DIFFERENCE EQUATIONS AND OMEGA FUNCTIONS

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Abstract. We introduce Omega functions that generalize Euler Gamma functions and study the functional difference equation they satisfy. Under a natural exponential growth condition, the vector space of meromorphic solutions of the functional equation is finite dimensional. We construct a basis of the space of solutions composed by Omega functions. Omega functions are defined as exponential periods. They have a meromorphic extension to the complex plane of order 1 with simple poles at negative integers. They are characterized by their growth property on vertical strips and their functional equation. This generalizes Wielandt's characterization of Euler Gamma function. We also introduce Incomplete Omega functions that play an important role in the proofs.

1. Introduction

1.1. **Difference equations.** We study in this article difference equations of the form

$$sf(s) = \sum_{k=1}^{d} \alpha_k f(s+k) \tag{1}$$

where $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$ and $\alpha_d \neq 0$. The simplest case is the functional equation satisfied by Euler Gamma function

$$s\Gamma(s) = \Gamma(s+1)$$

These equations are linear and we have a vector space of meromorphic solutions. A natural motivation for studying these functional equations comes from the study of subspaces generated by natural linear operators. For instance, we can consider, in the space of meromorphic functions, the shift (or integer translation) linear operator

$$T(f(s)) = f(s+1)$$

and the multiplication by s linear operator

$$S(f(s)) = sf(s)$$

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Observe that S has no eigenvectors and the minimal invariant subspace invariant by S containing the constant functions is the space of polynomials $\mathbb{C}[s]$. The space generated by S and the function f is the vector space $\mathbb{C}[s]f$

$$\langle f, S(f), S^2(f), \ldots \rangle = \mathbb{C}[s]f$$

It is natural to find the functions f such that the space $\mathbb{C}[s]f$ is generated by f and T. This happens if and only if f is solution of the difference equation (1).

Already, in the simplest case of the functional equation of Euler Gamma function, the space of solutions is infinite dimensional since any function of the form $e^{2\pi i n s}\Gamma(s)$ for an integer $n \in \mathbb{Z}$ is also a solution. It is classical to add conditions to characterize Euler Gamma function as the only normalized solution to this functional equation. One can mention Weierstrass characterization imposing some asymptotic behavior when $s \to +\infty$ (1856, [22]), or Wielandt's characterization (1939, [23], see also [20], [21]) requiring boundedness on vertical strips of width larger than 1, or, more recently, requiring finite order of the solutions and a right half plane free of zeros nor poles (2022, [17]). Wielandt's boundedness condition has been weakened by Fuglede to a moderate growth in the vertical strip (2008, [13]).

In the spirit of Wielandt, we search for solutions with some growth control on vertical strips. Under a suitable growth condition, we prove that the space of solutions is finite dimensional:

Theorem 1.1. *The space of meromorphic solutions f of the functional equation*

$$sf(s) = \sum_{k=1}^{d} \alpha_k f(s+k)$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, $\alpha_d \neq 0$, and f satisfies a growth condition, for $1 \leq \operatorname{Re} s \leq d$,

$$|f(s)| \leq C e^{-c \operatorname{Im} s}$$

for some constant $C > 0$ and $0 \leq c < 2\pi$, is finite dimensional of dimension d.

Moreover, we build an explicit basis of the vector space of solutions with Special Functions, that we call Omega functions, that generalize Euler Gamma function. There is a large classical literature on linear difference equations with polynomial coefficients by Poincaré [19], Birkhoff [3], Carmichael [8], Nörlund [18], and, more recently, solutions with vertical exponential growth have been studied by Barkatou [2] and Duval [10] following work of Ramis. The analysis of the equation of this article is self-contained and independent of the classical theory.

1.2. **Omega functions.** Historically, Euler Gamma function appears for the first time in a letter from Euler to Goldbach, dated January 8th 1730 ([12]). Euler defines the Gamma function for real values $s > 0$, by the integral formula

$$\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} dt \ .$$

which is also convergent for complex values of s with $\mathrm{Re}\, s > 0$. In this integral formula, the value $\Gamma(s)$ appears as an *exponential period*.

Algebraic periods are integrals of algebraic differential forms over cycles of an algebraic variety. In the special case of an algebraic curve, when we represent the curve as a Riemann domain over the complex plane or the Riemann sphere, algebraic periods are also the integrals of algebraic differential forms on paths joining two ramification points where we have singularities of the differential form. From the transalgebraic point of view, it is natural to consider exponential periods, where integrals involve exponential expressions, and the singularities can be exponential singularities. More general periods can be envisioned where the differential form has transcendental singularities with monodromy as t^s in a local variable (geometrically these corespond to differential forms living in a branched Riemann domain with an infinite ramification). There is a vast literature on classical algebraic periods, but almost none on the transalgebraic periods. We refer to [16] for a survey about classical periods, and to [6] and [7] for exponential periods and their relation with log-Riemann surfaces. Also we refer the reader to [17] for an historical survey of different definitions of Euler Gamma function and their generalizations, and to [24] for its classical properties.

We introduce (resp. Incomplete) Omega functions which are a natural generalization of the (resp. Incomplete) Gamma function. They are defined as exponential periods of the form

$$\Omega_k(s) = \int_0^{+\infty.\omega_k} t^{s-1} e^{P_0(t)} \, dt \quad , \quad \Omega_k(s,z) = \int_0^{z} t^{s-1} e^{P_0(t)} \, dt$$

where $P_0(t) \in \mathbb{C}[t]$ and ω_k is a root of unity pointing to a direction where the polynomial P_0 diverges to $-\infty$. Some critical computations in the proof of the main Theorem generalize computations carried out for exponential periods appearing in [7]. The generalization of the Ramificant Determinant is the key result for the proof of the linear independence of the Omega functions (Ω_k). In this magical calculation, we compute a determinant of a matrix of exponential periods which are individually not computable. Omega functions appeared before in the literature under the name of "modified Gamma functions" and their asymptotic behavior at infinite was studied by N. G. De Bruijn, [5] p.119, and A. Duval [10]. We know no earlier references for Incomplete Gamma functions.

