

ON PARETO THEORY OF CIRCULATION OF ELITES

RICARDO PÉREZ MARCO

ABSTRACT. We prove that Pareto theory of circulation of elites results from our wealth evolution model, Kelly criterion for optimal betting and Keynes' observation of "animal spirits" that drive the economy and cause that human financial decisions are prone to excess risk-taking.

1. INTRODUCTION.

At the end of the XIXth century, Vilfredo Pareto in his economic and social studies made some fundamental discoveries. Based on his encyclopedic economic knowledge, his most famous observation is the universality and stability of the wealth distribution in different countries (Pareto law). In [9] we proposed a simple model of wealth evolution that implies Pareto distribution. Another observation by Pareto did set the foundation and motivation for his Theory of Elites. He observed that elites in different societies along history do renew systematically. This is called Pareto's "Principle of Circulation of Elites". He formulated a more elaborate theory of elites, but stated as an open problem the reason for the circulation of elites. This is what he wrote about the distribution ("surface" in his terms is the graphical representation of the histogram) of wealth distribution (see [8], chapter VII, point 18, p. 386):

*"La surface nous donne une image de la société. La forme extérieure varie peu, la partie intérieure est, au contraire, en perpétuel mouvement; tandis que certains individus montent dans les régions supérieures, d'autres en descendent. Ceux qui arrivent dans la zone la plus pauvre disparaissent; de ce côté certains sont éliminés. Il est étrange, mais cela est certain, que ce même phénomène se reproduit dans les régions supérieures. L'expérience nous apprend que les aristocraties ne durent pas; les raisons du phénomène sont nombreuses et nous n'en connaisant que très peu, mais il n'y a aucun doute sur la réalité du phénomène."*¹

2010 *Mathematics Subject Classification.* Primary: 91D10. Secondary: 91A60, 91B30, 91B55.

Key words and phrases. Pareto, distribution, circulation, elite, Kelly, wealth.

¹*"The histogram gives an image of the society. The external shape changes little, but the internal structure, at the opposite, is in perpetual motion. At the same time that some individuals climb into the upper ladder, others fall down. Those arriving at the most poor region disappear and are eliminated. It is as strange as certain that the same phenomenon occurs in the upper regions.*

A more elaborate “Elite Theory” is a central subject in Sociology and Political Science and was later developed by G. Mosca, R. Michels and others. In these theories the “Elite” class is composed more about a ruling elite than about the wealthiest elite, but very often these two classes do coincide. In this article we only consider the elite class defined as the upper wealthy class.

The explanation of Pareto’s circulation of elites.

In the model of wealth evolution proposed in [9], the individual wealth follows a series of financial decision modelled as a betting game: In each round a fixed proportion of wealth is wagered, which is lost or doubled with some probability $0 < p < 1$. The main theorem in [9] is the solution of the functional equation for an invariant distribution. This yields that the only invariant distributions obey Pareto law. We prove also that this distribution are stable for bounded L^1 perturbations. Having an invariant distribution does not mean that the individual wealth is constant. On the contrary, as already announced by the above citation by Pareto, the wealth of individuals is in constant evolution. “en perpétuel mouvement” in his own words.

Paradoxically, a positive expectation of wealth increase, as one expects in a growing economy, does not result in an increase of the individual wealth. This is only true on average and individually only when the fraction of the wealth wagered is smaller than a fixed fraction $\gamma_0 > 0$ that depends on the probability $0 < p < 1$. This is a well known fact in gambling theory: The Kelly criterion gives the sharp fraction $0 < \gamma^* < \gamma_0$ that one should risk in order to maximize the long term exponential growth, and a side result is that if we risk a fraction $\gamma > \gamma_0$ then it is a mathematical theorem that we have almost sure ruin (with probability 1). This means in the wealth evolution model that if we do have $\gamma > \gamma_0$ then as corollary we get Pareto’s “circulation of elite”, and more precisely we have that this occurs at all wealth levels, exactly as Pareto conjectured. Periodically individuals get wealthier and later on lose they fortune.

