ON THE DEFINITION OF HIGHER GAMMA FUNCTIONS

RICARDO PÉREZ-MARCO

Abstract. We generalize the new definition of Euler Gamma function from [19] to higher Gamma functions. With this unified approach, we characterize Barnes higher Gamma functions, Mellin Gamma functions, Barnes multiple Gamma functions, Jackson q-Gamma function, and Nishizawa higher q-Gamma functions. This generalization reveals the multiplicative group structure of solutions of the functional equation that is a cocycle equation. We also generalize Barnes hierarchy of higher Gamma function and multiple Gamma functions. In this new approach, Barnes-Hurwitz zeta functions are no longer required for the definition of Barnes multiple Gamma functions. This simplifies the classical definition, without the necessary analytic preliminaries about the meromorphic extension of Barnes-Hurwitz zeta functions, and defines a larger class of Gamma functions. For some algebraic independence conditions on the parameters, we have uniqueness of the solutions, which implies the coincidence of our multiple Gamma functions with Barnes multiple Gamma functions.

1. Introduction

The first result is a new characterization and definition of Euler Gamma function that was already presented in the article [19] dedicated to Euler Gamma function. We can develop in a natural way the classical formulas in the theory from this new definition. We denote the right half complex plane by $\mathbb{C}_+ = \{s \in \mathbb{C}; \operatorname{Re} s > 0\}$.

Theorem 1.1. *There is one and only one finite order meromorphic function $\Gamma(s)$, $s \in \mathbb{C}$, without zeros nor poles in $\mathbb{C}_+$, with $\Gamma(1) = 1$, $\Gamma'(1) \in \mathbb{R}$, that satisfies the functional equation*

$$\Gamma(s+1) = s\,\Gamma(s)$$

Definition 1.2 (Euler Gamma function)**.** *The only solution to the above conditions is the Euler Gamma function.*

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Without the condition $\Gamma'(1) \in \mathbb{R}$ we don't have uniqueness, but we have the following result:

Theorem 1.3. *Let f be a finite order meromorphic function in $\mathbb{C}$, without zeros nor poles in $\mathbb{C}_+$, and satisfies the functional equation*

$$f(s+1) = s\, f(s) \ ,$$

then there exists $a \in \mathbb{Z}$ and $b \in \mathbb{C}$ such that

$$f(s) = e^{2\pi i a s + b}\Gamma(s) \ .$$

Moreover, if $f(1) = 1$ then we have

$$f(s) = e^{2\pi i a s}\Gamma(s) \ .$$

The proof can be found in [19] but we reproduce it here and as a preliminary result for the generalizations that are the core of this article. We refer to the companion article [19] for the various definitions of Euler Gamma function and the historical development of the subject of Eulerian integrals. We strongly encourage the reader to study first [19], and also the bibliograhic notes in [20] before going into the generalizations that we develop in this article.

In the proof we use the elementary theory of entire function and Weierstrass factorization that can be found in classical books as [5] (or in the Appendix of [19]).

Proof. We prove existence and then uniqueness.

Existence: If we have a function satisfying the previous conditions then its divisor must be contained in $\mathbb{C} - \mathbb{C}_+$, and the functional equation implies that it has no zeros and only simple poles at the non-positive integers. We can construct such a meromorphic function g with such divisor, for example,

$$g(s) = s^{-1} \prod_{n=1}^{+\infty} \left(1 + \frac{s}{n}\right)^{-1} e^{s/n} \tag{1}$$

which converges since $\sum_{n\geq 1} n^{-2} < +\infty$, and is of finite order. Now, we have that the meromorphic function $\frac{g(s+1)}{sg(s)}$ has no zeros nor poles and it is of finite order (as ratio of finite order meromorphic functions), hence there exists a polynomial P such that

$$\frac{g(s+1)}{sg(s)} = e^{P(s)} \ .$$

Consider a polynomial Q such that

$$\Delta Q(s) = Q(s+1) - Q(s) = P(s) \tag{2}$$

The polynomial Q is uniquely determined from P up to a constant, hence we can choose Q such that $e^{Q(0)} = g(1)^{-1}$. Now we have that $\Gamma(s) = e^{-Q(s)}g(s)$ satisfies the functional equation and all the conditions.

Uniqueness: Consider a second solution f. Let $F(s) = \Gamma(s)/f(s)$. Then F is an entire function of finite order without zeros, hence we can write $F(s) = \exp A(s)$ for some polynomial A. Moreover, the functional equation shows that F is $\mathbb{Z}$-periodic. Hence, there exists an integer $a \in \mathbb{Z}$, such that for any $s \in \mathbb{C}$,

$$A(s+1) = A(s) + 2\pi i a \ .$$

It follows that $A(s) = 2\pi i a s + b$ for some $b \in \mathbb{C}$. Since $F(1) = 1$, we have $e^b = 1$. Since $F'(1) \in \mathbb{R}$, and $F'(1) = F'(1)/F(1) = 2\pi i a \in \mathbb{R}$ we have $a = 0$, thus F is constant, $F \equiv 1$ and $f = \Gamma$. □

Remarks.

- Using the functional equation we can weaken the conditions and request only that the function is meromorphic only on $\mathbb{C}_+$ with the corresponding finite order growth. We can also assume that it is only defined on a cone containing the positive real axes, a vertical strip of width larger than 1, or in general with any region Ω which is a transitive region for the integer translations and f satisfies the finite order growth condition in Ω when $s \to +\infty$.

  Proposition 1.4. *Let $\Omega \subset \mathbb{C}$ a domain such that for any $s \in \mathbb{C}$ there exists an integer $n(s) \in \mathbb{Z}$ such that $s + n(s) \in \Omega$, and $|n(s)| \leq C|s|^d$, for some constants $C, d > 0$ depending only on Ω. Then any function $\tilde{\Gamma}$ satisfying a finite order estimate in Ω and the functional equation $\tilde{\Gamma}(s+1) = s\tilde{\Gamma}(s)$ when $s, s+1 \in \Omega$, extends to a finite order meromorphic function on $\mathbb{C}$.*

  Proof. Let $\tilde{\Gamma}$ be such a function. Let Ω be corresponding region. Iterating the functional equation we get that $\tilde{\Gamma}$ extends meromorphically to the whole complex plane. Then, if g is the Weierstrass product (1) and Q a polynomial given by (2), the function $h(s) = \tilde{\Gamma}(s)/(e^{-Q(s)}g(s))$ is a $\mathbb{Z}$-periodic entire function. Since $1/(e^{-Q}g)$ is an entire function of finite order, we have in Ω the finite order estimate for h. Using that $|n(s)| \leq C|s|^d$, we get that h is of finite order, hence $\tilde{\Gamma}$ is meromorphic and of finite order in the plane. □

- Assuming Γ real-analytic we get $\Gamma'(1) \in \mathbb{R}$, but this last condition is much weaker. Also, as it follows from the proof, we can replace this condition by $\Gamma(a) \in \mathbb{R}$ for some $a \in \mathbb{R} - \mathbb{Z}$, or only request that Γ is asymptotically real, $\lim_{x\in\mathbb{R}, x\to+\infty} \operatorname{Im}\Gamma(x) = 0$. Without the condition $\Gamma'(1) \in \mathbb{R}$ the proof shows that Γ is uniquely determined up to a factor $e^{2\pi iks}$ with $k \in \mathbb{Z}$.

2. General definition.

2.1. **General definition and characterization.** We first need to recall the notion of *"Left Located Divisor"* (LLD) function that is useful in the theory of Poisson-Newton formula for finite order meromorphic functions ([14], [15]).

Definition 2.1 (LLD function)**.** *A meromorphic function f in $\mathbb{C}$ is in the class LLD (Left Located Divisor) if f has no zeros nor poles in $\mathbb{C}_+$, i.e.* $\text{Div}(f) \subset \mathbb{C} - \mathbb{C}_+$.

The function is in the class CLD (Cone Located Divisor) if its divisor is contained in a closed cone in $\mathbb{C} - \mathbb{C}_+$.

The following Theorem is a generalization of Theorem 1.1 which results for the simple LLD function $f(s) = s$.

Theorem 2.2. *Let f be a real analytic LLD meromorphic function in $\mathbb{C}$ of finite order. There exists a unique function Γ^f, the Gamma function associated to f, satisfying the following properties:*

(1) $\Gamma^f(1) = 1$,
(2) $\Gamma^f(s+1) = f(s)\Gamma^f(s)$,
(3) Γ^f *is a meromorphic function of finite order,*
(4) Γ^f *is LLD,*
(5) Γ^f *is real analytic.*

If f is CLD then Γ^f is CLD.

Proof. The proof follows the same lines as the proof of Theorem 1.1. First, we prove that the functional equation (2) determines the divisor of Γ^f, then we construct a solution using a Weierstrass product, and finally we prove the uniqueness.

- **Determination of the divisor.**

As usual, we denote the divisor of f as

$$\text{Div}(f) = \sum_\rho n_\rho(f).(\rho)$$

where the sum is extended over $\rho \in \mathbb{C}$ and $n_\rho(f)$ is the multiplicity of the zero if ρ is a zero, the negative multiplicity of the pole if ρ is a pole, or $n_\rho(f) = 0$ if ρ is neither a zero or pole. A divisor is said to be LLD, resp. CLD, if it is the divisor of a LLD, resp. CLD, function.