2. Definition.

Let $P_0(t) \in \mathbb{C}[t]$ be a degree $d \geq 1$ polynomial such that $P_0(0) = 0$ and $\lim_{t\to+\infty} \operatorname{Re} P_0(t) = -\infty$, normalized by

$$P_0(t) = -\frac{1}{d}t^d + \sum_{k=1}^{d-1} a_k t^k$$

We also denote $a_d = -1/d$ and $a_0 = 0$. Let ω be the primitive d-th root of unity given by

$$\omega = e^{\frac{2\pi i}{d}}$$

and write $\omega_k = \omega^k$.

Definition 2.1. *Let $d \geq 1$. For $k = 0, 1, \ldots, d-1$, the Omega functions, or Ω-functions, associated to P_0, are defined for* $\operatorname{Re} s > 0$, *by*

$$\Omega_k(s) = \int_0^{+\infty.\omega_k} t^{s-1} e^{P_0(t)}\, dt$$

The roots ω_k point to the directions where the polynomial P_0 diverges exponentially to $-\infty$,

$$\lim_{t\to+\infty.\omega_k} \operatorname{Re} P_0(t) = -\infty$$

so the integral is converging and we have a sound definition. Usually, we spare the reference to P_0, but for some results it will be crucial to keep track of the dependence on parameters and we will write

$$\Omega(s) = \Omega(s|P_0) = \Omega(s|a_1, \ldots, a_{d-1}) \ .$$

For $d = 1$, we have $P_0(t) = -t$ and $\Omega_1 = \Gamma$ is Euler Gamma function.

If $P_0 \in \mathbb{R}[t]$, then Ω_0 is real analytic, and $\overline{\Omega_k(\bar{s}|P_0)} = \Omega_{d-k}(s|\bar{P}_0)$, where $\bar{P}_0$ is the conjugate polynomial

$$\bar{P}_0(t) = -\frac{1}{d}t^d + \sum_{k=1}^{d-1} \bar{a}_k t^k \ .$$

Sometimes we will be interested in the case where $\alpha_k \in \mathbb{K}$ where $\mathbb{K} \subset \mathbb{C}$ is a number field. In that case we say that the Omega functions are defined over $\mathbb{K}$.

3. Meromorphic extension, poles and residues.

Theorem 3.1. *The Omega functions* $(\Omega_k)_{0\leq k\leq d-1}$ *extend to the complex plane into meromorphic functions of order* 1 *satisfying the fundamental functional equation*

$$\Omega_k(s+d)+\alpha_{d-1}\Omega_k(s+d-1)+\ldots+\alpha_1\Omega_k(s+1)=s\,\Omega_k(s) \tag{2}$$

where $\alpha_l=-la_l$.

Moreover, the function Ω_k *is holomorphic in* $\mathbb{C}-\mathbb{N}$, *and has simple poles at the negative integers. The residue at* $s=0$ *is*

$$Res_{s=0}\,\Omega_k=1\ .$$

Observe that for $d=1$, $\Omega_0=\Gamma$ and the functional equation (2) is the classical functional equation $\Gamma(s+1)=s\Gamma(s)$.

Proof. We have for $\mathrm{Re}\,s>0$ and by integration by parts,

$$\begin{aligned}\Omega_k(s+d)+\sum_{l=1}^{d-1}\alpha_l\Omega_k(s+l)&=\int_0^{+\infty.\omega_k}t^s(-P_0'(t)).e^{P_0(t)}\,dt\\&=\left[-t^se^{P_0(t)}\right]_0^{+\infty.\omega_k}+s\int_0^{+\infty.\omega_k}t^{s-1}e^{P_0(t)}\,dt\\&=s\,\Omega_k(s)\end{aligned}$$

and we get the functional equation (2). Now, using once the functional equation we extend meromorphically Ω_k to $\{\mathrm{Re}\,s>-1\}$, and by induction to $\{\mathrm{Re}\,s>-n\}$, for $n=1,2,\ldots$, hence to all of $\mathbb{C}$. The only poles that can be introduced by this extension procedure using the functional equation are those created from the pole at $s=0$ at the negative integers. The functional equation shows that $s\Omega_k(s)$ is holomorphic at $s=0$, hence the pole at $s=0$ is simple. It follows from the functional equation and the extension procedure that the other poles are also simple. We compute the residue at $s=0$ using the functional equation,

$$\mathrm{Res}_{s=0}\,\Omega_k=\lim_{s\to 0}s\Omega_k(s)=\sum_{l=1}^{d}\alpha_l\Omega_k(l)=\int_0^{+\infty.\omega_k}(-P_0'(t)).e^{P_0(t)}\,dt=\left[-e^{P_0(t)}\right]_0^{+\infty.\omega_k}=1$$

□

More generally, we can compute the residues at the negative integers.

Theorem 3.2. *Let* $(\lambda_n)_{n\geq 0}$ *be the coefficients of the power series expansion of* $e^{P_0(t)}$,

$$e^{P_0(t)}=\sum_{n=0}^{+\infty}\lambda_nt^n\ .$$

Then the residue of Ω_k *at* $s=-n$ *is* λ_n,

$$Res_{s=-n}\,\Omega_k = \lambda_n \ .$$

Proof. For $n\geq 0$ let $r_n\in\mathbb{C}$ be the residue of Ω_k at $s=-n$, with $r_n=0$ if there is no pole, and $r_n=0$ for $n<0$. The functional equation (2) gives

$$r_n=\lim_{h\to 0} h\Omega_k(h-n)=\lim_{h\to 0}\frac{1}{h-n}\sum_{l=1}^{d}\alpha_l h\Omega_k(h-n+l)=-\frac{1}{n}\sum_{l=1}^{d}\alpha_l\, r_{n-l}$$

hence the recurrence relation

$$nr_n=-\sum_{l=1}^{d}\alpha_l\, r_{n-l} \tag{3}$$

Now, consider the generating power series

$$F(t)=\sum_{n=0}^{+\infty} r_n t^n$$

The recurrence relation (3) gives

$$\begin{aligned}
F'(t)=\sum_{n=0}^{+\infty} nr_n t^{n-1} &= -\sum_{l=1}^{d}\alpha_l\sum_{n=0}^{+\infty} r_{n-l}t^{n-1}\\
&= -\sum_{l=1}^{d}\alpha_l t^{l-1}\sum_{n=l}^{+\infty} r_{n-l}t^{n-l}\\
&= -\left(\sum_{l=1}^{d}\alpha_l t^{l-1}\right)F(t)\\
&= P_0'(t)F(t)
\end{aligned}$$

Since we have $F(0)=r_0=1$ from Theorem 3.1, we get

$$F(t)=e^{P_0(t)}$$

thus $r_n=\lambda_n$ as claimed. □

Example.

