bernstein lemma and $V_C(t) \sim \log \|t\|$ when $t \rightarrow \infty$, by

$$R(h_t) \geq \frac{C_0}{1 + \|t\|^d},$$

for some constant $C_0 > 0$.

Now, the proof also goes through in the case of Remark 4. In Lemma A the coefficient $h_i(t)$ is now a polynomial of degree $\varepsilon_0^{-1}|i|$, because $h_i(t)$ only depends on the coefficients of f_t of degree $\leq |i|$ (which are polynomials on t of degree $\mathcal{O}(\varepsilon_0^{-1}|i|)$). This only affects the bound on the radius of convergence (just replace d by ε_0^{-1}). \square

1.4. Proof of the corollaries. Corollary 1 is just a particular case of Theorem 1. Corollary 3 part (1) also. Now using Corollary 3 part (1) we prove

Corollary (J.-C. Yoccoz [Yo]). *The quadratic polynomial $P_\alpha(z) = e^{2\pi i\alpha}z + z^2$ is non-linearizable when α is not a Bruno number.*

Proof. For this, pick f non-linearizable (that exists from [Yo]) and consider $f_t(z) = tP_\alpha(z) + (1 - t)f(z)$. Since f_1 is not linearizable, all f_t except for a polar set E of values of t are not linearizable. By Douady–Hubbard straightening theorem $C - E$ is a neighborhood of 0 and 0 is not linearizable. \square

Now by the same argument, part (2) and (3) of Corollary 3 follow.

We prove:

Corollary (R. Pérez-Marco [PM1]). *If P is a structurally stable polynomial in the space*

$$\mathcal{P}_{\lambda,d} = \{P(z) = \lambda z + a_2 z^2 + \dots + a_d z^d; (a_2, \dots, a_d) \in \mathbb{C}^{d-1}\},$$

then P is not linearizable.

Proof. Just consider

$$f_t(z) = tP(z) + (1 - t)P_\alpha(z)$$

and do the same proof. \square

Now Corollary 2 follows from part (1) of Corollary 3.

Corollary 4 is not a strict corollary of Theorem 1 but of the improvement using Remark 4. One should observe that the coefficients of the linearization are polynomial functions of the coefficients of the rational function with appropriate degree. To see this, since 0 is not a pole of the rational function R , we can assume conjugating by a linear dilatation that the constant coefficient of the denominator is 1, i.e.

$$R(z) = \frac{P(z)}{1 - Q(z)}.$$

Now expanding

$$\frac{1}{1 - Q(z)} = \sum_i Q(z)^i$$

we see that the coefficients of the power series depend, as is power series on the coefficients of P and Q , with order bounded from below by a linear function.

For Corollary 5 only the last assertion is not immediate. If one of the eigenvalues of A violates the Bruno condition, λ_1 for example, then

$$(z_1, \dots, z_n) \mapsto (\lambda_1 z_1 + z_1^2, \lambda_2 z_2, \dots, \lambda_n z_n)$$

is not linearizable, thus the first part applies giving a rich family of polynomial non-linearizable examples.

2. Proof of the Other Theorems

2.1. Formal linearizations and Theorem 2. The proof of Theorem 2 is similar to the proof given in the previous section of Theorem 1. For a complete study of the resonant case we refer to [PM3]. We just mention here the new difficulties that appear.

Assume that we have a germ of holomorphic diffeomorphism

$$f : (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^n, 0)$$

with resonant linear part $A = D_0 f \in GL_n(\mathbf{C})$.

The formal linearization h is not always unique when the linear part A is resonant or not invertible. For example, for $n = 2$ and

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

then if h is a formal linearization then $l \circ h$ is also one, where

$$l(z_1, z_2) = (z_1 + k(z_2), z_2).$$

If we consider a polynomial family $(f_t)_{t \in \mathbf{C}^m}$, to request that h_t has coefficients depending polynomially on t does not improve things. One then can take various k_t depending polynomially on t . Thus the family (h_t) with this further restriction is not unique.

This presents a problem in order to prove the non-linearizability. Considering a polynomial parameter family of formal linearizations of a fixed map $f, (h_t)$, we may be in the second case, but this does not mean that f is not linearizable. For instance, if the exceptional set E is not empty, then f will be linearizable! The question of non-linearizability is harder to answer. In [PM3] we show that if the polynomial family of linearizations is chosen in a natural way, this difficulty does not arise.

2.2. *Other theorems.* The proofs are similar to Sect. 1. We just comment on the particularities of each problem.

For an explicit example where Theorem 3 applies one can workout the example of J. Poschel [Pos]. The polynomial dependence with the appropriate bound on the degrees follows from the formal computation of the formal equation of the invariant manifold.

Theorem 4 is proved in a similar way. We refer to [Sto] for the formal computation of a normalizing map with polynomial dependence on the parameter t with the appropriate degrees. One can workout in this situation similar results than in [PM3].

The linearization in Theorem 5 is unique and it is well known ([Ar]) that it depends polynomially on t with the appropriate degrees. Thus the same proof applies. Note that in \mathbb{C}^2 , by [PM-Y] one can realize any germ of holomorphic diffeomorphism in $(\mathbb{C}, 0)$ as holonomy of a singularity of holomorphic vector field of the type considered.

For the proof of Theorem 6, we give the induction formulas for the coefficients of g_β . Let $\mu = e^{2\pi i\alpha} = f_1$ and $\lambda = e^{2\pi i\alpha} = g_1$, and

$$f(z) = \sum_{n=1}^{+\infty} f_n z^n,$$

$$g(z) = \sum_{n=1}^{+\infty} g_n z^n.$$

Identifying terms of degree $n \geq 2$ in the equation $g \circ f = f \circ g$, we get for $n \geq 2$,

$$g_n = \frac{\mu^n - \mu}{\lambda^n - \lambda} f_n + \sum_{p=1}^{+\infty} f_p \sum_{\substack{i_1 + \dots + i_p = n \\ i_j \geq 1}} g_{i_1} \dots g_{i_p} + \sum_{p=1}^{+\infty} g_p \sum_{\substack{i_1 + \dots + i_p = n \\ i_j \geq 1}} f_{i_1} \dots f_{i_p}.$$

And by induction the coefficients of g_β depend polynomially on t and have the appropriate degrees.

Acknowledgements. This paper has benefit from remarks of several people. Many thanks to Rafael de La Llave who pointed out corrections and improvements, and for all the friendly and instructive discussions on Small Divisors. Also to the specialists in several complex variables, Nessim Sibony and independently Eric Bedford and Norm Levenberg, for pointing out very quickly that the natural notion of smallness is pluri-polarity and not Γ -capacity 0 used in the first version. Special thanks to Eric Bedford for explaining to me some basic notions of pluri-potential theory. Thanks to Giovanni Forni and Jean-Christophe Yoccoz for pointing out corrections, and also to Yulij Ilyashenko for exchanges on the subject.

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Communicated by Ya. G. Sinai