Lemma 2.3. *If* Γ^f *is LLD and satisfies the functional equation (2), then the divisor of* Γ^f *is*

$$\mathrm{Div}(\Gamma^f) = -\sum_{\rho, k\geq 0} n_\rho(f)\cdot(\rho - k)$$

where

$$\mathrm{Div}(f) = \sum_{\rho} n_\rho(f)\cdot(\rho)$$

and

$$n_\rho(\Gamma^f) = -\sum_{k=0}^{|\rho|} n_{\rho+k}(f) \tag{3}$$

If the divisor $\mathrm{Div}(f)$ *is LLD, resp. CLD, then* $\mathrm{Div}(\Gamma^f)$ *is LLD, resp. CLD.*

We allow ourselves the slight abuse of notation $\mathrm{Div}(\Gamma^f)$ to denote the divisor of a potential solution Γ^f when we have not yet proved the existence of Γ^f.

Proof. For any $\rho \in \mathbb{C}$, the functional equation gives

$$n_{\rho+1}(\Gamma^f) = n_\rho(f) + n_\rho(\Gamma^f)\ ,$$

or equivalently

$$n_\rho(\Gamma^f) = -n_\rho(f) + n_{\rho+1}(\Gamma^f)\ ,$$

Hence, by induction, we have

$$n_\rho(\Gamma^f) = -\sum_{k=0}^{m} n_{\rho+k}(f) + n_{\rho+m}(\Gamma^f)$$

and since Γ^f is LLD, for $m \geq -\operatorname{Re}\rho \geq |\rho|$ we have $n_{\rho+m}(\Gamma^f) = 0$, so

$$n_\rho(\Gamma^f) = -\sum_{k=0}^{|\rho|} n_{\rho+k}(f) = -\sum_{k=0}^{+\infty} n_{\rho+k}(f) \tag{4}$$

and we get, with $\rho' = \rho + k$,

$$\mathrm{Div}(\Gamma^f) = -\sum_{k\geq 0}\sum_{\rho} n_{\rho+k}\cdot(\rho) = -\sum_{\rho', k\geq 0} n_{\rho'}\cdot(\rho' - k)$$

which gives the formula for $\mathrm{Div}(\Gamma^f)$. □

• **Convergence exponent of a divisor.**

Definition 2.4. *The divisor of f has exponent of convergence $\alpha > 0$ if*

$$||\operatorname{Div}(f)||_\alpha = \sum_{\rho\neq 0} |n_\rho(f)|.|\rho|^{-\alpha} < +\infty \ .$$

We recall that a meromorphic function of finite order has a divisor with some finite exponent of convergence. More precisely, if $o(f) < +\infty$ is the order of f, then for any $\epsilon > 0$, $\alpha = o(f) + \epsilon$ is an exponent of convergence of its divisor.

Proposition 2.5. *If $\operatorname{Div}(f)$ is LLD of finite order, then $\operatorname{Div}(\Gamma^f)$ given by Lemma 2.3 is LLD and of finite order.*

Proof. We already known from Lemma 2.3 that $\operatorname{Div}(\Gamma^f)$ is LLD. We prove that if α is an exponent of convergence of $\operatorname{Div}(f)$, then $2\alpha + 1$ is an exponent of convergence of $\operatorname{Div}(\Gamma^f)$ (we don't try to be sharp here).

First, observe that since α is an exponent of convergence for f, then

$$||\operatorname{Div}(f)||_\alpha = \sum_{\rho\neq 0} |n_\rho(f)|.|\rho|^{-\alpha} \leq C(\alpha) < +\infty \ ,$$

so we get

$$|n_\rho(f)| \leq C(\alpha)|\rho|^\alpha \ .$$

Using equation (3) we have, for $|\rho| \geq 1$, and $C_0(\alpha) = C(\alpha)(1 + 2^{\alpha+1}/(\alpha + 1))$,

$$\begin{aligned}
|n_\rho(\Gamma^f)| \leq \sum_{k=0}^{|\rho|} |n_{\rho+k}(f)| &\leq C(\alpha) \sum_{k=0}^{|\rho|} |\rho + k|^\alpha \\
&\leq C(\alpha) \sum_{k=0}^{|\rho|} (|\rho| + k)^\alpha \\
&\leq C(\alpha) \left(|\rho|^\alpha + \int_0^{|\rho|} (|\rho| + x)^\alpha \, dx \right) \\
&\leq C(\alpha) \left(|\rho|^\alpha + \frac{2^{\alpha+1} - 1}{\alpha + 1} |\rho|^{\alpha+1} \right) \\
&\leq C_0(\alpha)|\rho|^{\alpha+1}
\end{aligned}$$

Therefore, we have

$$||\operatorname{Div}(\Gamma^f)||_{2\alpha+1} = \sum_{\rho, |\rho|\geq 1} |n_\rho(\Gamma^f)|.|\rho|^{-(2\alpha+1)} < +\infty \ .$$

□

We prove a more precise result when f is in the class CLD.

Proposition 2.6. *If $\alpha > 0$ is an exponent of convergence for f in the class CLD, then Γ^f is CLD and $\alpha+1$ is an exponent of convergence for Γ^f. More precisely, there exists a constant $C > 0$ such that*

$$|| \operatorname{Div}(\Gamma^f)||_{\alpha+1} \leq C \, || \operatorname{Div}(f)||_{\alpha+1} + \frac{C}{\alpha} \, || \operatorname{Div}(f)||_{\alpha}$$

Proof. Lemma 2.3 proves that Γ^f is CLD if we start with f CLD. Now, if f is CLD, there is a constant $C > 0$ such that for any $k \geq 1$ and ρ in the left cone (the constant C depends on the cone)

$$|\rho - k|^{-1} \leq C(|\rho| + k)^{-1} \ .$$

Then we have, with $\rho' = \rho + k$,

$$\begin{aligned}
|| \operatorname{Div}(\Gamma^f)||_{\beta} &= \sum_{\rho \neq 0} |n_\rho(\Gamma^f)|.|\rho|^{-\beta} \\
&= \sum_{\rho \neq 0} \sum_{k=0}^{|\rho|} |n_{\rho+k}(f)|.|\rho|^{-\beta} \\
&= \sum_{\rho' \notin \mathbb{N}} |n_{\rho'}(f)| \sum_{k=0}^{+\infty} |\rho' - k|^{-\beta} \\
&\leq C \sum_{\rho' \neq 0} |n_{\rho'}(f)| \sum_{k=0}^{+\infty} (|\rho'| + k)^{-\beta} \\
&= C \sum_{\rho' \neq 0} |n_{\rho'}(f)|.|\rho'|^{-\beta} + C \sum_{\rho' \neq 0} |n_{\rho'}(f)| \int_0^{+\infty} (|\rho'| + x)^{-\beta} \, dx \\
&\leq C \sum_{\rho' \neq 0} |n_{\rho'}(f)|.|\rho'|^{-\beta} + \frac{C}{\beta - 1} \sum_{\rho' \neq 0} |n_{\rho'}(f)|.|\rho'|^{-\beta+1} \\
&= C \, || \operatorname{Div}(f)||_{\beta} + \frac{C}{\beta - 1} \, || \operatorname{Div}(f)||_{\beta-1}
\end{aligned}$$

hence, for $\beta = \alpha + 1$ the sum is converging and we prove the Lemma. □

• **Existence of Γ^f.**

Since f has finite order, the divisor of f has a finite convergence exponent. Hence, $\operatorname{Div}(\Gamma^f)$ determined by Lemma 2.3 has a finite exponent of convergence. Let $d \geq 1$

be an integer that is an exponent of convergence for this divisor (the case $d = 0$ only occurs for a finite divisor). We consider the Weierstrass product,

$$g(s) = s^{-n_0(f)} \prod_{\rho \neq 0} E_d(s/\rho)^{n_\rho(\Gamma^f)}$$

where

$$E_d(x) = (1-x) \exp\left(x + \frac{x^2}{2} + \ldots + \frac{x^d}{d} \right) .$$

Then g has order d and $\text{Div}(g) = \text{Div}(\Gamma^f)$. Therefore the meromorphic function

$$\frac{g(s+1)}{f(s)g(s)}$$

is of finite order and has no zeros nor poles. So, it is an entire function of finite order without zeros. Therefore, there exists a polynomial ϕ such that

$$\frac{g(s+1)}{f(s)g(s)} = e^{\phi(s)} \tag{5}$$

There is a unique polynomial ψ such that $\psi(0) = 0$ and

$$\psi(s+1) - \psi(s) = \phi(s). \tag{6}$$

We can obtain ψ directly by developing ϕ on the bases of falling factorial polynomials, $s^{\underline{k}} = s(s-1)\ldots(s-k+1)$, that diagonalize the difference operator, $\Delta s^{\underline{k}} = k\, s^{\underline{k-1}}$,

$$\phi(s) = \sum_{k=0}^{n} \frac{a_k}{k!}\, s^{\underline{k}}$$

then

$$\psi(s) = \sum_{k=0}^{+\infty} \frac{a_k}{(k+1)!}\, s^{\underline{k+1}} .$$

Now, considering a constant c such that $e^c = g(0)^{-1}$ the meromorphic function

$$\Gamma^f(s) = e^{\psi(s)+c} g(s), \tag{7}$$

satisfies $\Gamma^f(1) = 1$ (condition (1)), the functional equation (2) and all the other conditions in Theorem 2.2, and we have proved the existence.

- **Uniqueness of Γ^f.**

Consider a second solution G. Let $F(s) = \Gamma^f(s)/G(s)$. Then F is an entire function of finite order without zeros, hence we can write $F(s) = \exp A(s)$ for some polynomial

A. Moreover, the functional equation shows that F is $\mathbb{Z}$-periodic. Therefore, there exists an integer $a \in \mathbb{Z}$, such that for any $s \in \mathbb{C}$,

$$A(s+1) = A(s) + 2\pi i a \ .$$

It follows that $A(s) = 2\pi i a s + b$ for some $b \in \mathbb{C}$. Since $F(1) = 1$, we have $e^b = 1$. Since $F'(1) \in \mathbb{R}$, and $F'(1) = F'(1)/F(1) = 2\pi i a \in \mathbb{R}$ we have $a = 0$, thus F is constant, $F \equiv 1$ and $G = \Gamma^f$.