For $d=1$, the generating power series is

$$F(t)=e^{-t}=\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}\,t^n$$

and we recover the classical result that

$$\mathrm{Res}_{s=-n}\,\Gamma = \frac{(-1)^n}{n!} \ .$$

Note that the residues at the simple poles at the negative integers are the same for all the functions $\Omega_0, \ldots, \Omega_{d-1}$. Indeed, for $k \neq l$, we can check directly that $\Omega_k - \Omega_l$ is an entire function because of the convergence for all $s \in \mathbb{C}$ of the integral

$$\Omega_k(s) - \Omega_l(s) = \int_{+\infty.\omega_l}^{+\infty.\omega_k} t^{s-1} e^{P_0(t)}\, dt = \int_{\gamma_{lk}} t^{s-1} e^{P_0(t)}\, dt$$

where the integral can be taken over any path γ_{lk} assymptotic to $+\infty.\omega_l$ and $+\infty.\omega_k$ in the proper direction and with 0 winding number around 0. This integral depends holomorphically on the parameter $s \in \mathbb{C}$.

Observe also that if the coefficients of P_0 belong to a number field $\mathbb{K}$, $P_0(t) \in \mathbb{K}[t]$, then the residues of Ω_k belong also to $\mathbb{K}$. Another arithmetical observation is the following:

Corollary 3.3. *We assume that the only non-zero coefficients of P_0 are for powers divisible by an integer $n_0 \geq 2$, that is, if $a_k \neq 0$ then $n_0 | k$.*

Then, if n_0 does not divide n, we have $r_n = 0$.

Proof. From the previous Theorem we have

$$e^{P_0(t)} = \prod_{k=1}^{d} e^{a_k t^k} = \prod_{k=1}^{d} \left(\sum_{m \geq 0} \frac{a_k^m}{m!} t^{mk} \right)$$

and when we expand the last product we get the result. □

We have a more precise result than just the computation of the residues. We can determine the Mittag-Leffler decomposition of Ω_k.This is an analytic result that requires some estimates.

Theorem 3.4. *The Omega function Ω_k has the Mittag-Leffler decomposition:*

$$\Omega_k(s) = \sum_{n=0}^{+\infty} \frac{\lambda_n}{s+n} + \int_1^{+\infty.\omega_k} t^{s-1} e^{P_0(t)}\, dt$$

where the integral is an entire function of order 1.

Observe that this Theorem shows that the Omega function Ω_k is a meromorphic function of order 1.

Corollary 3.5. *The Omega functions Ω_k are meromorphic functions of order* 1.

Proof. We write

$$\Omega_k(s) = \int_0^1 t^{s-1} e^{P_0(t)}\, dt + \int_1^{+\infty.\omega_k} t^{s-1} e^{P_0(t)}\, dt$$

and we compute the first integral expanding the exponential in power series (uniformly convergent in $[0,1]$),

$$\int_0^1 t^{s-1} e^{P_0(t)}\, dt = \sum_{n=0}^{+\infty} \lambda_n \int_0^1 t^{s+n-1}\, dt = \sum_{n=0}^{+\infty} \lambda_n \left[\frac{t^{s+n}}{s+n}\right]_0^1 = \sum_{n=0}^{+\infty} \frac{\lambda_n}{s+n}$$

The second integral can be bounded by the next Lemma that shows that it is an entire function of order 1 (using that Euler Gamma function is of order 1). □

Lemma 3.6. *We have the estimate*

$$\left|\int_1^{+\infty.\omega_k} t^{s-1} e^{P_0(t)}\, dt\right| \leq e^{-2\pi \frac{k}{d} \operatorname{Im} s} \left(C_0 + C_1 d^{\operatorname{Re} s/d}\, \Gamma\left(\frac{\operatorname{Re} s}{d}\right)\right)$$

Proof. We make the change of variables $t = \omega_k u$

$$\begin{aligned}\int_1^{+\infty.\omega_k} t^{s-1} e^{P_0(t)}\, dt &= \omega_k^s \int_{\omega_k^{-1}}^{+\infty} u^{s-1} e^{-\frac{1}{d}u^d(1+\mathcal{O}(u^{-1}))}\, du \\ &= e^{2\pi i \frac{k}{d} s} \int_{\omega_k^{-1}}^{+\infty} u^{s-1} e^{-\frac{1}{d}u^d(1+\mathcal{O}(u^{-1}))}\, du\end{aligned}$$

This gives the bound

$$\left|\int_1^{+\infty.\omega_k} t^{s-1} e^{P_0(t)}\, dt\right| \leq e^{-2\pi \frac{k}{d} \operatorname{Im} s} \left|\int_{\omega_k^{-1}}^{+\infty} u^{s-1} e^{-\frac{1}{d}u^d(1+\mathcal{O}(u^{-1}))}\, du\right|$$

Now, taking an integration path of finite length from 1 to ω_k and bounded away from 0, we get (using the same letter C to denote several universal constants $C > 0$)

$$\left|\int_1^{+\infty.\omega_k} t^{s-1} e^{P_0(t)}\, dt\right| \leq e^{-2\pi \frac{k}{d} \operatorname{Im} s} \left(C + \left|\int_1^{+\infty} u^{s-1} e^{-\frac{1}{d}u^d(1+\mathcal{O}(u^{-1}))}\, du\right|\right)$$

