DEGENERATE CONFORMAL STRUCTURES

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Abstract. We present new rectification theorems of degenerate quasi-conformal structures that give a meaning to quotients of Riemann surfaces with empty interior "fundamental domains". These techniques are used to define the unique renormalization of polynomials with Cantor set Julia sets.

The spirit of Riemann will move future generations as it has moved us (L.V. Ahlfors)

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1. Introduction

A quotient of a topological space by an arbitrary equivalent relation has a natural topological space structure. The quotient of a Hausdorff topological space by a Hausdorff equivalence relation is a Hausdorff topological space. Manifolds are defined as Hausdorff topological spaces endowed with a *local* smooth structure determined by an atlas. Therefore, classically, in order to quotient manifolds we need an equivalence relation with some non-local structure: *A fundamental domain.*

Even when the quotient is a topological manifold, without a fundamental domain, it seems impossible from the classical point of view to recover a natural smooth structure inherited from the original one. The main purpose of this article is to overcome this difficulty in natural specific situations *for a complex structure*. We show that some natural Hausdorff quotients of the Riemann sphere without fundamental domain led to a topological two dimensional sphere with a canonical complex structure. Some of these quotients are dynamically motivated and natural, but not all of them. In these quotients of the Riemann sphere, an open dense set of total measure can be collapsed into a set of zero measure. More precisely, the topological quotient is determined by an infinite number of holes in the Riemann sphere. These holes have piecewise analytic boundaries and they can be dense in the sphere. The quotient is then obtained by pasting analytically the boundaries of the holes in this "butchered Riemann sphere". Similar quotients exist in general Riemann surfaces, but we restrict to consider the Riemann sphere in this article that contains all the analytic complexity.

One of the main heuristic ideas behind these techniques is the belief that complex structures can be though of as a more basic structure that the topological one. This seems paradoxical. A classical complex structure determines an underlying topological structure. The heuristics is that the classical definition is only a restricted definition of complex structures to topological manifolds. The general unrestricted definition has still to be found. Many results in dynamics and geometry hint that such a general definition may exist. This philosophy is very useful in Holomorphic Dynamics, in particular in Hedgehog Theory (hedgehogs are topologically complex invariant compacts associated to indifferent periodic orbits discovered by the author [16] whose dynamics exhibits remarkable rigidity properties that are only to be expected when one thinks to the hedgehog as possesing an intrinsic complex structure). The general definition should be well behaved, i.e. immersions will induce complex structures. Also should enjoy compactness properties as those of Hausdorff-Gromov length spaces. This circle of ideas can be traced back to Riemann where the fundamental entity of complex structures permeates everywhere in the "Inaugural Dissertation".

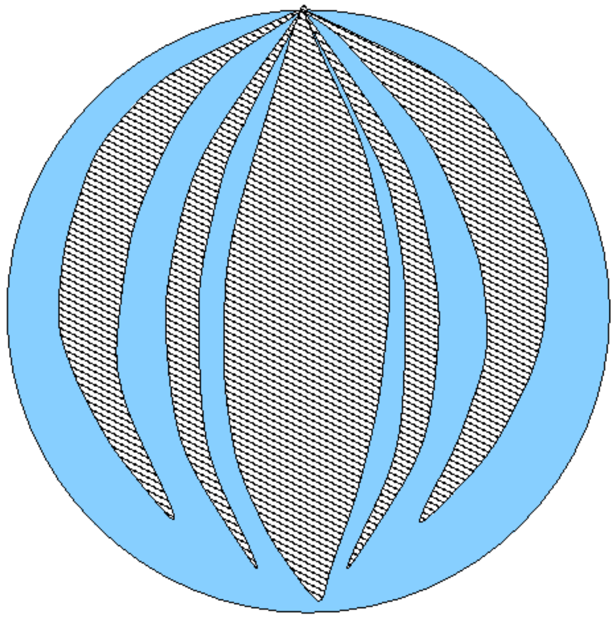

FIGURE 1. The butchered Riemann sphere.

This philosophy is a valuable guide in our problem of quotienting the butchered Riemann sphere. We have a compact set in the sphere K (the complement of the open holes), that plays the role of "fundamental domain" (but may have empty interior...) The complex structure on the Riemann sphere should induce an intrinsic complex structure on K that will induce a complex structure on the quotiented sphere. So one might expect to have a natural complex structure in the quotient. Thus also a natural absolutely continuous class in the quotiented topological sphere. Note that the absolutely continuous class will be determined by the conformal structure (and not the other way around as classically!). There seems to be a priori no natural way to recover directly the absolutely continuous class without going through the complex structure.