□

2.2. **Uniqueness results.** It is interesting to note, following the argument for uniqueness, that we can drop the normalisation condition (1) and the real-analyticity condition (5) and we obtain the following Theorem (this is similar to Theorem 1.3),

Theorem 2.7. *Let f be a LLD meromorphic function in $\mathbb{C}$ of finite order. We consider a function g satisfying*

1. $g(s+1) = f(s)g(s)$,
2. *g is a meromorphic function of finite order,*
3. *g is LLD,*

Then there is always a solution $\Gamma^f(s)$ and any other solution g is of the form $g(s) = e^{2\pi i a s + b}\Gamma^f(s)$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{C}$. If f is CLD then the solutions are CLD.

Moreover, we have possible further normalizations:

- *If we add the condition $g(1) = 1$, or $g(k) = 1$ for some $k \in \mathbb{N}^*$, then all solutions are of the form $g(s) = e^{2\pi i a s}\Gamma^f(s)$.*
- *If f^{-1} has a pole at 0 and we add the condition $Res_{s=0}\, g = 1$ then all solutions are of the form $g(s) = e^{2\pi i a s}\Gamma^f(s)$.*
- *If f has no zero at 0 then we can add the condition $g(0) = 1$ and all solutions are of the form $g(s) = e^{2\pi i a s}\Gamma^f(s)$.*
- *If we add the conditions $g(1) = 1$ and $g(\omega) \in \mathbb{R}$ where $\omega \in \mathbb{R}_+ - \mathbb{Q}$ then $g = \Gamma^f$ is unique.*
- *If we add the conditions $g(1) = 1$ and $g'(1) \in \mathbb{R}$ then the solution $g = \Gamma^f$ is unique.*
- *If we add the hypothesis that f is real analytic and the condition that g is real analytic then all solutions are of the form $g(s) = c.\Gamma^f(s)$ with $c \in \mathbb{R}^*$.*

Proof. With the same proof as before we get the existence of a solution $\Gamma^f(s)$ and that any other solution is of the form $g(s) = e^{2\pi i a s + b}\Gamma^f(s)$ (note that the constant 0 function is not LLD). For another solution g, the condition $g(k) = 1$ for $k \in \mathbb{Z}$ implies $e^b = 1$, hence the first normalization result. For the second statement we observe that

$$\mathrm{Res}_{s=0} g = e^b \mathrm{Res}_{s=0} \Gamma^f$$

hence $e^b = 1$. The third statement is similar to the first one observing that g has no pole at $s = 0$. The fourth normalization condition forces $b = 0$ (first statement) and

$$e^{2\pi i a \omega} = 1$$

which implies $a = 0$ because ω is irrational. For the fifth statement, for a second solution we have, from $g(1) = 1$, $g(s) = e^{2\pi i a s} \Gamma^f(s)$. Differentiate and set $s = 1$, then we get

$$g'(1) = 2\pi i a g(1) + \left(\Gamma^f\right)'(1) = 2\pi i a + 1 \in \mathbb{R}$$

hence $a = 0$ and the solution is unique. For the last statement, $g(s) = e^{2\pi i a s + b} \Gamma^f(s)$ and g and Γ^f real analytic forces $a = 0$, and $e^b \in \mathbb{R}^*$. □

Example 2.8. For $f(s) = s$ and the conditions g real analytic and $g(1) = 1$, this Theorem is just Theorem 1.1 and the only solution $g(s) = \Gamma(s)$ is Euler Gamma function.

Let $\omega \in \mathbb{C}_+$ and consider $f(s) = \omega s$. Then $g(s) = \omega^s \Gamma(s)$ is a solution and all the solutions are of the form

$$g(s) = e^{2\pi i a s + s \log \omega + b} \Gamma(s) =$$

for $a \in \mathbb{Z}$ and $b \in \mathbb{C}$ (note that the choice of the branch of $\log \omega$ is irrelevant).

If $\omega \in \mathbb{C}^*$ and we request $g(1) = 1$, then all solutions are of the form, with $a \in \mathbb{Z}$,

$$g(s) = e^{(s-1)(2\pi i a + \log \omega)} \, \Gamma(s) \tag{8}$$

If $\omega \in \mathbb{R}_+$, then $f(s) = \omega s$ is real analytic, and if we request g to be real analytic and $g(1) = 1$, then, taking the real branch of $\log \omega$, we must have $a = 0$ and

$$g(s) = e^{(s-1) \log \omega} \, \Gamma(s) \tag{9}$$

Example 2.9. Another particular example that is worth noting in this Theorem is when $f(s) = e^{P(s)}$. Then the solutions are of the form $g(s) = e^{Q_k(s)}$ where

$$\Delta Q_k = P + 2\pi i a$$

for $a \in \mathbb{Z}$, where Δ is the difference operator. This means that $Q_k(s) = Q_0(s) + 2\pi i a s + b$, where $b \in \mathbb{C}$. If we want solutions normalized such that $g(1) = 1$ then $e^b = 1$ and $b \in 2\pi i \mathbb{Z}$.

2.3. **A continuity result.** We prove the continuity of the operator $\Gamma : f \mapsto \Gamma^f$ for the appropriate natural topology.

Theorem 2.10. *Let $(f_n)_{n\geq 0}$ be a sequence of meromorphic functions with uniformly bounded convergence exponent $\alpha > 0$ and such that*

$$|| \operatorname{Div}(f_n)||_\alpha = \sum_{\rho \in \operatorname{Div}(f_n), \rho\neq 0} |n_\rho||\rho|^{-\alpha} \leq M < +\infty$$

for a uniform bound $M > 0$. We assume that the functions (f_n) satisfy the hypothesis of Theorem 2.2 and that $f_n \to f$ when $n \to +\infty$, where f is a meromorphic function and the convergence is uniform on compact sets outside the poles of f. Then f has convergence exponent bounded by $\alpha > 0$,

$$|| \operatorname{Div}(f)||_\alpha \leq M < +\infty$$

and satisfies the hypothesis of Theorem 2.2, and also we have, uniformly outside the poles,

$$\lim_{n\to+\infty} \Gamma^{f_n} = \Gamma^f$$

Proof. We can read the divisors $\operatorname{Div}(f_n)$ as an integer valued functions with discrete support which are converging to $\operatorname{Div}(f)$ uniformly on compact sets. By uniform boundedness of the sums

$$|| \operatorname{Div}(f_n)||_\alpha = \sum_{\rho \in \operatorname{Div}(f_n), \rho\neq 0} |n_\rho||\rho|^{-\alpha}$$

we can pass to the limit and

$$|| \operatorname{Div}(f)||_\alpha = \lim_{n\to+\infty} || \operatorname{Div}(f_n)||_\alpha \leq M \ .$$

Therefore f has finite order. The class of LLD real analytic functions is closed. The class of functions satisfying the functional equation is also closed, hence f satisfies the hypothesis of Theorem 2.2, so Γ^f is well defined.

Now, since $\operatorname{Div}(f_n) \to \operatorname{Div}(f)$, we have using Lemma 2.3 that $\operatorname{Div}(\Gamma^{f_n}) \to \operatorname{Div}(\Gamma^f)$. On compact sets outside of the support of $\operatorname{Div}(\Gamma^f)$, the sequence of meromorphic functions $(\Gamma^{f_n})_{n\geq 0}$ is uniformly bounded (otherwise we would have a subsequence with poles out of the limit that would contradict the convergence of the divisor). Hence, we can extract converging subsequences. But any limit is identified by the uniqueness of Theorem 2.2, and we have convergence. □

2.4. **Multiplicative group property.** Consider the space $\mathcal{E}$ of LLD finite order meromorphic functions in the plane. We have that

$$\mathcal{E} = \bigcup_{n>0} \mathcal{E}_n$$

where $\mathcal{E}_n$ is the subgroup of meromorphic functions of order $\leq n$. On $\mathcal{E}_n$ we consider the topology given by convergence of the divisor on compact sets and the convergence of functions on compact sets outside the limit divisor. On $\mathcal{E}$ we consider the inductive topology from the exhaustion by the $\mathcal{E}_n$ spaces. Also $\mathcal{E}$ and $\mathcal{E}_n$ are stable under multiplication, and $(\mathcal{E}, .)$ and $(\mathcal{E}_n, .)$ are multiplicative topological group. Consider the closed subgroup $\mathcal{E}_0 \subset \mathcal{E}$ of real-analytic functions f normalized such that $f(1) = 1$.

Theorem 2.11. *The map* $\Gamma : \mathcal{E}_0 \to \mathcal{E}_0$ *such that*

$$\Gamma(f) = \Gamma^f$$

is an continuous injective group morphism.

Proof. Continuity results from Theorem 2.10. We observe that from

$$\Gamma^f(s+1) = f(s)\Gamma^f(s)$$
$$\Gamma^g(s+1) = g(s)\Gamma^g(s)$$

we get

$$\Gamma^f(s+1)\Gamma^g(s+1) = f(s)g(s)\Gamma^f(s)\Gamma^g(s)$$

and by uniqueness of Theorem 2.2 we get

$$\Gamma^f.\Gamma^g = \Gamma^{fg} \ .$$

Also, if $\Gamma^f = 1$, then directly from the functional equation we get that $f = 1$, and $\mathrm{Ker}(\Gamma) = \{1\}$. □

This Theorem justifies using Euler Gamma function as building block of the general solution by decomposing along the divisor.

Remark.