The last integral can be estimated by

$$\begin{aligned}\left|\int_1^{+\infty} u^{s-1} e^{-\frac{1}{d}u^d(1+\mathcal{O}(u^{-1}))}\, du\right| &\leq (1+C) \int_1^{+\infty} u^{s-1} e^{-\frac{1}{d}u^d}\, du \\ &\leq (1+C) \int_0^{+\infty} u^{s-1} e^{-\frac{1}{d}u^d}\, du \\ &\leq (1+C) d^{\operatorname{Re} s/d}\, \Gamma\left(\frac{\operatorname{Re} s}{d}\right)\end{aligned}$$

(for the computation of the last integral we use the change of variable $v = u^d/d$). □

4. Incomplete Omega functions.

We define the Incomplete Omega functions that generalize the Incomplete Gamma function.

Definition 4.1. *For $z, s \in \mathbb{C}$, $\operatorname{Re} s > 0$, the Incomplete Omega function $\Omega(s, z)$ is defined by*

$$\Omega(s, z) = \int_0^z t^{s-1} e^{P_0(t)} \, dt$$

For $P_0(t) = -t$ this is the classical Incomplete Gamma function. Observe that we recover all the (Ω_k) functions by taking the appropriate limit of $\Omega(s, z)$ as $z \to \infty$,

$$\Omega_k(s) = \lim_{z \to +\infty . \omega_k} \Omega(s, z)$$

For the particular values of $s = 1, 2, \ldots, d-1$ these are the transcendental entire functions of the variable $z \in \mathbb{C}$ studied in [7] which form a basis of the fundamental for the transcendental vector space of functions on a simply connected log-Riemann surface with exactly d infinite ramification points. Some of the results proved here generalize some results from [7]. Following the same Abel's philosophy that inspires [7], we prove that we only need to use a finite number of transcendentals $(\Omega(s + k, z))_{0 \leq k \leq d-1}$ to compute integrals of the form

$$\int_0^z Q(t, t^s) e^{P_0(t)} \, dt$$

where $Q(x, y) \in \mathbb{C}[x, y]$ is a polynomial. We start by studying the simpler case when $Q(t) \in \mathbb{C}[t]$.

Proposition 4.2. *Let $Q(t) \in \mathbb{C}[t]$. For $d \geq 2$, the integral*

$$\int_0^z t^s Q(t) e^{P_0(t)} \, dt$$

is of the form

$$\int_0^z t^s Q(t) e^{P_0(t)} \, dt = z^s A(s, z) \, e^{P_0(z)} + \sum_{k=0}^{d-1} c_k(s) \, \Omega(s + k, z)$$

where $A \in \mathbb{C}[s, z]$, and the polynomial coefficients $c_k(s) \in \mathbb{C}[s]$ have coefficients depending polynomially on the coefficients of P_0, $(a_1, \ldots, a_{d-1})$.

Proof. First we consider the case $d=1$. We prove the result for $Q(t)=t^n$ integrating by parts $n+1$ times

$$\int_0^z t^{s+n}e^{-t}\,dt = \left[-t^{s+n}e^{-t}\right]_0^z + (s+n-1)\int_0^z t^{s+n-1}e^{-t}\,dt$$
$$= -z^{s+n}e^{-z} + (s+n)\int_0^z t^{s+n-1}e^{-t}\,dt$$
$$= -(z^{s+n}+(s+n)z^{s+n-1})e^{-z} + (s+n)(s+n-1)\int_0^z t^{s+n-2}e^{-t}\,dt$$
$$\vdots$$
$$= -z^s(z^n+(s+n)z^{n-1}+\ldots)e^{-z} + (s+n)(s+n-1)\ldots s\,\Omega(s,z)$$

For a general polynomial $Q(t)$ we have the the result by linear decomposition of the integral.

In the rest of the proof we assume $d\geq 2$. If $q=\deg Q\leq d-2$, by linearity, the integral is a linear combination (with the coefficients of $tQ(t)$) of $(\Omega(s+k,z))_{0\leq k\leq d-1}$ and the result follows.

If $\deg Q\geq d-1$, then we consider the Euclidean division of $Q(t)$ by $P_0'(t)$,

$$Q(t)=A_1(t)P_0'(t)+B_1(t)$$

with $A_1,B_1\in\mathbb{C}[t]$, $\deg B_1\leq d-2$ and $\deg A_1=\deg Q-(d-1)=q-(d-1)\leq q-1$. We proceed splitting the integral:

$$\int_0^z t^sQ(t)e^{P_0(t)}\,dt = \int_0^z t^sA_1(t)P_0'(t)e^{P_0(t)}\,dt + \int_0^z t^sB_1(t)e^{P_0(t)}\,dt$$

Since $\deg B_1\leq d-2$, the second integral is a linear combination of $\Omega(s,z),\Omega(s+1,z),\ldots,\Omega(s+d-1,z)$, thus of the desired form, and we can forget about it. We work on the first integral integrating by parts,

$$\int_0^z t^sA_1(t)P_0'(t)e^{P_0(t)}\,dt = \left[t^sA_1(t)e^{P_0(t)}\right]_0^z - \int_0^z (t^sA_1(t))'\,e^{P_0(t)}\,dt$$

Then we get:

$$\int_0^z t^sA_1(t)P_0'(t)e^{P_0(t)}\,dt = z^sA_1(z)e^{P_0(z)} - \int_0^z t^sA_1'(t)e^{P_0(t)}\,dt - s\int_0^z t^{s-1}A_1(t)e^{P_0(t)}\,dt$$

The first integral in the right hand side is of the same form as the initial one with $Q(t)$ but with $\deg A_1'=\deg A_1-1\leq\deg Q-(d-1)-1=q-d\leq q-2$ (using here $d\geq 2$), hence by descending induction we can forget about it. For the second integral, we can write

$$s\int_0^z t^{s-1}A_1(t)e^{P_0(t)}\,dt = A_1(0)s\,\Omega(s,z) + s\int_0^z t^s\left(\frac{A_1(t)-A_1(0)}{t}\right)e^{P_0(t)}\,dt$$

and $t^{-1}(A_1(t) - A_1(0))$ is a polynomial of degree $\deg A_1 - 1 \leq q - 2$. Then the descending induction gives the expression for the integral as announced.