Now, the question reduces to why it is the rule that the fraction risked at each stage is superior to what Kelly criterion dictates. This cannot be a mathematical theorem but a common behavioural argument. It is a common observation that Kelly fraction is far more conservative than what the natural instincts dictate. Probability is not intuitive for the human mind, and we experience difficulties grasping the true nature of randomness. Even for profesional gamblers it requires hard discipline to adjust to Kelly fraction. For example, a well known quotation by Jessy May, poker player and commentator, that applies to many other gambling or trading activities with advantage [6]

Experience tells us that the aristocracies do not last. The reasons are numerous but we know little, but there no doubt about the existence of the phenomenon.”

"Poker is a combination of luck and skill. People think mastering the skill is hard, but they're wrong. The trick to poker is mastering the luck."

In economics and finance, a celebrated quotation of probabilist and economist J.M. Keynes argues into the same direction ([2] p. 161):

"Even apart from the instability due to speculation, there is the instability due to the characteristic of human nature that a large proportion of our positive activities depend on spontaneous optimism rather than mathematical expectations, whether moral or hedonistic or economic. Most, probably, of our decisions to do something positive, the full consequences of which will be drawn out over many days to come, can only be taken as the result of animal spirits a spontaneous urge to action rather than inaction, and not as the outcome of a weighted average of quantitative benefits multiplied by quantitative probabilities."

An attempt to explain these natural inclinations for risk-taking can be found is proposed by J. Coates, neuroscientist and former Wall Street trader, in his interesting book [1] "The hour between dog and wolf. Risk-taking, gut feelings and the biology of boom and bust" where a biological origin is proposed for this type of behaviour. The point of view of J.M. Keynes is exposed very clearly ([1] p.141):

John Maynard Keynes, more than any other economist, understood these subterranean urges to explore, calling them "animal spirits" (...) He considered them the pulsing heart of the economy.(...) He suspected that business enterprise is no more driven by the calculation of odds than is an expedition to the South Pole. Enterprise is driven to a great extent by a pure love of risk-taking."

The inadequacy of human mind to comprehend probability and randomness, and to calibrate risk properly, in particular hidden risks, permeates Nassim Taleb's "risk philosophy", and his "black swans" and "antifragile" theory. This was already masterly described in his first book "Fooled by randomness: The Hidden Role of Chance in Life and in the Markets" [11].

There is overwhelming evidence that the risk-taking in financial decisions by average individuals go far beyond what Kelly criterion dictates, thus this explains Pareto's "circulation of elite".

In the next sections we review the mathematical technical parts needed in our argument.

2. DYNAMICAL MODEL OF WEALTH EVOLUTION.

A family of models of wealth evolution are presented in [9], and for the simplest one a complete analysis is carried on. In order to explain the circulation of elites we need to use a more realistic model where the fraction of the wealth at risk is totally lost in case of unfavorable outcome. For this model the Kelly criterion applies and this is key to our argument.

1. Setup. Let $f(x)$ be the wealth distribution, i.e. $df = f(x) dx$ is the number of individuals with wealth in the infinitesimal interval $[x, x + dx]$. The distribution function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous, positive and decreasing and $\lim_{x \rightarrow +\infty} f(x) = 0$. A distribution is of Pareto type if it presents a power law decay $x^{-\alpha}$ at $+\infty$, that is

$$\lim_{x \rightarrow +\infty} -\frac{\log f(x)}{\log x} = \alpha > 0 .$$

The exponent $\alpha > 0$ is *Pareto exponent*. A distribution of the form $f(x) = C.x^{-\alpha}$ is called a Pareto distribution. Smaller values of α indicate larger inequalities in wealth distribution. Notice that $\alpha > 1$ is necessary for the distribution to be summable at $+\infty$, i.e. finite wealth at infinite (finiteness near 0 is not significant since the model aims to explain the tail behaviour at $+\infty$).

2. Wealth dynamics. We focus on the evolution of individual wealth. We assume that the evolution is based on two main factors: Financial decisions, that we model as a betting game, and by public redistribution of wealth, that absorbs part of the individual wealth into public wealth.

For the first factor we model the financial decisions of each individual by a sequence of bets. Each financial decision turns out to be a bet, wagering a proportion of his wealth. As a first approximation, we assume that the probability of success is the same for all agents and bets $0 < p < 1$ (this is the average probability). At each round, each agent risks the same percentage of his wealth, a fraction $\gamma > 0$ (that is also an average). If he wins, his wealth is multiplied by the factor $1 + \gamma$ and if he loses, with probability $q = 1 - p$, it is subtracted the fraction of wealth put at risk, that is a percentage γ of his wealth, that is, his wealth ² goes from x to $x(1 - \gamma)$.