Because we don't have a general definition of intrinsic complex structures, we will construct the complex structure in the quotient in an indirect way. In dimension 2 complex structures can be identified with conformal structures. One of the main ideas consists in getting the wild topological quotient as a limit of quasi-conformal deformations which degenerate into the quotient. Necessarily the quasi-conformal distorsion blows out since at the limit we get topological collapsing. These quasi-conformal deformations are defined by Beltrami coefficients that can degenerate everywhere (i.e. $||\mu_n||_{L^\infty} \to 1$ locally everywhere.) Classical quasi-conformal theory is unable to provide the existence of a limit map of these rectification mappings for this wild type of collapsing. The second fundamental ingredient consists in combining classical quasi-conformal theory with classical potential theory. Roughly speaking the quasi-conformal deformations will be compatible with the potential of a compact set in the Riemann sphere (the holes in the butchered Riemann sphere are Green lines for this potential). Compactness coming from classical potential theory shows the existence of a unique topological collapsing that is the limit of the quasi-conformal deformations. The mixture of classical quasi-conformal theory and classical potential theory in order to obtain unique limits for degenerating quasi-conformal mappings is the new ingredient that provides the compactness that is lacking in the classical theory.

D. Sullivan already pointed out the analogy between Beltrami coefficients parametrizing quasi-conformal mappings and L^∞ densities parametrizing Lipschitz mappings (see [21] p.468) Pushing this analogy one might expect that degenerate Beltrami coefficients will parametrize non-quasiconformal homeomorphisms in the same way that non-atomic measures parametrize general homeorphisms. The notion of virtual conformal structures (see section 2.4) explains very accurately this analogy.

New rectifications theorems.

The first results in section 3 consist on new rectification theorems for degenerate conformal structures. In this article a "conformal structure" in a domain Ω of the Riemann sphere is defined in a generalized form: It consists of a field of conformal classes in tangent spaces at almost all points in Ω. No assumption of quasi-conformality with respect to the standard quasi-conformal structure is made. Conformal structures can be transported by absolutely continuous almost everywhere differentiable homeomorphisms, or even continuous maps. This is a more natural category than the one of quasi-conformal homeomorphisms. The natural objection we can think of is the lack of rectification theorems. We prove different sorts of rectification theorems for degenerate quasi-conformal conformal structures (we label them "degenerate" to stress that they are not quasi-conformal). First we start considering conformal structures that are quasi-conformal on compact subsets of the complement of a given compact, then potential conformal structures and finally virtual conformal structures. The definitions of this notions and necessary background is contained in section 2.4.

Without more notations we can already state a first rectification theorem. Consider a compact set K in the Riemann sphere $\overline{\mathbb{C}}$ and a conformal structure ξ in $\overline{\mathbb{C}} - K$ which is locally quasi-conformal in $\overline{\mathbb{C}} - K$. Let μ be the Beltrami coefficient of ξ ($\mu = 0$ on K).

Theorem 1.1. *If the Riemann surface $(\overline{\mathbb{C}} - K, \xi)$ is in the class O_{AD} then, up to a Moëbius composition to the left, there exists a unique continuous map $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$*

such that h is locally quasi-conformal on $\overline{\mathbb{C}} - K$, and on $\overline{\mathbb{C}} - K$,

$$\bar{\partial} h = \mu \, \partial h.$$

Note that the rectification h in this theorem is not even a homeomorphism in general.

The Riemann surface $(\overline{\mathbb{C}} - K, \xi)$ is the Riemann surface structure induced by the locally quasi-conformal structure ξ (hence ξ defines locally a complex structure by classical quasi-conformal theory). A Riemann surface is in the class O_{AD} if there no non-constant holomorphic functions with finite Dirichlet integrals. There is a geometric criterium due to Sario that can be used to recognize when a Riemann surface is in the class O_{AD}. This is exploited to give effective versions of this rectification theorems and to prove the rectifications theorems for potential conformal structures in section 3.3 and 3.4.