The coefficients of P_0 appear first linearly in the Euclidean divisions by P_0' then, by repeated Euclidean divisions the dependence of the $c_k(s)$ is polynomial on the coefficients of P_0. □

Now, we can get easily the general result for $Q(t, t^s)$.

Corollary 4.3. *Let $Q(t, t^s) \in \mathbb{C}[t, t^s]$. For $d \geq 2$, the integral*

$$\int_0^z Q(t, t^s) e^{P_0(t)} \, dt$$

is of the form

$$\int_0^z t^{s-1} Q(t, t^s) e^{P_0(t)} \, dt = A(s, z, z^s) e^{P_0(z)} + \sum_{k=0}^{d-1} c_k(s) \Omega(s + k, z)$$

where $A \in \mathbb{C}[s, z, z^z]$ and the coefficients $c_k(s) \in \mathbb{C}[s]$ have coefficients that are polynomials on the coefficients of P_0 $(a_1, \ldots, a_{d-1})$.

Proof. We write $Q(t, t^s) = Q(t)(t^s)$ as a polynomial on the variable t^s with coefficients polynomials in t, and we split linearly the integral, observing that the part corresponding to each monomial is like the integral when Q is a polynomial in t as before, but with s shifted into $s + l$ by some positive integer l. The result follows from the previous Proposition. □

Now, we can prove that the Omega functions $(\Omega_k(s))_{0 \leq k \leq d-1}$ generate a large class of exponential periods:

Corollary 4.4. *Let $Q(t, t^s) \in \mathbb{C}[t, t^s]$ and $0 \leq n \leq d - 1$. The exponential period*

$$\int_0^{+\infty . \omega_n} Q(t, t^s) \, e^{P_0(t)} \, dt$$

is a linear combination of the exponential periods $(\Omega_k(s))_{0 \leq k \leq d-1}$

$$\int_0^{+\infty . \omega_n} Q(t, t^s) \, e^{P_0(t)} \, dt = \sum_{k=0}^{d-1} c_k(s) \, \Omega_k(s)$$

where the coefficients $c_k(s)$ are polynomials on s and on the coefficients $(a_1, \ldots, a_{d-1})$.

Proof. For $d \geq 2$, we have from the previous Proposition that

$$\int_0^z Q(t, t^s) e^{P_0(t)} \, dt = A(s, z, z^s) e^{P_0(z)} + \sum_{k=0}^{d-1} c_k(s) \Omega(s + k, z)$$

When $z \to +\infty.\omega_n$ the terms in the first sum vanish, since $A(s,z,z^s)e^{P_0(z)} \to 0$ for a polynomial $A(s,z,z^s) \in \mathbb{C}[z]$ (the exponential decay of $e^{P_0(z)}$ takes over the polynomial divergence of $A(z)$), and we get

$$\int_0^{+\infty.\omega_n} Q(t,t^s)e^{P_0(t)}\,dt = \sum_{k=0}^{d-1} c_k(s)\Omega_n(s+k)$$

□

5. Linear independence.

The row vector build with Omega functions $\mathbf{\Omega}(s) = (\Omega_k(s))_{0\leq k\leq d-1}$ has the following important linear independence property:

Theorem 5.1. *For any $s \in \mathbb{C} - \mathbb{N}_-^*$, the vectors $\mathbf{\Omega}(s+1), \mathbf{\Omega}(s+2), \ldots, \mathbf{\Omega}(s+d)$ are linearly independent,*

$$\Delta(s) \neq 0$$

where

$$\Delta(s|a_1,\ldots,a_{d-1}) = \det \begin{bmatrix} \Omega_{11} & \Omega_{12} & \ldots & \Omega_{1d} \\ \Omega_{21} & \Omega_{22} & \ldots & \Omega_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{d1} & \Omega_{d2} & \ldots & \Omega_{dd} \end{bmatrix} .$$

where $\Omega_{kl} = \Omega_{k-1}(s+l)$.

More precisely, we can compute

$$\Delta(s|a_1,\ldots,a_{d-1}) = \Delta(s|0,\ldots,0)\exp\left(\Pi_d(s,a_1,\ldots,a_{d-1})\right)$$

where $\Pi_d(s,a_1,\ldots,a_{d-1})$ is a universal polynomial with rational coefficients.

In view of the last formula, the result follows from $\Delta(s|0,\ldots,0) \neq 0$. We will prove the last formula and compute explicitly the determinant $\Delta(s|0,\ldots,0)$. These computations are similar to the ones for the Ramificant Determinant (see [7]) that corresponds to the special case $s = 0$.

We can compute $\Omega_k(s+l|0,\ldots,0)$ using Euler Gamma function.

Lemma 5.2. *We have*

$$\Omega_k(s+l|0,\ldots,0) = \omega^{k(s+l)} d^{\frac{s+l}{d}-1}\Gamma\left(\frac{s+l}{d}\right)$$

Proof. We first make the change of variables $t = \omega^k u$, and then $v = u^d/d$,

$$\begin{aligned}
\Omega_k(s+l|0,\ldots,0) &= \int_0^{+\infty.\omega^k} t^{s+l-1} e^{-\frac{1}{d}t^d}\, dt \\
&= \omega^{k(s+l)} \int_0^{+\infty} u^{s+l-1} e^{-\frac{1}{d}u^d}\, du \\
&= \omega^{k(s+l)} d^{\frac{s+l}{d}-1} \int_0^{+\infty} v^{\frac{s+l}{d}-1} e^{-v}\, dv \\
&= \omega^{k(s+l)} d^{\frac{s+l}{d}-1} \Gamma\left(\frac{s+l}{d}\right)
\end{aligned}$$

□

Now we recall the following well known elementary Vandermonde Lemma:

Lemma 5.3. *If $\xi_1, \ldots, \xi_d$ are the d roots of a monic polynomial $Q(X)$, then we can compute the Vandermonde determinant $V(\xi_1, \ldots, \xi_d)$ of the $(\xi_1, \ldots, \xi_d)$ as*

$$V(\xi_1,\ldots,\xi_d) = \begin{vmatrix} 1 & \xi_1 & \xi_1^2 & \ldots & \xi_1^{d-1} \\ 1 & \xi_2 & \xi_2^2 & \ldots & \xi_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \xi_d & \xi_d^2 & \ldots & \xi_d^{d-1} \end{vmatrix} = \prod_{i\neq j}(\xi_i - \xi_j) = \prod_{i=1}^{d} Q'(\xi_i) \ .$$

Using this Lemma with $Q(X) = X^d - 1$ we compute the Vandermonde determinant:

$$V_d = \begin{vmatrix} 1 & \omega_1 & \omega_1^2 & \ldots & \omega_1^{d-1} \\ 1 & \omega_2 & \omega_2^2 & \ldots & \omega_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_d & \omega_d^2 & \ldots & \omega_d^{d-1} \end{vmatrix} = \prod_{i\neq j}(\omega_i - \omega_j) = \prod_i (d\omega_i^{d-1}) = d^d \left(\prod_i \omega_i\right)^{d-1} = (-1)^{d-1} d^d \ .$$

We use this result to compute $\Delta(s|0,\ldots,0)$.

Lemma 5.4. *We have*

$$\Delta(s|0,\ldots,0) = \frac{(2\pi d)^{\frac{d}{2}}}{\sqrt{2\pi}} \omega^{\frac{d(d-1)}{2}s} s\,\Gamma(s)$$

and in particular $\Delta(s|0,\ldots,0) \neq 0$ for $s \neq -1, -2, \ldots$.

Taking the limit $s \to 0$ we recover the formula from Lemma 3.5 from [7].

Proof. Using Lemma 5.2 we have

$$\Delta(s|0,\ldots,0) =$$

$$= \begin{vmatrix} \omega^{0.(s+1)} d^{\frac{s+1}{d}-1}\Gamma\left(\frac{s+1}{d}\right) & \omega^{0.(s+2)} d^{\frac{s+2}{d}-1}\Gamma\left(\frac{s+2}{d}\right) & \ldots & \omega^{0.(s+d)} d^{\frac{s+d}{d}-1}\Gamma\left(\frac{s+d}{d}\right) \\ \omega^{1.(s+1)} d^{\frac{s+1}{d}-1}\Gamma\left(\frac{s+1}{d}\right) & \omega^{1.(s+2)} d^{\frac{s+2}{d}-1}\Gamma\left(\frac{s+d}{d}\right) & \ldots & \omega^{1.(s+d)} d^{\frac{s+d}{d}-1}\Gamma\left(\frac{s+d}{d}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \omega^{(d-1).(s+1)} d^{\frac{s+1}{d}-1}\Gamma\left(\frac{s+1}{d}\right) & \omega^{(d-1).(s+2)} d^{\frac{s+2}{d}-1}\Gamma\left(\frac{s+2}{d}\right) & \ldots & \omega^{(d-1).(s+d)} d^{\frac{s+d}{d}-1}\Gamma\left(\frac{s+d}{d}\right) \end{vmatrix}$$

$$= \omega^{\frac{d(d-1)}{2}s} d^s d^{\frac{d-1}{2}-d}\Gamma\left(\frac{s}{d}+\frac{1}{d}\right)\Gamma\left(\frac{s}{d}+\frac{2}{d}\right)\ldots\Gamma\left(\frac{s}{d}+\frac{d}{d}\right)\begin{vmatrix} \omega_0^1 & \omega_0^2 & \ldots & \omega_0^d \\ \omega_1^1 & \omega_1^2 & \ldots & \omega_1^d \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{d-1}^1 & \omega_{d-1}^2 & \ldots & \omega_{d-1}^d \end{vmatrix}$$

$$= \omega^{\frac{d(d-1)}{2}s} d^{-\frac{d}{2}} s (2\pi)^{\frac{d-1}{2}}\Gamma(s) d^d$$

$$= d^{\frac{d}{2}}\,(2\pi)^{\frac{d-1}{2}}\; s\omega^{\frac{d(d-1)}{2}s}\,\Gamma(s)$$

$$= \frac{(2\pi d)^{\frac{d}{2}}}{\sqrt{2\pi}}\omega^{\frac{d(d-1)}{2}s} s\,\Gamma(s)$$

where we have used Gauss multiplication formula in the second line (that is in fact due to Euler and not to Gauss, see [1]) with $z = s/d$,

$$\Gamma(z).\Gamma\left(z+\frac{1}{d}\right)\ldots\Gamma\left(z+\frac{d-1}{d}\right) = (2\pi)^{\frac{d-1}{2}} d^{\frac{1}{2}-dz}\Gamma(dz)\ .$$

and that the determinant in the fourth line is equal to $(-1)^{d-1}V_d$ where V_d is the Vandermonde determinant computed previously.

