A SIMPLE PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

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Abstract. We present a simple short proof of the Fundamental Theorem of Algebra, without complex analysis and with a minimal use of topology.

1. Statement.

Theorem 1.1. *A non constant polynomial $P(z) \in \mathbb{C}[z]$ with complex coefficients has a root.*

The proof is based only on the following elementary facts:

- A polynomial has at most a finite number of roots.
- The Implicit Function Theorem.
- Removing from $\mathbb{C}$ a finite number of points leaves an open connected space.

2. The proof.

It is enough to consider a monic polynomial P. We denote by $\mathcal{C} = (P')^{-1}(0)$ the finite set of critical points of P, and by $\mathcal{D} = P(\mathcal{C})$ the finite set of critical values of P.

- Let $R = \{c \in \mathbb{C};$ the polynomial $P(z) - c$ has at least a simple root and no double roots$\}$.

- $R \subset \mathbb{C} - \mathcal{D}$. This is because if $c \in \mathcal{D}$, then $c = P(z_0)$ for some critical point $z_0 \in \mathcal{C}$, hence $P'(z_0) = 0$ and $P(z) - c = 0$ has a double root at z_0. Note that $\mathbb{C} - \mathcal{D}$ is open and connected ($\mathcal{D}$ being finite).

- R is open. This is an application of the Implicit Function Theorem. Let $c_0 \in R \subset \mathbb{C} - \mathcal{D}$, and $z_0 \in \mathbb{C}$ be a root of $P(z) - c_0$. We apply the Implicit Function Theorem to the equation $F(z, c) = P(z) - c = 0$. Since $\frac{\partial F}{\partial z}(z_0, c_0) = P'(z_0) \neq 0$, there is a neighborhood U of c_0 such that for $c \in U$ we have a root $z(c)$ of $P(z) - c$. Taking U small enough, by continuity of P' and $c \mapsto z(c)$, we have $P'(z(c)) \neq 0$ and the root $z(c)$ is simple. Since $\mathbb{C} - \mathcal{D}$ is open we can take $U \subset \mathbb{C} - \mathcal{D}$ and $P(z) - c$ does not have any double root, thus $U \subset R$.

- R is closed in $\mathbb{C} - \mathcal{D}$. Because P is monic, if c is uniformly bounded then any root of $P(z) - c$ is uniformly bounded (since $P(z)/z^n \to 1$ uniformly when $z \to \infty$, if n is the degree). We can take a subsequence of $c_n \to c_\infty \in \mathbb{C} - \mathcal{D}$ and a converging subsequence of roots of $P(z) - c_n$. By continuity, the limit is a root of $P(z) - c_\infty$, so this polynomial has roots. Moreover, all roots of $P(z) - c_\infty$ are simple since $c_\infty \in \mathbb{C} - \mathcal{D}$.

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• R is non-empty. For any $a \in \mathbb{C}$ we have that for $c = P(a)$, $P(z) - c$ has at least $z = a$ as root. If we choose $a \in \mathbb{C} - P^{-1}(\mathcal{D})$, then for any root z_0 of $P(z) - c$ with $c = P(a)$, we have $P(z_0) = P(a) \notin \mathcal{D}$, so $z_0 \notin P^{-1}(\mathcal{D})$, but $\mathcal{C} \subset P^{-1}(\mathcal{D})$, and $z_0 \notin \mathcal{C}$, and the root z_0 is simple.

The above proves that $R = \mathbb{C} - \mathcal{D}$. Now, if $0 \in \mathcal{D}$, then $0 = P(z_0)$ for a critical point z_0 of P that is also a root of P. If $0 \notin \mathcal{D}$, then $0 \in R = \mathbb{C} - \mathcal{D}$ and the equation $P(z) - 0 = 0$ has a simple root. In all cases P has a root. $\diamond$

3. Comment.

The above proof is inspired from a beautiful proof by Daniel Litt [1]. He works in the global space of monic polynomials of degree $n \geq 1$ (biholomorphic to $\mathbb{C}^n$), and removes the algebraic locus $\mathcal{D}_n$, defined by the discriminant, of polynomials with a double root. He uses that the complement of an algebraic variety in $\mathbb{C}^n$ is connected. Essentially the proof above achieves the same goal in a more elementary way working with $n = 1$. In particular, we only need the simpler fact that the complement of a finite set in the plane is connected (which for $n = 1$ is the same as the connectedness of the complement of an algebraic variety in $\mathbb{C}^n$). We also avoid the use of discriminants.

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References

[1] LITT, D.; *Yet another proof of the Fundamental Theorem of Algebra*, Manuscript, 2011. (www.daniellitt.com/blog/2016/10/6/a-minimal-proof-of-the-fundamental-theorem-of-algebra)

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