The rectification theorem for virtual conformal structures is given in section 3.4. The notion of solution to the Beltrami equation has to be reinterpreted. We obtain "rectifications" that are totally degenerate and realize the collapsing of the butchered Riemann sphere into a new Riemann sphere.

In section 3.5 we introduce generalized rectifications of a different kind. We use the potential theory outside a connected compact set and make use of Rickman's removability theorem. These rectifications are useful in the new theory of renormalization of polynomials with connected Julia sets.

The motivation for this article comes from problems in Holomorphic Dynamics and Renormalization Theory. Here we present some of the analytic tools that are necessary to a new approach to the renormalization of polynomials. The main interest of this new renormalization is that it is canonical and, contrary to the classical Douady-Hubbard approach, natural in the class of polynomials since it remembers the polynomial structure of the dynamics by using the combinatorical structure of external rays (and not only the polynomial-like structure). In particular, it allows to define uniquely the renormalization of polynomial combinatorically renormalizable and with Cantor set Julia set for the quadratic polynomials (and for higher degree if one considers removable Julia sets) (see [17]).

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2. Preliminaries.

2.1. **Potential theory.**

For basic background on potential theory we refer to [18] and [22]. We give a more geometric point of view than in the classical expositions, that are more "function theoretic" oriented.

We consider a compact and full set $K \subset \overline{\mathbb{C}}$ containing at least two points and $\infty \notin K$. We denote $\Omega = \overline{\mathbb{C}} - K$ the complementary domain. We assume that K has positive capacity and, more precisely, that all points of K are regular points for the Dirichlet problem. This is the case for a uniformly perfect set, in particular for Julia sets of rational functions ([5] p.64).

Let $z \mapsto g(z, \infty) = G(z)$ be the Green function in Ω (the potential of K.) The potential $G(z)$ converges to 0 when $z \to K$ and $G_K(z) = \log|z| + u(z)$ where u is a harmonic function, and

$$\lim_{z\to\infty} G_K(z) - \log|z| = \gamma,$$

where γ is Robin's constant of Ω. The capacity of K is $e^{-\gamma} = \mathrm{Cap}(K)$. These properties and the fact that $G : \mathbb{C} - K \to \mathbb{R}_+^*$ is harmonic and positive determine G uniquely. We denote $G = G_K$.

Definition 2.1. *An equipotential of K is a level sets of G_K.*

The following Proposition is elementary:

Proposition 2.2. *An equipotential is a finite union of analytic Jordan curves if and only if it does not contain critical points of G_K. Each component of an equipotential is endowed with a natural parametrization (or measure) generated by integration of the conjugate function G_K^* (the harmonic parametrization or measure.)*

Definition 2.3. *An external ray or a Green line is an orthogonal trajectory to equipotentials.*

An we have another elementary proposition:

Proposition 2.4. *External rays are analytic arcs if and only if they do not contain critical points of G_K. Otherwise they split at critical points into several analytic arcs and they are called critical external rays.*

External rays are endowed with a natural parametrization determined by the potential. The first critical point in a critical external ray is the critical point with higher potential. The sub-critical part of a critical external ray is the closed subset of points with potential below the first critical point.

Definition 2.5. *The skeleton $\Gamma(K)$ is the union of sub-critical parts of critical external rays.*

The skeleton of a connected set K is empty. The skeleton of K has always 0 Lebesgue measure since it is the union of at most a countable number of segments, . The union $K \cup \Gamma(K)$ is a full connected set.

Let G_K^* be a conjugate function of G_K in a neighborhood of ∞. The conjugate function is determined up to an additive constant. We choose the additive constant so that $\varphi_K(z) = e^{G_K(z)+iG_K^*(z)}$ defines a univalent map angularly tangent to the identity in a neighborhood of ∞ (this means that the derivative of φ_K at the fixed point ∞ is real and positive, equal to $1/\operatorname{Cap}(K)$.) Note that a preimage of a circle of large radius centered at 0 is an *equipotential* of K, i.e. a constant locus for G_K. The holomorphic germ φ_K defines by analytic continuation a multivalued function. The monodromy around the critical points of G_K introduces an indeterminacy in the argument. The analytic continuation of φ_K is well defined outside the skeleton. At points of the skeleton we obtain different limits depending where we approach the point from one side or the other. The image skeleton $\Gamma'(K) \subset \overline{\mathbb{C}} - \overline{\mathbb{D}}$ is the set of all these limits. The image skeleton is composed by at most a countable number of radial semi-open segments having the open end-point on the unit circle.