□

Proof of Theorem 5.1. Consider the entire function of several complex variables $\Delta(s|a_1,a_2,\ldots,a_{d-1})$ on the variables $(a_1,a_2,\ldots,a_{d-1})$. Observe that Corollary 4.4 proves that each integral

$$\int_0^{+\infty.\omega_k} t^{s+n-1}e^{P_0(t)}\ dt\ ,$$

is a linear combination with coefficients that are polynomial on s and the (a_k) of the integrals $\Omega_k(s)$ for $k=0,1,\ldots,d-1$, Therefore, differentiating column by column, we observe that for each $k=0,1,\ldots,d-1$, we have

$$\partial_{a_k}\Delta = Q_k\,\Delta$$

where Q_k is a polynomial on s and the (a_k). We conclude that the logarithmic derivative of Δ with respect to each variable a_k is a universal polynomial on the variables s and (a_k). This gives the existence of the universal polynomial Υ_d such that

$$\Delta(s|a_1, a_2, \dots, a_{d-1}) = c(s).e^{\Upsilon_d(s;a_1,a_2,\dots,a_{d-1})} ,$$

with $c(s).e^{\Upsilon_d(s;0,0,\dots,0)} = \Delta(s|0,\dots,0) \in \mathbb{C}$. Then if we define $\Pi_d(s, a_1, \dots, a_{d-1}) = \Upsilon_d(s; a_1, a_2, \dots, a_{d-1}) - \Upsilon_d(s; 0, 0, \dots, 0)$ we get the result

$$\Delta(s|a_1, a_2, \dots, a_{d-1}) = \Delta(s|0,\dots,0)e^{\Pi_d(s,a_1,\dots,a_{d-1})}$$

□

Corollary 5.5. *The functions $\Omega_0, \dots, \Omega_{d-1}$ do not have a common zero in $\mathbb{C} - \mathbb{N}_-$.*

Proof. Otherwise, if $s_0 \in \mathbb{C} - \mathbb{N}_-$ is a common zero, then $s_0 + 1 \in \mathbb{C} - \mathbb{N}_-^*$ and the functional equation shows that the non-zero vector $(1, \alpha_{d-1}, \dots, \alpha_1)$ is in the kernel of the matrix $[\Omega_{kl}(s_0 + 1)]$, which contradicts that it has non-zero determinant by Theorem 5.1. □

Observe that this simultaneously non-vanishing result relies on the fact that Euler Gamma function has no zeros. This is something that was explained to be a "mini-Riemann hypothesis" in [17], and was the subject of correspondence between Hermite and Stieltjes [15]. Although used in the proof, the non-vanishing of Euler Gamma function is a particular case of this general result for Omega functions.

Corollary 5.6. *The functions $\Omega_0, \dots, \Omega_{d-1}$ are $\mathbb{C}$-linearly independent.*

Proof. Otherwise there will be a non-trivial null linear combination of the rows of the matrix $[\Omega_{kl}]$ and the determinant will be zero. □

6. Solutions of the functional equation.

Observe that the functional equation (1) reduces to the functional equation (2) by dividing the equation by α_d that is assumed to be non-zero. We can make a first observation that the space of solutions of the functional equation (2) is an infinite dimensional vector space.

Proposition 6.1. *The space of meromorphic solutions f of the functional equation*

$$f(s + d) + \alpha_{d-1}f(s + d - 1) + \ldots + \alpha_1 f(s + 1) = s\, f(s) \tag{4}$$

is an infinite dimensional vector space.

Proof. The functional equation is linear and there are non-zero solutions (the Ω_k functions). Given a non-zero meromorphic solution $f(s)$, we can construct an infinite number of linear independent solutions

$$g(s) = e^{2\pi i n s} f(s)$$

where $n \in \mathbb{Z}$ is any integer. $\square$

If we restrict to solutions with a controlled growth, the situation the space of solutions is finite dimensional.

Definition 6.2. *We consider the $\mathbb{C}$-vector space $\mathbb{V}$ of meromorphic functions f satisfying the functional equation (2) and the estimate in the vertical strip $S_{1,d} = \{1 \leq \operatorname{Re} s \leq d\}$, for $s \in S_{1,d}$,*

$$|f(s)| \leq C e^{-c \operatorname{Im} s} \tag{5}$$

for some constant $0 \leq c < 2\pi$.

It is clear that the space $\mathbb{V}$ is a subspace of the vector space of general solutions (without a prescribed growth condition). We prove first that $\mathbb{V}$ is non-empty by proving the estimates for the functions Ω_k for $k = 0, 1, \ldots d-1$.

Proposition 6.3. *For $k = 0, 1, \ldots, d-1$, for any strip $S_{a,b} = \{a \leq \operatorname{Re} s \leq b\}$ with $0 < a < b$, there exists a constant $C = C(a, b, P_0) > 0$, depending only on $a, b > 0$ and the polynomial P_0, such that for $s \in S_{a,b}$, we have*

$$|\Omega_k(s)| \leq C e^{-\frac{2\pi k}{d} \operatorname{Im} s}$$

Obviously we can take $a = 1$ and $b = d$ and since $0 \leq c = \frac{2\pi k}{d} < 2\pi$ we get that Ω_k satisfies the estimate (5).

Proof. We make the change of variables $t = \omega_k u$

$$\begin{aligned}
\Omega_k(s) &= \int_0^{+\infty.\omega_k} t^{s-1} e^{P_0(t)} \, dt \\
&= \omega_k^s \int_0^{+\infty} u^{s-1} e^{-\frac{1}{d} u^d (1 + \mathcal{O}(u^{-1}))} \, du \\
&= e^{2\pi i \frac{k}{d} s} \int_0^{+\infty} u^{s-1} e^{-\frac{1}{d} u^d (1 + \mathcal{O}(u^{-1}))} \, du
\end{aligned}$$

so, we get for $0 < a \leq \operatorname{Re} s \leq b$

$$\begin{aligned} |\Omega_k(s)| &\leq e^{-2\pi\frac{k}{d}\operatorname{Im} s}(1+C_1)\int_0^{+\infty} u^{\operatorname{Re} s-1} e^{-\frac{1}{d}u^d}\, du \\ &\leq e^{-2\pi\frac{k}{d}\operatorname{Im} s}(1+C_1) d^{\operatorname{Re} s/d}\Gamma\left(\frac{\operatorname{Re} s}{d}\right) \\ &\leq C e^{-\frac{2\pi k}{d}\operatorname{Im} s} \end{aligned}$$

where $C, C_1 > 0$ are constants depending only on $a, b > 0$ and P_0. □

The growth condition on the strip $S(1,d)$ and the functional equation implies a control of f in the halph plane $\{\operatorname{Re} s \geq 1\}$. More precisely, we have

Proposition 6.4. *Let $f \in \mathbb{V}$. Then f there exists constants $C_0, \tau > 0$ such that for* $\operatorname{Re} s \geq 1$

$$|f(s)| \leq C_0 e^{\tau |s|}\, \Gamma\left(\frac{\operatorname{Re} s}{d}\right)$$