Only considering this first factor, one round evolution the distribution transforms into the new distribution

$$(1) \quad \mathcal{W}(f)(x) = \frac{p}{1 + \gamma} f\left(\frac{x}{1 + \gamma}\right) + \frac{q}{1 - \gamma} f\left(\frac{x}{1 - \gamma}\right) .$$

²This is what is different from the simplest model in [9] where after a loss, wealth was divided by $1 + \gamma$. This allowed a complete computation of invariant distributions which is no longer the case here. The present model is more realistic and has the same asymptotic properties.

The operator \mathcal{W} is “wealth preserving”, that is, in terms of L^1 -norm we have

$$\|\mathcal{W}(f)\|_{L^1} = \|f\|_{L^1} .$$

The agents will only risk their capital if there is a positive expectation of gain. Giving a capital x , the expected gain at each round is

$$\mathbb{E}(\Delta x) = p\gamma x - q\gamma x = (p - q)\gamma x = (2p - 1)\gamma x ,$$

thus we should assume that $p > 1/2$.

There are other mechanisms that affect wealth evolution that we should consider, as for example inheritances that divide wealth, taxes, etc. As in [9] it is natural to consider a broader class of operators \mathcal{W} with a dissipative parameter $\kappa \geq 1$, the *dissipative coefficient*,

$$\mathcal{W}_\kappa(f)(x) = \frac{1}{\kappa} \mathcal{W}(f)(x) = \frac{p}{\kappa(1+\gamma)} f(x/(1+\gamma)) + \frac{q}{\kappa(1-\gamma)} f(x/(1-\gamma)) .$$

This operator is L^1 contracting for $\kappa > 1$.

3. Invariant distributions. Distributions invariant by the evolution operator \mathcal{W}_κ must satisfy the fixed point functional equation $\mathcal{W}_\kappa(f) = f$, that is,

$$(2) \quad f(x) = \frac{p}{\kappa(1+\gamma)} f(x/(1+\gamma)) + \frac{q}{\kappa(1-\gamma)} f(x/(1-\gamma)) .$$

4. Transformation of the functional equation. Considering the change of variables $F(x) = f(e^x)$, equation (3) becomes a linear functional equation for $F : \mathbb{R} \rightarrow \mathbb{R}$

$$(3) \quad a F(x + \lambda) - F(x) + b F(x - \mu) = 0 ,$$

where $\lambda = -\log(1 - \gamma)$, $\mu = \log(1 + \gamma)$, $a = q/(\kappa(1 - \gamma))$ and $b = p/(\kappa(1 + \gamma))$, with $a, b, \lambda, \mu > 0$.

We have a general theory of these type of functional equations. L. Schwartz (see [10], and also [5], [3]) studied more general “mean periodic” smooth functions F that satisfy a convolution functional equation of the form

$$(4) \quad \omega \star F = 0 ,$$

where ω is a compactly supported distribution. In our case, $\omega = a \delta_\lambda - \delta_0 + b \delta_{-\mu}$. The way to study these equations is by Fourier transforming it (à la Carleman using hyperfunctions in order to work in sufficient generality). Taking the Fourier transform of equation (4), we get

$$\hat{\omega} \cdot \hat{F} = 0 .$$

Thus if \hat{F} is a Schwartz distributions then its support is contained in the set where $\hat{\omega} = 0$, and if ω is a finite sum of atomic masses as in our model, this is the set of ρ 's

which are roots of a Dirichlet polynomials. In our specific model we are led to solve the *characteristic equation* in ρ :

$$(5) \quad ae^{\lambda\rho} - 1 + be^{-\mu\rho} = 0 .$$

This equations yields particular solutions of the form $F(x) = e^{\rho x}$ (note that $\hat{F} = \delta_\rho$). In general the zeos of such a Dirichlet polynomial are contained in a vertical strip, thus the real part being bounded, and those with negative real parts give the admissible solutions all of them with Pareto assymptotics.