Consider the shift operator $T : \mathcal{E} \to \mathcal{E}$, $f(s) \mapsto T(f) = f(s+1)$ and the associated multiplicative cohomological equation in g with f given,

$$T(g).g^{-1} = f \ .$$

We have proved that the cohomological equation can be solved in $\mathcal{E}$ by the group morphism Γ, $g = \Gamma^f$. For $f \in \mathcal{E}_\alpha$ it can be solved in $\mathcal{E}_{\alpha+1}$. We observe a similar phenomenon of "loss of regularity" as in "Small Divisors" problems than in our setting can be interpreted as "loss of transalgelbraicity".

3. Application: Barnes higher Gamma functions.

We generalize the classical hierarchy of Barnes Gamma functions.

Definition 3.1. *Let f be a real analytic LLD meromorphic function of finite order such that $f(1) = 1$. The higher Gamma functions associated to f is a family $(\Gamma_N^f)_{N\geq 0}$ satisfying the following properties:*

(1) $\Gamma_0^f(s) = f(s)$,
(2) $\Gamma_N^f(1) = 1$,
(3) $\Gamma_{N+1}^f(s+1) = \Gamma_N^f(s)^{-1}\,\Gamma_{N+1}^f(s)$, *for* $N \geq 0$,
(4) Γ_N^f *is a meromorphic function of finite order,*
(5) Γ_N^f *is LLD,*
(6) Γ_N^f *is real analytic.*

Theorem 3.2. *Let f be a real analytic LLD meromorphic function of finite order such that $f(1) = 1$. There exists a unique family of higher Gamma functions $(\Gamma_N^f)_N$ associated to f. If f is CLD then the Γ_N^f are CLD.*

Proof. We set $\Gamma_0^f(s) = f(s)$, and for $N \geq 0$, the function Γ_{N+1}^f is constructed from $1/\Gamma_N^f$ using Theorem 2.2, and is unique. ☐

The uniqueness property implies the following multiplicative group morphism property:

Corollary 3.3. *For $N \geq 0$, we consider the map $\Gamma_N : \mathcal{E}_0 \to \mathcal{E}_0$ defined by $\Gamma_N(f) = \Gamma_N^f$. Then Γ_N is a continuous injective group morphism.*

Proof. Given $f, g \in \mathcal{E}_0$, it is clear that the sequence of functions $\Gamma_N^f.\Gamma_N^g$ satisfy all the properties of higher Gamma functions associated to fg, hence, by uniqueness, we have $\Gamma_N^{fg} = \Gamma_N^f\Gamma_N^g$, hence the group morphism property. The kernel is reduced to the constant function 1 by uniqueness, hence the injectivity. The continuity follows as before from Theorem 2.10. ☐

Definition 3.4 (Barnes higher Gamma functons Γ_N)**.** *The higher Gamma functions associated to $f(s) = s$ is the family of higher Barnes Gamma functions $(\Gamma_N)_{N\geq 0}$, and Γ_1 is Euler Gamma function.*

Note that Vignéras' normalization (1979, [28]) is slightly different and defines (for $f(s) = s$) a hierarchy of functions $(G_N^f)_{N\geq 0}$ as in Definition 3.1 but with the functional equation replaced by

$$G_{N+1}^f(s+1) = G_N^f(s)G_{N+1}^f(s)$$

We have a simple direct relation between the two hierarchies

$$G_N^f = (\Gamma_N^f)^{(-1)^{N+1}} \ .$$

For $f(s) = s$ we obtain $G_2^f = G$ which is Barnes G-function (Barnes, 1900, [3]). The convention in Definition 3.1 is compatible with Barnes multiple Gamma functions that generalize the (Γ_N) (Barnes, 1904, [4], and Section 6).

Proposition 3.5. *The higher Barnes Gamma function Γ_N is CLD of order N, and*

$$\mathrm{Div}(\Gamma_N) = -\sum_{n=0}^{+\infty} \binom{n+N-1}{N-1} . (-n)$$

Proof. The function Γ_N is in the class CLD by induction since f is in this class. Any $\alpha > 0$ is exponent of convergence for $f(s) = s$, so by Proposition 2.6 we have by induction that any $\alpha > N$ is exponent of convergence for Γ_N. We can check this directly using the formula for the divisor that follows by induction from Lemma 2.3 and the combinatorial identity

$$\binom{n+N}{N} = \sum_{k=0}^{n} \binom{k+N-1}{N-1}$$

If we write the Weierstrass factorization and Q_N denotes the Weierstrass polynomial, we have that $\deg Q_1 = 1$, and by induction the same proof gives that $\deg Q_N = N$. □

When we drop the real analyticity condition, there is no longer uniqueness, but we can prove the following Theorem,

Theorem 3.6. *Let f be a LLD meromorphic function of finite order such that $f(1) = 1$. Consider a family $(g_N^f)_{N\geq 0}$ satisfying the following properties:*

1. (1) $g_0^f(s) = f(s)$,
2. (2) $g_N^f(1) = 1$,
3. (3) $g_{N+1}^f(s+1) = g_N^f(s)^{-1} g_{N+1}^f(s)$, *for* $N \geq 0$,
4. (4) g_N^f *is a meromorphic function of finite order,*
5. (5) g_N^f *is LLD,*

Then there exists an integer sequence $(a_k)_{k\geq 0}$, such that

$$g_N^f(s) = \exp\left(2\pi i \sum_{k=0}^{N} a_{N-k} \binom{s}{k}\right) \Gamma_N^f(s)$$

Proof. This follows by induction from Theorem 2.7. We can also give a direct argument using the group structure. For any solution $(g_N^f)_{N\geq 0}$, the functions $h_N^f = \Gamma_N^f / g_N^f$

are solution for $f = 1$. The case $f = 1$ is easily resolved. By induction, the solutions have no zeros nor poles, and finite order, so we have

$$h_N^f(s) = e^{2\pi i A_N(s)}$$

where the $(A_N)_{N\geq 0}$ is a sequence of polynomials satisfying

$$\Delta A_{N+1} = -A_N$$

and $A_0(s) = a_0 \in \mathbb{Z}$. The difference equation and the sequence $a_N = (-1)^N A_N(0)$ determines the sequence of polynomials $(A_N)_{N\geq 0}$ that are given by the explicit formula

$$A_N(s) = \sum_{k=0}^{N} a_{N-k} \binom{s}{k}$$

□

4. Application: Jackson q-Gamma function.

For $0 < q < 1$, Jackson (1905, [10], [11]) (see also the precursor work by Halphen [7], vol. 1, p. 240; and Hölder [8]) defined the q-Gamma function Γ_q by the product formula

$$\Gamma_q(s) = \frac{(q;q)_\infty}{(q^s;q)_\infty}(1-q)^{1-s}$$

where the ∞-Pochhammer symbol is

$$(z;q)_\infty = \prod_{k=0}^{+\infty}(1-zq^k) \ .$$

The q-Gamma function satisfies the functional equation

$$\Gamma_q(s+1) = \frac{1-q^s}{1-q}\,\Gamma_q(s)$$

and Euler Gamma function appears as the limit when $q \to 1$,

$$\Gamma(s) = \lim_{q\to 1-0} \Gamma_q(s)$$

Askey ([1], 1980) proved a q-analog of the Bohr-Mollerup theorem characterizing Γ_q by its functional equation, the normalization $\Gamma_q(1) = 1$, and the real log-convexity of Γ_q. It is natural to investigate if we can use our approach. The answer is affirmative as shows the next Theorem.

Theorem 4.1. *The q-Gamma function is the only real analytic, finite order meromorphic function such that $\Gamma_q(1) = 1$ and satisfying the functional equation,*

$$\Gamma_q(s+1) = \frac{1-q^s}{1-q}\,\Gamma_q(s)$$

Proof. This is an application of our general Theorem 2.2 with

$$f(s) = \frac{1-q^s}{1-q}$$

which is an order 1 real analytic function in the class LLD (but not CLD), $f(1) = 1$, and

$$\text{Div}(f) = \sum_{k\in\mathbb{Z}} 1. \left(\frac{2\pi i k}{\log q}\right) .$$

□

An application of the continuity Theorem 2.10 shows:

Proposition 4.2. *We have*

$$\lim_{q\to 1-0} \Gamma_q = \Gamma$$

uniformly on compact sets of $\mathbb{C}$*.*

Proof. Uniformly on compact sets of $\mathbb{C}$ we have

$$\lim_{q\to 1-0} \frac{1-q^s}{1-q} = s$$

and we use Theorem 2.10. □

Nishiwaza (1996, [16]) has defined the q-analog $\Gamma_{N,q}$ of Barnes higher Gamma functions Γ_N following the Bohr-Mollerup approach. With our methods we can obtain Nishiwaza's $\Gamma_{N,q}$ functions directly from the higher hierarchy generated by f using Definition 3.1 and Theorem 3.2 using the uniqueness of the solution.

Theorem 4.3. *Nishiwaza's higher q-Gamma functions $\Gamma_{N,q}$ are obtained by the higher hierarchy from Theorem 3.2*

$$\Gamma_{N,q} = \Gamma_N^f$$

associated to the real analytic function

$$f(s) = \frac{1-q^s}{1-q} .$$

5. Application: Mellin Gamma functions.

Mellin (1897, [13]) considered general Gamma functions satisfying the functional equation

$$F(s+1) = R(s)F(s)$$

where R is a rational function. He constructs solutions by using Euler Gamma function as building block along the divisor. An application of the extension of our general

Theorem 2.7, and the group structure Theorem 2.11, gives the precise existence characterization of Mellin Gamma functions.