Proof. More precisely, we prove that for $\operatorname{Re} s \geq 1$

$$|f(s)| \leq C_0 \left| d^{2\frac{s}{d}} \Gamma(s/d) \right| \leq C_0 d^{2\frac{\operatorname{Re} s}{d}}\, \Gamma\left(\frac{\operatorname{Re} s}{d}\right)$$

The functional equation gives, for $\operatorname{Re} s \geq s_0$, for $s_0 > 0$ large enough,

$$|f(s+d)| \leq d\, |s f(s)|$$

We can take the constant $C_0 > 0$ large enough to have, for $\operatorname{Re} s \leq s_0$,

$$f(s)| \leq C_0 \left| d^{2\frac{s}{d}} \Gamma(s/d) \right|$$

then

$$\begin{aligned} |f(s+d)| &\leq d|s| C_0 \left| d^{2\frac{s}{d}} \Gamma(s/d) \right| = C_0 \left| d^{2\frac{s+d}{d}}\, \frac{s}{d}\, \Gamma(s/d) \right| \\ &\leq C_0 \left| d^{2\frac{s+d}{d}} \Gamma\left(\frac{s+d}{d}\right) \right| \end{aligned}$$

Thus the estimate holds for $\operatorname{Re} s \geq s_0 + d$ and by induction for $\operatorname{Re} s \geq s_o + kd$ for all $k \geq 1$. □

Now we prove the main Theorem:

Theorem 6.5. *The space of solutions $\mathbb{V}$ is a finite dimensional vector space generated by the basis $(\Omega_k)_{0 \leq k \leq d-1}$.*

We recall Carlson's Theorem [8]:

Theorem 6.6 (Carlson, 1914)**.** *Let $\mathbb{C}_+ = \{s \in \mathbb{C}; \operatorname{Re} s > 0\}$ and $f : \mathbb{C}_+ \to \mathbb{C}$ be a holomorphic function extending continuously to $\overline{\mathbb{C}_+}$. We assume that f is of exponential type, that is, there is $C, \tau > 0$ such that for all $s \in \mathbb{C}_+$,*

$$|f(s)| \leq Ce^{\tau |s|}$$

We assume that on the imaginary axes we have a more precise control, for $y \in \mathbb{R}$,

$$|f(iy)| \leq Ce^{c|y|}$$

for some constant $c < \pi$.

If $f(n) = 0$ for all $n \in \mathbb{N}$, then f is identically 0.

We use Carlson's Theorem in the half plane $\{\operatorname{Re} s > 1\}$ to prove the main Theorem.

Proof. We consider a meromorphic solution $f(s)$ of the functional equation and satisfying the estimate (5). The matrix $[\Omega_{kl}(1)]$ being invertible, we have a linear combination $g(s) = c_0\Omega_0(s) + \ldots + c_{d-1}\Omega_{d-1}(s)$ with $c_0, \ldots, c_{d-1} \in \mathbb{C}$ such that $g(l) = f(l)$ for $l = 1, 2, \ldots, d$. Since g satisfies also the functional equation, we get by induction using the functional equation that f and g take the same values at all the positive integers $s \in \mathbb{N}^*$. So the function $f - g$ vanish at all positive integers and satisfies the estimate (5). Therefore, the function $h(s) = e^{-i\pi s}(f(s) - g(s))/\Gamma(s/d)$ satisfies on $\operatorname{Re} s = 1$ (recall that Γ is bounded on vertical strips),

$$|h(s)| \leq Ce^{-(c-\pi) \operatorname{Im} s}$$

with $0 \leq c < 2\pi$. Therefore we have on $\operatorname{Re} s = 1$

$$|h(s)| \leq Ce^{c' |\operatorname{Im} s|}$$

with $0 \leq c' < \pi$.

Also using Proposition 6.4 the function h has exponential growth in the right half plane $\{\operatorname{Re} s \geq 1\}$. Therefore using Carlson's Theorem we conclude that h is identically 0, thus $f(s) = g(s)$ for all values s in this half plane, hence in $\mathbb{C}$. □

We have proved that the vector space generated by Omega functions can be characterized by the functional equation (2) and the growth property (5). This generalizes to Omega functions Wielandt's characterization for Euler Gamma function (1939, [23], [20], [21]).

We also observe that Omega functions provide the general solutions of the functional equation (2) with estimates (5) since given such a functional equations with coefficients (α_l) we can build the coefficients $a_l = -l^{-1}\alpha_l$, then the polynomial P_0 and the Omega functions $(\Omega_k)_{0 \leq k \leq d-1}$ that form a basis for the space of solutions.

It is also easy to see that we can replace the (5) by an estimate of the form, for $s \in S(1, b)$,

$$|f(s)| \leq C e^{-c \operatorname{Im} s}$$

with $2\pi n \leq c < 2\pi(n+1)$ for an integer $n \in \mathbb{Z}$. Then the space of solutions is also finite dimensional as the map $f(s) \mapsto e^{-2\pi i n s} f(s)$ provides an isomorphism of the space of solutions with $\mathbb{V}$.

The structure of the space of solutions is interesting. The space of holomorphic solutions is a subspace of dimension $d-1$.

Proposition 6.7. *The subspace of holomorphic solutions in* $\mathbb{V}$ *is a subspace of dimension* $d-1$ *generated by the entire functions*

$$\Omega_l(s) - \Omega_0(s) = \int_{\gamma_{0l}} t^{s-1} e^{P_0(t)} \, dt$$

Proof. As observed before, the functions $\Omega_l(s) - \Omega_0(s)$ are entire functions and are linearly independent. □

Some of the results in [7] can be generalized. In particular the Integrability criterion and Abel-like Theorem (Theorems 4.2 and 4.3). This will be studied in a separate article. K. Biswas has extended results from [7] to curves of higher genus [4]. It is interesting to speculate on the extension of the results for Omega functions in higher genus.

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