

A SIMPLE PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

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ABSTRACT. We present a simple short proof of the Fundamental Theorem of Algebra, with minimal use of topology.

1. STATEMENT.

Theorem 1.1. *A non constant polynomial $P(z) \in \mathbb{C}[z]$ with complex coefficients has a root.*

The proof is based on the following elementary facts:

- A polynomial has at most a finite number of roots.
- A polynomial is proper.
- The Inverse Function Theorem.
- Removing from \mathbb{C} a finite number of points leaves a connected space.

2. THE PROOF.

It is enough to consider a monic polynomial P . We denote by $\mathcal{C} = (P')^{-1}(0)$ the finite set of critical points of P , and by $\mathcal{D} = P(\mathcal{C})$ is the finite set of critical values of P .

- Let $R = \{c \in \mathbb{C}; \text{ the polynomial } P(z) - c \text{ has a simple root and no double roots}\}$.
- $R \subset \mathbb{C} - \mathcal{D}$. This is because if $c \in \mathcal{D}$, then $c = P(z_0)$ for some critical point $z_0 \in \mathcal{C}$, hence $P'(z_0) = 0$ and $P(z) - c = 0$ has a double root at z_0 . Note that $\mathbb{C} - \mathcal{D}$ is connected (\mathcal{D} being finite).
- R is open. By application of the Inverse Function Theorem at the simple roots of $P(z) - c$ we can locally follow each root for c' near c and they remain simple. Moreover, by properness of P the number of roots remains locally constant and we don't have any other roots (in particular double ones) for c' near c .
- R is closed in $\mathbb{C} - \mathcal{D}$. Since P is monic, if c is uniformly bounded then any root of $P(z) - c$ is uniformly bounded. We can take a subsequence of $c_n \rightarrow c_\infty \in \mathbb{C} - \mathcal{D}$ such that the limit of roots of $P(z) - c_n$ converge to roots of $P(z) - c_\infty$. We have at least one such limit so $P(z) - c_\infty$ has roots. Moreover, all roots of $P(z) - c_\infty$ are all simple since $c_\infty \in \mathbb{C} - \mathcal{D}$.
- R is non-empty. For any $a \in \mathbb{C}$ we have that for $c = P(a)$, $P(z) - c$ has at least $z = a$ as root. If we choose $a \in \mathbb{C} - P^{-1}(\mathcal{D})$, then for any root z_0 of $P(z) - c$ with $c = P(a)$, we have $P(z_0) = P(a) \notin \mathcal{D}$, so $z_0 \notin P^{-1}(\mathcal{D})$, but $\mathcal{C} \subset P^{-1}(\mathcal{D})$, and $z_0 \notin \mathcal{C}$, and the root z_0 is simple.

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The above proves that $R = \mathbb{C} - \mathcal{D}$. Now, if $0 \in \mathcal{D}$, then $0 = P(z_0)$ for a critical point z_0 of P that is also a root of P . If $0 \notin \mathcal{D}$, then $0 \in R = \mathbb{C} - \mathcal{D}$ and the equation $P(z) - 0 = 0$ has a simple root. In all cases P has a root. \diamond

3. COMMENT.

The above proof is inspired from a beautiful proof by Daniel Litt [1]. He works in the global space of monic polynomials of degree $n \geq 1$ (biholomorphic to \mathbb{C}^n), and removes the algebraic locus \mathcal{D}_n , defined by the discriminant, of polynomials with a double root. He uses that the complement of an algebraic variety in \mathbb{C}^n is connected. Essentially the proof above achieves the same goal in a more elementary way working with $n = 1$. In particular, we only need the simpler fact that the complement of a finite set in the plane is connected (which for $n = 1$ is the same as the connectedness of the complement of an algebraic variety in \mathbb{C}^n). We also avoid the use of discriminants.

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REFERENCES

- [1] LITT, D.; *Yet another proof of the Fundamental Theorem of Algebra*, Manuscript, 2011.
(www.daniellitt.com/blog/2016/10/6/a-minimal-proof-of-the-fundamental-theorem-of-algebra)

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