Definition 5.1. *A meromorphic function f is LLD at infinite if $f(s+a)$ is LLD for some $a \in \mathbb{R}$.*

Since $\mathrm{Div}(f(s+a)) = \mathrm{Div}(f) - a$ this means that the divisor of f is in some left half plane (not necessarily $\mathbb{C}_+$).

Theorem 5.2. *Let R be a rational function,*

$$R(s) = a\frac{(s-\alpha_1)\dots(s-\alpha_n)}{(s-\beta_1)\dots(s-\beta_m)}$$

where $a \in \mathbb{C}^*$, and (α_k) and (β_k) are the zeros, resp. the poles, of R counted with multiplicity.

Consider the finite order meromorphic functions, LLD at infinite, that are solutions of the functional equation

$$F(s+1) = R(s)F(s) \ . \tag{10}$$

They are of the form

$$F(s) = a^s \frac{\Gamma(s-\alpha_1)\dots\Gamma(s-\alpha_n)}{\Gamma(s-\beta_1)\dots\Gamma(s-\beta_m)} e^{2\pi iks}$$

for some $k \in \mathbb{Z}$.

In particular, if $R(1) = 1$ and R is real analytic there is only one real analytic solution such that $F(1) = 1$.

Proof. Let α be a zero or pole. We consider the linear function $f_\alpha(s) = s - \alpha$ and a solution Γ^{f_α} to

$$F_\alpha(s+1) = f_\alpha(s)F_\alpha(s+1) \ .$$

Also a^s is a solution to $F(s+1) = aF(s)$. Then, Theorem 2.7 and the group structure of the solutions, Theorem 2.11, shows that the general solutions of the functional equation (10) are of the form

$$\begin{aligned} F(s) &= a^s e^{2\pi ins} \frac{\Gamma^{f_{\alpha_1}}(s)e^{2\pi ik_1 s}\dots\Gamma^{f_{\alpha_n}}(s)e^{2\pi ik_n s}}{\Gamma^{f_{\beta_1}}(s)e^{2\pi il_1 s}\dots\Gamma^{f_{\beta_m}}(s)e^{2\pi il_m s}} \\ &= a^s \frac{\Gamma^{f_{\alpha_1}}(s)\dots\Gamma^{f_{\alpha_n}}(s)}{\Gamma^{f_{\beta_1}}(s)\dots\Gamma^{f_{\beta_m}}(s)} e^{2\pi iks} \end{aligned}$$

where $n, k_1, \dots, k_n, l_1, \dots, l_m \in \mathbb{Z}$, and $k = n + k_1 + \dots k_n + l_1 + \dots + l_m$.

We finish the proof by observing that we can take $\Gamma^{f_\alpha}(s) = \Gamma(s-\alpha)$. When R is real analytic, $a \in \mathbb{R}^*$, the set of roots (α_j) and poles (β_j) are self-conjugated, and we must have $k = 0$ to have F real analytic. □

Considering a LLD rational function R, real analytic and such that $R(1) = 1$, we can define the unique associated higher Gamma functions $(\Gamma_N^R)_{N\geq 0}$ given by Theorem 3.2. These higher Mellin Gamma functions do not seem to appear in the literature.

6. Application: Barnes multiple Gamma functions.

For $N \geq 1$ and parameters $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) \in \mathbb{C}_+^n$, Barnes multiple Gamma functions $\Gamma(s|\omega_1, \ldots, \omega_N) = \Gamma(s|\boldsymbol{\omega})$ are a generalization by Barnes (1904, [4]) of Barnes higher Gamma functions Γ_N studied in section 3. When $\omega_1 = \ldots = \omega_N = 1$ we recover Γ_N as

$$\Gamma_N(s) = \Gamma(s|1, \ldots, 1)$$

Barnes only considers the apparently more general case where $\omega_1, \ldots, \omega_n$ all belong to a half plane limited by a line through the origin ([4] p.387). This situation that can be reduced to our case by a rotation. Also, he assumes $\dim_{\mathbb{Q}}(\omega_1, \ldots, \omega_N) \geq 3$ to have an essentially different situation from the double Gamma function G that he studied previously, although this condition is not the appropriate one. Barnes defines these multiple Gamma functions *à la Lerch*. First, Barnes defines the Barnes-Hurwitz zeta functions, a multiple version of Hurwitz zeta function, as

$$\zeta(t, s|\omega_1, \ldots, \omega_N) = \sum_{k_1, \ldots k_N \geq 0} (s + k_1\omega_1 + \ldots + k_N\omega_N)^{-t} ,$$

which is converging for $\operatorname{Re} s > N$, and symmetric on $\omega_1, \ldots, \omega_N$. This multiple zeta function reduces to Hurwitz zeta function for $N = 1$ (Hurwitz, 1882, [9]). Its analytic continuation and Lerch formula (Lerch, 1894, [12])

$$\log \Gamma(s) = \left[\frac{\partial}{\partial t}\zeta(t, s)\right]_{t=0} - \zeta'(0) \tag{11}$$

allows to define Euler Gamma function. Barnes generalizes this approach and he shows, using a Hankel type integral, that $\zeta(s, t|\omega_1, \ldots, \omega_N)$ has a meromorphic extension in (s, t). Then he defines

$$\Gamma_B(s|\boldsymbol{\omega}) = \rho_N(\boldsymbol{\omega}) \exp\left(\left[\frac{\partial}{\partial t}\zeta(t, s|\boldsymbol{\omega})\right]_{t=0}\right)$$

where $\rho_N(\boldsymbol{\omega})$ is Barnes modular function, and is defined to provide the normalization such that $\Gamma_B(s|\boldsymbol{\omega})$ has residue 1 at $s = 0$,

$$\text{Res}_{s=0}\Gamma_B(s|\boldsymbol{\omega}) = \lim_{s\to 1} s\Gamma_B(s|\boldsymbol{\omega}) = 1 \tag{12}$$

From the definition we get that both $\rho_N(\boldsymbol{\omega})$ and $\Gamma_B(s|\boldsymbol{\omega})$ are necessarily symmetric on $\omega_1, \ldots, \omega_N$. Note that for Euler Gamma function, because of the form of the functional equation, the normalization $\Gamma(1) = 1$ is equivalent to $\text{Res}_{s=0}\Gamma = 1$. In general, for Γ^f the normalization $\Gamma^f(1) = 1$ is equivalent to

$$\text{Res}_{s=0}\Gamma^f = \text{Res}_{s=0} f^{-1} \ .$$

For Barnes higher Gamma functions Γ_N discussed in section 3, we see that the normalization $\Gamma_N(1) = 1$ is equivalent to $\text{Res}_{s=0}\Gamma_N = 1$ when we make $s \to 0$ in

$$\Gamma_{N+1}(s+1) = (s\Gamma_N(s))^{-1} \ s\Gamma_{N+1}(s)$$

we get

$$\Gamma_{N+1}(1) = \text{Res}_{s=0}\Gamma_{N+1} = 1 \ .$$

and the result follows by induction.

Barnes ([4], p.397) observes that $\log \rho(\boldsymbol{\omega})$ plays the role of Stirling's constant of the asymptotic expansion when $k \to +\infty$ of the divergent sum

$$\sum_{\omega \in \Omega^*, |\omega| \leq k} \log |\omega|$$

where $\Omega^* = \mathbb{N}.\omega_1 + \mathbb{N}.\omega_2 + \ldots + \mathbb{N}.\omega_N - \{0\}$. In this way, $\log \rho(\boldsymbol{\omega})$ can also be defined.

Modern presentations ([22]) Later applications to number theory of Barnes multiple Gamma functions and generalizations by Shintani (1976,[25], [26], [27]) drop Barnes normalization. They define multiple Gamma functions directly by the formula

$$\Gamma(s|\boldsymbol{\omega}) = \exp\left(\left[\frac{\partial}{\partial t}\zeta(t, s|\boldsymbol{\omega})\right]_{t=0}\right)$$

We keep Shintani's normalization that has become the usual one in recent articles. This modern normalization has the advantage to yield a simpler functional equation not involving Barnes modular function $\rho(\boldsymbol{\omega})$. For $\omega = \omega_k$, we denote $\hat{\boldsymbol{\omega}}$ the $N-1$ dimensional vector obtained from $\boldsymbol{\omega}$ removing the k-th coordinate. Then we have the following ladder functional equation for the zeta function,

$$\zeta(t, s+\omega|\boldsymbol{\omega}) - \zeta(t, s|\boldsymbol{\omega}) = -\zeta(t, s|\hat{\boldsymbol{\omega}}) \tag{13}$$

where we start with

$$\zeta(t, s|\emptyset) = s^{-t} \ .$$

From the zeta function functional equation we get the functional equation for the multiple Gamma functions,

$$\Gamma(s+\omega|\boldsymbol{\omega}) = \Gamma(s|\hat{\boldsymbol{\omega}})^{-1}\Gamma(s|\boldsymbol{\omega}) \tag{14}$$

with the convention $\Gamma(s|\emptyset) = s$. Note that the functional equation for Barnes normalized multiple Gamma functions is different:

$$\Gamma_B(s+\omega|\boldsymbol{\omega}) = \rho(\hat{\boldsymbol{\omega}})\Gamma_B(s|\hat{\boldsymbol{\omega}})^{-1}\Gamma_B(s|\boldsymbol{\omega}) \ . \tag{15}$$

Example 6.1. For $N = 1$, $\Gamma(s|\omega)$ can be computed explicitly from Euler Gamma function (see [24], p.203).