The function $h(\rho) = ae^{\lambda\rho} - 1 + be^{-\mu\rho}$ is convex since $h''(\rho) = a\lambda^2e^{\lambda\rho} + b\mu^2e^{-\mu\rho} > 0$, and h' has a unique zero at

$$\rho_0 = \frac{1}{\lambda + \mu} \log \left(\frac{\mu b}{\lambda a} \right) .$$

In order to have admissible solutions with Pareto assymptotics at $+\infty$ converging to 0, we need to have a negative solution to the characteristic equation. The condition $h(0) = a + b - 1 < 0$ is equivalent to the condition on $\kappa \geq 1$,

$$(6) \quad \kappa > \kappa_0 = \frac{1 - (p - q)\gamma}{1 - \gamma^2} .$$

Thus we only have Pareto solutions with the right assymptotics if we have this condition on κ . It is interesting to note that $\kappa_0 > 1$ is equivalent to $\gamma > p - q = 2p - 1 = \gamma^*$, and as we see in the next section, this is the Kelly condition.

3. KELLY CRITERION.

1. Setup. We assume that we are playing a game with repetition. At each round we risk a fraction $0 \leq \gamma \leq 1$ of our capital X . With probability $0 < p < 1$ we win and the pay-off is the fraction of the capital we bet. This means that if X is our current capital, if we lose the bet (which happens with probability $q = 1 - p$) we substract fX to our capital, and if we win, we add fX to our capital . As we have already seen the condition that the game has positive expectation is $p > 1/2$. If we play a game with advantage, then we expect an exponential growth of our initial bankroll X_0 if we follow a reasonable betting strategy. We assume that there is no minimal unit of bet. By homogeneity of the problem, a sharp strategy must consist in betting a proportion $\gamma(p)$ of the total bankroll. Our bankroll after having played n rounds of the game have is

$$X_n = X_0 \prod_{i=1}^n (1 + \epsilon_i \gamma(p))$$

where $\epsilon_i = +1$ if we won in the i -th round, and $\epsilon_i = -1$ if we lose in the i -th round. The exponential rate of growth of the bankroll is

$$G_n = \frac{1}{n} \log \frac{X_n}{X_0} = \frac{1}{n} \sum_{i=1}^n \log(1 + \epsilon_i \gamma(p)).$$

The Kelly criterion maximizes the expected value of the exponential rate of growth:

Theorem 1. (*Kelly criterion*) *For a game with advantage, that is $p > 1/2$, the expected value of the exponential rate of growth G_n is maximized for*

$$\gamma^*(p) = p - q = 2p - 1 .$$

The argument is straightforward. Observe that the expected value is

$$\mathbb{E}(G_n) = \mathbb{E}(G_1) = p \log(1 + \gamma) + q \log(1 - \gamma) = g(\gamma).$$

This function of the variable γ has a derivative,

$$g'(\gamma) = \frac{p}{1 + \gamma} - \frac{q}{1 - \gamma} = \frac{(2p - 1) - \gamma}{1 - \gamma^2} ,$$

and $g(\gamma) \rightarrow 0$ when $\gamma \rightarrow 0^+$, and $g(\gamma) \rightarrow -\infty$ when $\gamma \rightarrow 1^-$. Also $g'(\gamma) > 0$ near 0, and g' is decreasing, so g is concave. Thus g increases from 0 to its maximum attained at

$$\gamma^* = \gamma^*(p) = 2p - 1$$

and then decreases to $-\infty$.

A corollary of this analysis is:

Theorem 2. (*Kelly ruin threshold*) *Under the same conditions as before, there is a threshold value $\gamma_0(p) > \gamma^*(p)$, that is the only solution of the equation*

$$p \log(1 + \gamma_0) + (1 - p) \log(1 - \gamma_0) = 0 ,$$

such that for $\gamma > \gamma_0(p)$ we have almost sure ruin.

Indeed the equation is $g(\gamma_0) = 0$ and since g is decreasing for $\gamma > \gamma^*$ we have that for $\gamma > \gamma_0$, $g(\gamma) < 0$, and the expected exponential growth is negative.

Behavioural considerations of Keynes “animal spirits”, as discussed in the preliminaries, indicate that most players take more risks than allowed by the Kelly criterion and that the average γ is larger than γ_0 , which completes our argument.

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CNRS, LAGA UMR 7539, UNIVERSITÉ PARIS XIII, 99, AVENUE J.-B. CLÉMENT, 93430-VILLETANEUSE, FRANCE

E-mail address: ricardo.perez.marco@gmail.com