Definition 2.6. *The Green map of K is the holomorphic diffeomorphism*

$$\varphi_K : \overline{\mathbb{C}} - (K \cup \Gamma(K)) \to \overline{\mathbb{C}} - (\overline{\mathbb{D}} \cup \Gamma'(K)).$$

When K is connected, the map φ_K is a conformal representation. Observe that the function φ_K is onto because $G_K(z) \to 0$ when $z \to K$ and $|\varphi_K| = e^{G_K}$. Observe that we can transport by φ_K conformal structures on $\overline{\mathbb{C}} - \overline{\mathbb{D}}$ defined almost everywhere.

When K is a Cantor set, $K \cup \Gamma(K)$ is connected and locally connected (in the region of potential larger than $\epsilon > 0$, $\Gamma(K)$ consists of a finite number of analytic arcs). Thus when K is a Cantor set, the univalent map φ_K has a radial extension at all points, i.e all non-critical external rays do land at some point.

Sometimes it is more natural to work in $\mathbb{H}/\mathbb{Z}$ instead of $\mathbb{C} - \overline{\mathbb{D}}$. We denote $E : \mathbb{H}/\mathbb{Z} \to \mathbb{C} - \overline{\mathbb{D}}$ defined by

$$E(z) = e^{-2\pi i z}.$$

The image $E^{-1}(\Gamma'(K))$ is at most a countable union of vertical segments in $\mathbb{H}/\mathbb{Z}$. These vertical segments can be grouped in groups with the same length so that their tip corresponds to the same critical point of the Green function. For each group we can cut these vertical segments and we repaste the boundaries respecting the imaginary coordinate. We do the pasting according to the identifications of the corresponding parts of the skeleton $\Gamma(K)$ so that the Riemann surface $\mathcal{S}_K$ obtained in that way is biholomorphic to $\mathbb{C} - K$. Observe that the complex structure in that Riemann surface is generated by a flat metric inherited from the flat metric of $\mathbb{H}/\mathbb{Z}$ because we did a geodesic gluing. The surface $\mathcal{S}_K$ endowed with this metric structures is the *cylindrical model* of $\mathbb{C} - K$.

The capacity (and the Lebesgue measure as well) is an upper-semi-continuous function in the space of compact subsets of the plane endowed with the Hausdorff topology (see [22] p.57).

Proposition 2.7. *Let $(K_n)_{n\geq 0}$ be a sequence of compact subsets in $\mathbb{C}$ converging in Hausdorff topology to a compact set $K \subset \mathbb{C}$. We have*

$$\limsup_{n\to+\infty} \mathrm{Cap}(K_n) \leq \mathrm{Cap}(K).$$

Given a compact set $K \subset \mathbb{C}$ with positive capacity and $\epsilon > 0$ we denote by

$$W_\epsilon(K) = \{z \in \mathbb{C}; G_K(z) \leq \epsilon\}$$

the ϵ-potential closed neighborhood of K. Observe that in Hausdorff topology

$$\lim_{\epsilon\to 0} W_\epsilon(K) = K^*,$$

where $K^* \subset K$ is the *carrier* of K, i.e. the subset of regular points, or the support of the equilibrium measure of K (see [Ra] Thm. 4.2.4 p.93, Thm. 4.3.14 p.105.)

Lemma 2.8. *Let K_1 and K_2 be two compact sets in the plane with positive capacity, and $\epsilon > 0$ such that*

$$K_1 \subset W_\epsilon(K_2).$$

Then we have on $\mathbb{C} - W_\epsilon(K_2)$,

$$G_{K_2} \leq G_{K_1} + \epsilon.$$

Proof. On the boundary of $W_\epsilon(K_2)$ we have

$$G_{K_2} - \epsilon \leq 0 \leq G_{K_1},$$

thus the maximum principle on the domain $\overline{\mathbb{C}} - W_\epsilon(K_2)$ applied to $G_{K_2} - G_{K_1} - \epsilon$ gives the result. □