Lemma 6.2. *We have*

$$\begin{aligned}\Gamma(s|\omega) &= (2\pi)^{-1/2} e^{\left(\frac{s}{\omega}-\frac{1}{2}\right)\log\omega}\,\Gamma\left(\frac{s}{\omega}\right)\\ \rho_1(\omega) &= \sqrt{\frac{\omega}{2\pi}}\end{aligned}$$

and therefore

$$\Gamma_B(s|\omega) = \sqrt{\frac{2\pi}{\omega}}\,\Gamma(s|\omega) = e^{\left(\frac{s}{\omega}-1\right)\log\omega}\,\Gamma\left(\frac{s}{\omega}\right)$$

and

$$\begin{aligned}\Gamma(\omega|\omega) &= \sqrt{\frac{\omega}{2\pi}}\\ Res_{s=0}\Gamma(s|\omega) &= \sqrt{\frac{\omega}{2\pi}}\\ \Gamma_B(\omega|\omega) &= 1\\ Res_{s=0}\Gamma_B(s|\omega) &= 1\end{aligned}$$

In particular,

$$\begin{aligned}\Gamma(s|1) &= \frac{\Gamma(s)}{\sqrt{2\pi}}\\ \Gamma_B(s|1) &= \Gamma(s)\end{aligned}$$

Proof. For $\omega = 1$, $\zeta(t,s|1) = \zeta(t,s)$ is the original Hurwitz zeta function that generalizes Riemann zeta function $\zeta(t) = \zeta(t,1)$,

$$\zeta(t,s) = \sum_{k\geq 0}(s+k)^{-t} \ .$$

Making $t = 0$ in the first formula from Lemma 3.18 from [19] we have the classical result (see also [29] p.267)

$$\zeta(0,s) = \frac{1}{2} - s \ . \tag{16}$$

Observe now that we have $\zeta(t,s|\omega)=\omega^{-t}\zeta\left(t,\frac{s}{\omega}\right)$, hence

$$\frac{\partial}{\partial t}\zeta(t,s|\omega)=-(\log\omega)\omega^{-t}\zeta\left(t,\frac{s}{\omega}\right)+\omega^{-t}\frac{\partial}{\partial t}\zeta(t,s)$$

and making $t=0$, using formula (16) and Lerch formula (11), we get

$$\log\Gamma(s|\omega)=\left(\frac{s}{\omega}-\frac{1}{2}\right)\log\omega+\log\Gamma\left(\frac{s}{\omega}\right)+\zeta'(0)\ .$$

Now, $\zeta'(0)=-\frac{1}{2}\log(2\pi)$ gives the first formula. Then using this formula we get

$$\mathrm{Res}_{s=0}\Gamma(s|\omega)=\lim_{s\to 1}s\Gamma(s|\omega)=(2\pi)^{-1/2}e^{-\frac{1}{2}\log\omega}\omega=\sqrt{\frac{\omega}{2\pi}}\ .$$

□

For $N\geq 2$ we create new transcendentals $\Gamma(s|\boldsymbol{\omega})$, which are not generated from Euler Gamma function. For example for $N=2$, if ω_1 and ω_2 are $\mathbb{Q}$-independent we get new transcendentals. When the parameters are $\mathbb{Q}$-dependent then $\Gamma(s|\omega_1\omega_2)$ can be expressed from Barnes G-function, $G_2=\Gamma_2^{-1}$.

From the functional equations, and from our point of view, it is natural to aim to characterize $\Gamma(s|\boldsymbol{\omega})$ by solving a tower of difference equations corresponding to the sequence $(\omega_k)_{1\leq k\leq n}$. Our approach leads to a new definition, not needing Barnes-Hurwitz zeta functions. We start by considering real analytic multiple zeta functions that are those relevant in Shintani's applications to real quadratic number fields (1978, [26]). The following result follows from Theorem 2.7.

Theorem 6.3. *Let $\omega\in\mathbb{R}_+$. Let f be a real analytic LLD meromorphic function in $\mathbb{C}$ of finite order. There exists a unique function $\Gamma^f(s|\omega)$ satisfying the following properties:*

(1) $\Gamma^f(1|\omega)=1$,
(2) $\Gamma^f(s+\omega|\omega)=f(s)\Gamma^f(s|\omega)$,
(3) $\Gamma^f(s|\omega)$ *is a meromorphic function of finite order,*
(4) $\Gamma^f(s|\omega)$ *is LLD,*
(5) Γ^f *is real analytic.*

If f is CLD then Γ^f is CLD.

If we drop condition (1) then $\Gamma^f(s|\omega)$ is unique up to multiplication by a constant $c\in\mathbb{R}^*$.

If $Res_{s=0}f^{-1}=1$, we can replace condition (1) by the condition $Res_{s=0}\Gamma^f=1$.

Proof. We make the change of variables $t = \omega^{-1}s$. The application of Theorem 2.7 to the real analytic function $h(t) = f(\omega t)$ gives a unique real analytic solution $\Gamma^h(t)$ such that $\Gamma^h(1) = 1$ and

$$\Gamma^h(t+1) = h(t)\Gamma^h(t) \ .$$

If we set $\Gamma^f(s|\omega) = \Gamma^h(\omega^{-1}s)$, this equation becomes

$$\Gamma^f(s+\omega|\omega) = \Gamma^h(\omega^{-1}s+1) = h(\omega^{-1}s)\Gamma^h(\omega^{-1}s) = f(s)\Gamma^f(s|\omega)$$

and $\Gamma^f(s|\omega)$ satisfies all conditions. Furthermore, $\Gamma^f(s|\omega)$ is unique from the uniqueness of Γ^h that follows from the last uniqueness condition in Theorem 2.7. In view of this uniqueness result, the two last statement are clear. Also if f is CDL then $\Gamma^f(s|\omega)$ is CDL. □

Example 6.4. For $f(s) = s$ the proof gives $h(t) = \omega t$ and a solution $\Gamma^f(s|\omega) = \Gamma^h\left(\frac{t}{\omega}\right)$. The condition $\Gamma^f(1|\omega) = 1$ is equivalent to $\Gamma^h\left(\frac{1}{\omega}\right) = 1$, then according to Example 2.8 there is a unique real analytic solution

$$\Gamma^h(t) = e^{(t-1)\log\omega} \, \frac{\Gamma(t)}{\Gamma(\omega^{-1})}$$

and it follows that

$$\Gamma^f(s|\omega) = e^{\left(\frac{s}{\omega}-1\right)\log\omega} \, \frac{\Gamma\left(\frac{s}{\omega}\right)}{\Gamma(\omega^{-1})}$$

Therefore, by uniqueness of the normalization,

$$\Gamma_B(s|\omega) = \Gamma(\omega^{-1})\Gamma^f(s|\omega)$$

and we recover the formula for $\Gamma_B(s|\omega)$ from Lemma 6.2

$$\Gamma_B(s|\omega) = e^{\left(\frac{s}{\omega}-1\right)\log\omega} \, \Gamma\left(\frac{s}{\omega}\right)$$

Then the formula for $\Gamma(s|\omega)$ follows from

$$\Gamma(s|\omega) = \sqrt{\frac{\omega}{2\pi}}\,\Gamma_B(s|\omega) = (2\pi)^{-1/2} e^{\left(\frac{s}{\omega}-\frac{1}{2}\right)\log\omega}\,\Gamma\left(\frac{s}{\omega}\right) \ .$$

We have established,

Proposition 6.5. *For $f(s) = s$ we have*

$$\Gamma(s|\omega) = \sqrt{\frac{\omega}{2\pi}}\,\Gamma(\omega^{-1})\,\Gamma^f(s|\omega)$$

where $\Gamma^f(s|\omega)$ is the unique solution in Theorem 6.3.

Using similar ideas, the general version of Theorem 2.7 for $\omega \in \mathbb{C}_+$ and without the hypothesis of f being real analytic is the following:

Theorem 6.6. *Let $\omega \in \mathbb{C}_+$. Let f be a LLD meromorphic function in $\mathbb{C}$ of finite order. We consider a function g satisfying*

(1) $g(1) = 1$,
(2) $g(s+\omega) = f(s)g(s)$,
(3) *g is a meromorphic function of finite order,*
(4) *g is LLD,*

Then there is a solution $\Gamma^f(s|\omega)$. Any other solution g is of the form $g(s) = e^{2\pi i a \frac{s-1}{\omega}} \Gamma^f(s|\omega)$ for some $a \in \mathbb{Z}$.

If we remove condition (1) then all solutions are of the form the form $g(s) = e^{b+2\pi i a \frac{s}{\omega}} \Gamma^f(s|\omega)$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{C}$.

Proof. As before, we make the change of variables $t = \omega^{-1}s$ and apply Theorem 2.7 to the function $h(t) = f(\omega t)$ gives an unconditional solution $\Gamma^f(s|\omega) = \Gamma^h(t)/\Gamma^h(\omega^{-1})$. From the general uniqueness statement in 2.7 we know that all the other solutions removing condition (1) are of the form $g(s) = e^{2\pi i a \frac{s}{\omega} + b} \Gamma^f(s|\omega)$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{C}$. Condition (1) is then equivalent to $2\pi i a/\omega + b = 2\pi i k$ with $k \in \mathbb{Z}$, hence the general form. □

Therefore, in general for $\omega \in \mathbb{C}^*$, Γ^f is not uniquely determined, but its values on $1 + \mathbb{Z}.\omega$ are well determined. More precisely, we have

Proposition 6.7. *The values taken by solutions at the points $1 + k\omega$ for $k \in \mathbb{Z}$ are uniquely determined and do not depend on the solution chosen.*

If $\omega \in \mathbb{C}_+$, any solution g is uniquely determined by $\operatorname{Im} g'(1)$, in particular, if f is real analytic then there is a unique real analytic solution.

If $\dim_{\mathbb{Q}}(1,\omega) = 2$, any solution g is uniquely determined by its value $g(k)$ for some integer $k \geq 2$.

Proof. From the functional equation we have

$$g(1+k\omega) = g(1)\prod_{j=0}^{k-1} f(1+j\omega) = \prod_{j=0}^{k-1} f(1+j\omega)$$

hence the first claim.

Now, consider two solutions g_1 and g_2 such that $\operatorname{Im} g_1'(1) = \operatorname{Im} g_2'(1)$. Since they are of the form $g_j(s) = e^{2\pi i a_j \frac{s-1}{\omega}} \Gamma^f(s|\omega)$ for some $a_j \in \mathbb{Z}$, taking logarithmic derivatives we have

$$g_j'(1) = \frac{g_j'(1)}{g_j(1)} = 2\pi i \frac{a_j}{\omega} + \frac{\left(\Gamma^f\right)'(1|\omega)}{\Gamma^f(1|\omega)} = 2\pi i \frac{a_j}{\omega} + \left(\Gamma^f\right)'(1|\omega)$$

hence

$$g_1'(1) - g_2'(1) = 2\pi i \, \frac{a_1 - a_2}{\omega} \in \mathbb{R}$$

and the condition $\omega \in \mathbb{C}_+$ forces $a_1 = a_2$.

Now assume $\dim_{\mathbb{Q}}(1, \omega) = 2$ and consider two solutions g_1 and g_2 such that $g_1(k) = g_2(k)$ for some integer $k \geq 2$. Then, since $s = k$ is neither a zero nor a pole, we have for some $l \in \mathbb{Z}$

$$\frac{g_1(k)}{g_2(k)} = e^{2\pi(k-1)\frac{a_1-a_2}{\omega}} = 1$$

thus, for some integer $l \in \mathbb{Z}$, we have

$$(k-1)(a_1 - a_2) - l\omega = 0$$

and, by $\mathbb{Q}$-independence, we must have $l = 0$ and $(k-1)(a_1 - a_2) = 0$, thus, since $k \geq 2$, $a_1 = a_2$ and $g_1 = g_2$. □

General Multiple Gamma Hierarchies.

Now, we can iterate Theorem 6.3 to define new real-analytic multiple Gamma function corresponding to f and positive real parameters $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_N) \in \mathbb{R}_+^N$

For a sequence of parameters $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots) \in \mathbb{C}_+^\infty$, we can now define a generalization of Barnes multiple Gamma hierarchy. We denote $\boldsymbol{\omega}_N = (\omega_1, \ldots, \omega_N) \in \mathbb{C}_+^N$.

Definition 6.8 (General Multiple Gamma Hierarchy)**.** *Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots) \in \mathbb{C}_+^\infty$ and f be a LLD meromorphic function in $\mathbb{C}$ of finite order. A general multiple Gamma hierarchy $(\Gamma_N^f(s|\boldsymbol{\omega}_N))_{N\geq 0}$ associated to f is a sequence of functions satisfying:*

(1) $\Gamma_0^f(s) = f(s)$,
(2) $\Gamma_{N+1}^f(s + \omega_{N+1}|\boldsymbol{\omega}_{N+1}) = \Gamma_N^f(s|\boldsymbol{\omega}_N)^{-1}\, \Gamma_{N+1}^f(s|\boldsymbol{\omega}_{N+1})$, *for* $N \geq 0$,
(3) $\Gamma_N^f(s|\boldsymbol{\omega}_N)$ *is a meromorphic function of finite order,*
(4) $\Gamma_N^f(s|\boldsymbol{\omega}_N)$ *is LLD.*

Next we show that, with some simple normalization, General Multiple Gamma Hierarchies are unique for real parameters and f real analytic.

Theorem 6.9. *Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots) \in \mathbb{R}_+^\infty$ and f a real analytic LLD meromorphic function of finite order, such that $f(1) = 1$. There exists a unique General Multiple Gamma Hierarchy $(\Gamma_N^f(s|\boldsymbol{\omega}_N))_{N\geq 1}$ associated to f, and normalized such that*

$$\Gamma_N^f(1|\boldsymbol{\omega}_N) = 1 \ .$$

If f is CLD then the $\Gamma_N^f(s|\boldsymbol{\omega}_N)$ are CLD.

Proof. The existence and uniqueness is proved by induction on $N \geq 0$. For $N = 0$, $\Gamma_0^f(s) = f(s)$. We assume that the result has been proved for $N \geq 0$. Then we construct $\Gamma_{N+1}^f(s|\boldsymbol{\omega}_{N+1})$ by using Theorem 6.3 using the function $f = \Gamma_N^f(s|\boldsymbol{\omega}_N)^{-1}$. □

The particular case $f(s) = s$, using uniqueness, yields Barnes multiple Gamma functions for real parameters $\boldsymbol{\omega}$.

Definition 6.10 (Barnes multiple Gamma functions)**.** *For $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots) \in \mathbb{R}_+^\infty$ the General Multiple Gamma Hierarchy associated to $f(s) = s$ is Barnes Multiple Gamma Hierarchy $(\Gamma_N^f(s|\boldsymbol{\omega}_N))_{N\geq 1}$ with the normalization $\Gamma_N^f(1|\boldsymbol{\omega}_N) = 1$. We simplify the notation and we denote $\Gamma_N^f(s|\boldsymbol{\omega}_N) = \Gamma(s|\boldsymbol{\omega}_N)$.*

We observe that since the Barnes multiple Gamma functions $\Gamma(s|\boldsymbol{\omega}_N)$ are symmetric on the real parameters $(\omega_1, \ldots, \omega_N)$ then, by uniqueness, the solutions of Theorem 6.9 for $f(s) = s$ must also be symmetric on the parameters. This is general when we can define the Gamma functions à la Lerch, including the case of complex parameters $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots) \in \mathbb{C}_+^\infty$. Consider f a real analytic LLD meromorphic function of finite order, such that,

$$f(1) = 1$$

and $\operatorname{Re} f(s) > 0$ for $s \in \mathbb{C}_+$. These conditions are sufficient to define $f(s)^{-t}$ for $s \in \mathbb{C}_+$ by taking the principal branch of log in $\mathbb{C}_+$, $f(s)^{-t} = \exp(-t \log f(s))$. We assume that the multiple Barnes-Hurwitz multiple zeta function associated to f,

$$\zeta^f(t, s|\omega_1, \ldots, \omega_N) = \sum_{k_1, \ldots k_N \geq 0} f(s + k_1\omega_1 + \ldots + k_N\omega_N)^{-t} \ ,$$

is well defined and holomorphic in a right half plane $\operatorname{Re} t > t_0$ for all $s \in \mathbb{C}_+$, and has a meromorphic extension to $t \in \mathbb{C}$. We define $\Gamma^f(s|\emptyset) = f(s)^{-t}$, and, à la Lerch, for $s \in \mathbb{C}_+$,

$$\Gamma_L^f(s|\boldsymbol{\omega}_N) = \exp\left(\left[\frac{\partial}{\partial t}\zeta^f(t, s|\boldsymbol{\omega}_N)\right]_{t=0} - \left[\frac{\partial}{\partial t}\zeta^f(t, s|\boldsymbol{\omega}_N)\right]_{t=0, s=1}\right)$$

Note that we have normalized these functions such that $\Gamma_L^f(1|\boldsymbol{\omega}_N) = 1$. By construction, these functions are obviously symmetric on the parameters $\omega_1, \ldots, \omega_N$. As before, these functions satisfy the functional equations,

$$\Gamma_L^f(s + \omega_N|\boldsymbol{\omega}_N) = \Gamma_L^f(s|\boldsymbol{\omega}_{N-1})^{-1}\Gamma_L^f(s|\boldsymbol{\omega}_N) \tag{17}$$

which show that they have a meromorphic extension to all $s \in \mathbb{C}$. now, using the uniqueness from Theorem 6.9 we get for real parameters:

Theorem 6.11. *Let* $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots) \in \mathbb{R}_+^\infty$. *When* $\Gamma_L^f(s|\boldsymbol{\omega}_N)$ *is well defined, we have*

$$\Gamma_L^f(s|\boldsymbol{\omega}_N) = \Gamma_N^f(s|\boldsymbol{\omega}_N)$$

where the $(\Gamma_N^f(s|\boldsymbol{\omega}_N))_{N\geq 0}$ *are the solutions of Theorem 6.9.*

Corollary 6.12. *Let* $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots) \in \mathbb{R}_+^\infty$. *The Barnes multiple Gamma hierarchy defined by Theorem 6.9,* $\Gamma_N^f(s|\boldsymbol{\omega}_N)$ *are symmetric on the parameters* $\boldsymbol{\omega}_N = (\omega_1, \omega_2, \ldots, \omega_N)$.

We should note that our definition of the hierarchies using the functional equation is more general than the definition à la Lerch, since we need conditions on f so that the multiple f-Barnes-Hurwitz zeta function is well defined and holomorphic in a half plane. If we don't add the normalization condition

$$\Gamma_N^f(1|\boldsymbol{\omega}_N) = 1$$

then there are solutions that are non-symmetric on the parameters. As we see next, this is even more evident for complex parameters since in that case, without further hypothesis, there is no symmetry on the parameters $\boldsymbol{\omega}$. This shows that our functional equation approach defines a larger class of functions.

We observe also that the existence and uniqueness of Theorem 6.9 implies the morphism property. Let $\mathcal{E}^\mathbb{R}$ be the multiplicative group of real-analytic LLD meromorphic functions of finite order and

$$\mathcal{E}^\mathbb{R} = \bigcap_{n\geq 0} \mathcal{E}_n^\mathbb{R}$$

and $\mathcal{E}_0^\mathbb{R}$ the subgroup of functions f such that $f(1) = 1$. With the same arguments as before, we have

Theorem 6.13. *For* $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots) \in \mathbb{R}_+^\infty$ *and* $N \geq 0$, *we consider the map*

$$\Gamma_N(\boldsymbol{\omega}_N) : \mathcal{E}_0^\mathbb{R} \to \mathcal{E}_0^\mathbb{R}$$

defined by $\Gamma_N(\boldsymbol{\omega}_N)(f) = \Gamma_N^f(.|\boldsymbol{\omega}_N)$. *Then* $\Gamma_N(\boldsymbol{\omega}_N)$ *is a continuous injective group morphism.*

Complex parameters.

We study now the non-real-analytic case for complex parameters $\omega_1, \ldots, \omega_N \in \mathbb{C}_+$. In general we don't have uniqueness as in Theorem 6.9. We consider f a LLD meromorphic function in $\mathbb{C}$ of finite order with $f(1) = 1$ and study the question of existence and uniqueness of a general multiple Gamma functions hierarchy as in Definition 6.8 with the normalization

$$\Gamma_N^f(1|\boldsymbol{\omega}_N) = 1 \ .$$

We have the following result without the real analyticity condition:

Theorem 6.14. *Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots) \in \mathbb{C}_+^\infty$ and f a LLD meromorphic function of finite order such that $f(1) = 1$. There exists General Multiple Gamma Hierarchy $(\Gamma_N^f(s|\boldsymbol{\omega}_N))_{N\geq 0}$ associated to f, and for any other hierarchy $(\tilde{\Gamma}_N^f(s|\boldsymbol{\omega}_N))_{N\geq 0}$ there exists a sequence of polynomials $(P_N)_{N\geq 1}$ such that*

$$\tilde{\Gamma}_N^f(s|\boldsymbol{\omega}_N) = \exp\left(2\pi i P_N(s)\right) \Gamma_N^{\boldsymbol{f}}(s|\boldsymbol{\omega}_N)$$

with $P_N(1) \in \mathbb{Z}$, P_0 is a constant integer, and for $N \geq 0$ we have

$$\Delta_{\omega_{N+1}} P_{N+1} = -P_N$$

where Δ_ω is the ω-difference operator $\Delta_\omega P = P(s+\omega) - P(s)$. The space of polynomials P_N is isomorphic to $\mathbb{Z}^{N+1}$.

If the functions f is CLD then $\Gamma_N^{\boldsymbol{f}}(s|\boldsymbol{\omega}_N)$ and all the other solutions are CLD.

Proof. For the existence result, it is the same proof by induction as for Theorem 6.9 (without the normalization condition) and using Theorem 6.6. If a second solution $(\tilde{\Gamma}_N^f(s|\boldsymbol{\omega}_N))_{N\geq 1}$ exists, then $(\tilde{\Gamma}_N^f(s|\boldsymbol{\omega}_N)/\Gamma_N^f(s|\boldsymbol{\omega}_N))_{N\geq 1}$ is a solution of the problem for the constant function $f(s) = 1$. The solution for $f(s) = 1$ has no divisor and is of finite order, hence they are of the form $\exp(P_N)$ where P_N are polynomial which satisfy the above difference equations. Next in what follows, we discuss uniqueness conditions and the structure of the general polynomials P_N will become clear. □

We observe that the integer sequence $(P_N(1))_{N\geq 1}$ and the difference equation determine uniquely the sequence of polynomials $(P_N)_{N\geq 1}$. To simplify the recurrence, we write $Q_N(s) = (-1)^N P_N(s-1)$ and $a_N = Q_N(0)$. The polynomials (Q_N) satisfy the difference equations

$$\Delta_{\omega_{N+1}} Q_{N+1} = Q_N \ .$$

We define the ω-descending factorial that form a triangular bases for the action of the operator Δ_ω on polynomials.

Definition 6.15. *Let $\omega \in \mathbb{C}^*$. For $s \in \mathbb{C}$ and for an integer $k \geq 1$, we define the ω-descending factorial as*

$$s^{[k,\omega]} = s(s-\omega)\ldots(s-(k-1)\omega)$$

For $\omega = 1$ we get the usual descending factorial. A simple computation shows:

Proposition 6.16. *We have*

$$\Delta_\omega s^{[k+1,\omega]} = (k+1)\omega s^{[k,\omega]}$$

Now we can give the general structure of the solutions (Q_N).

Proposition 6.17. *For $N \geq 1$, we have*

$$Q_N(s) = \sum_{k=0}^{N} \frac{a_{N-k}}{\omega_1\omega_2\dots\omega_k}\left[\frac{s^{[k,\omega_N]}}{k!} + A_{N,k}(\omega_1,\dots,\omega_N,s)\right]$$

where the $A_{N,k}$ are polynomials in $N+1$ variables and their total degree in the first N variables is strictly less than k. The coefficient a_0 is an arbitrary integer.

From this Proposition it is clear that the space of solutions Q_N, and P_N, is isomorphic to $\mathbb{Z}^{N+1}$ by the one-to-one correspondence $Q_N \mapsto (a_0, a_1, \dots, a_N) \in \mathbb{Z}^{N+1}$. The proof of this Proposition follows by induction on $N \geq 1$, solving the difference equation

$$\Delta_{\omega_{N+1}} Q_{N+1} = Q_N \ .$$

For this, we develop the polynomials

$$\frac{s^{[k,\omega_N]}}{k!} + A_{N,k}(\omega_1,\dots,\omega_N,s)$$

in the bases (s^k), then we change to the bases $(s^{[\omega_{N+1},k]})$ using the following Lemma:

Lemma 6.18. *For $n \geq 1$,*

$$s^n = \sum_{k=0}^{n} B_{n,k}(\omega) s^{[\omega,k]}$$

where $B_{n,n} = 1$, $B_{n,k} \in \mathbb{Z}[X]$ and $\deg B_{n,k} \leq n-k$.

Proof. We proceed by induction. The result is clear for $n = 1$, and developing $s^{[\omega,n]} = s(s-\omega)\dots(s-(n-1)\omega)$ we get

$$s^n = s^{[\omega,n]} - \sum_{k=1}^{n} a_k \omega^k s^{n-k}$$

and the induction hypothesis proves the result. □

Now we can study uniqueness conditions. A first result is a straightforward generalization by induction of the uniqueness result from Proposition 6.7.

Proposition 6.19. *Under the conditions as in Theorem 6.14, and if we assume that for $1 \leq n \leq N-1$, ω_{n+1} and ω_n are $\mathbb{Q}$-independent, then the hierarchy up to $N \geq 1$, $(\Gamma_n^f(s|\boldsymbol{\omega}_n))_{1\leq n\leq N}$ is uniquely determined by its values $(\Gamma_n^f(k|\boldsymbol{\omega}_n))_{1\leq n\leq N}$ at some integer $k \geq 2$.*

If we assume some algebraic independence of the parameters, we have a much stronger result.

Theorem 6.20. *Under the same conditions as in Theorem 6.14, and if we assume that for* $1 \leq n \leq N$,

$$[\mathbb{Q}[\omega_1, \ldots, \omega_n] : \mathbb{Q}[\omega_1, \ldots, \omega_{n-1}]] \geq n+1 \tag{18}$$

then the hierarchy $(\Gamma_n^f(s|\boldsymbol{\omega}_n))_{1\leq n\leq N}$ *is uniquely determined by any value* $\Gamma_N^f(k|\boldsymbol{\omega}_N)$ *at some integer point* $k \geq 2$.

Proof. If we have two solutions $(g_n(s|\boldsymbol{\omega}_n))_{1\leq n\leq N}$ and $(\tilde{g}_n(s|\boldsymbol{\omega}_n))_{1\leq n\leq N}$, the equality, $g_N(k|\boldsymbol{\omega}_N) = \tilde{g}_N(k|\boldsymbol{\omega}_N)$, at the integer $k \in \mathbb{C}_+$, that is neither a zero nor pole of the functions, shows that the corresponding polynomials Q_N and $\tilde{Q}_N$ satisfy

$$Q_N(k) - \tilde{Q}_N(k) = a \in \mathbb{Z}$$

Then, using Proposition 6.17, this gives

$$\sum_{k=0}^{N} \frac{a_{N-k} - \tilde{a}_{N-k}}{\omega_1\omega_2\ldots\omega_k} \left[\frac{s^{[k,\omega_N]}}{k!} + A_{N,k}(\omega_1, \ldots, \omega_N, s)\right] = 0$$

or, multiplying by $\omega_1\omega_2\ldots\omega_N$, we get the algebraic relation

$$(a_0 - \tilde{a}_0)\frac{s^{[N,\omega_N]}}{N!} + \ldots + (a_N - \tilde{a}_N) = 0$$

where the dots are of degre $< N$ in ω_N. The degree assumption proves that $a_0 = \tilde{a}_0$. Using the induction hypothesis on N (replacing f by $g_1(s|\omega_1) = \tilde{g}_1(s|\omega_1)$, etc), we get $a_1 = \tilde{a}_1$,..., $a_N = \tilde{a}_N$. □

Using Proposition 6.17 we can give other uniqueness results and characterizations.

To conclude this section, we note that Ruijsenaars (2000, [22]) exploited also the difference equations and their minimal solutions to prove numerous properties of Barnes multiple Gamma functions. Shintani (1976, [25]) extended Barnes approach to multiple Gamma functions to a several variable setting. Friedman and Ruijsenaars (2004, [6]) extended Shintani'smltiple Gamma functions. We can also apply our functional equation approach to define these several variables Gamma functions without Barnes-Hurwitz zeta functions and we will treat this case in a forthcoming article.

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CNRS, IMJ-PRG, Univ. Paris 7, Paris, France

E-mail address: ricardo.perez-marco@imj-prg.fr