Consider now a sequence of compact sets $(K_n)_{n\geq 0}$ in the plane with all their points regular,

$$\mathrm{Cap}(K_n) \geq \epsilon_0 > 0,$$

and converging in Hausdorff topology to a compact set K. Then we have $\mathrm{Cap}(K) \geq \epsilon_0$. Not all points of K are necessarily regular, in general $K^* \neq K$. We have $\mathrm{Cap}(K^*) = \mathrm{Cap}(K) \geq \epsilon_0$ ([22] p.56.) By Harnack's principle ([18] p.16) we can extract a converging subsequence of the sequence of positive harmonic functions $(G_{K_n})_{n\geq 0}$ converging uniformly on compact subsets of $\mathbb{C} - K$ to a positive harmonic function G. It follows from the lemma that $G = G_K = G_{K^*}$. We conclude:

Proposition 2.9. *Let $(K_n)_{n\geq 0}$ be a sequence of compact sets with all points regular and*

$$\mathrm{Cap}(K_n) \geq \epsilon_0.$$

We assume that

$$\lim_{n\to+\infty} K_n = K.$$

Then uniformly on compact subsets of $\mathbb{C} - K$,

$$\lim_{n\to+\infty} G_{K_n} = G_K.$$

Thus

$$\lim_{n\to+\infty} \varphi_{K_n} = \varphi_{K^*},$$

in the Caratheodory kernel of the domain of definition of these Green mappings.

2.2. **The analytic tree.** We consider in this section a disconnected compact set $K \subset \mathbb{C}$ with positive capacity and with all its points regular. The complement $\Omega = \overline{\mathbb{C}} - K$ is a planar Riemann surface. As described in section I.1.a there exists a Green function G_K with logarithmic singularity at ∞ which tends to 0 in K. The Riemann surface Ω is hyperbolic, i.e. Green functions exist.

We study the conformal invariants of Ω. Let $\mathcal{C}(K)$ be the set whose elements are critical equipotentials, i.e. those containing critical points of G_K. In the generic situation, all critical equipotential contain only one critical point with multiplicity one (this is not the case for the Julia set of a generic polynomial of degree larger than 3). We say that K is generic if this happens. It can be proved that this holds for generic K in the Baire sense for the Hausdorff metric (Sketch of the proof: The capacity function is upper semi-continuous, thus the space of compact sets of $\mathbb{C}$ with capacity larger than $r > 0$ is a closed set, thus a Baire space. For $N > 0$ the set of those K whose first N critical equipotentials contain only one critical point with multiplicity one is open and dense). It is sufficient for our purposes to study the generic case. The modifications to treat the general case are straightforward, but the combinatorics is cumbersome. We will sketch the few modifications necessary to treat the general case and will consider only the generic situation.

The connected components of $\overline{\mathbb{C}} - (\{\infty\} \cup \mathcal{C}(K) \cup K)$ is a family of annuli (A_i). We have an order relation $A_2 < A_1$ if A_2 is contained in the bounded component of the complement of A_1. For this order relation, the unique annulus A_0 in the family such that $\infty \in \partial A_0$ is a maximal element. The family (A_i) with the order $<$ is a tree.

Definition 2.10. ***(Combinatorial tree).*** *The combinatorial tree associated to K or Ω is*

$$\mathcal{T}_c(K) = \mathcal{T}_c(\Omega) = ((A), <).$$

In the generic case, this tree is always a dyadic tree, i.e. the valence of each vertex is 2. In the general case the combinatorial tree has finite valence at each vertex because the harmonic function G_K has only isolated critical points and each critical equipotential contains only a finite number of critical points.

The combinatorial tree is obviously a conformal invariant of Ω but it is not the only one. We label each vertex A of $\mathcal{T}_c(K)$ with the modulus $\mod A \in]0, +\infty]$ of the corresponding annulus. These are the *modular invariants*.

Next we have combinatorial invariants that describe how to paste the annulus of the tree. The combinatorics is much simpler in the generic situation : It only depends on two angles (and in the symmetric situation, that will be specially important for the applications to the quadratic Julia sets, it depends only on one angle). We treat the generic situation. Each annulus of the tree that is not the root ($A \neq A_0$) or an end contains one critical point in its outer boundary component and one critical point in its inner boundary component that corresponds to two points of its prime-end circle. Given such an annulus A that is not the root ($A \neq A_0$), we map conformally A into a round annulus (i.e. bordered by circles) centered at the origin of the complex plane. We map the outer (resp. inner) boundary component into the outer (resp. inner) boundary component. We denote z_+ the image of the critical point in the outer boundary component and $(z_-^{(1)}, z_-^{(2)})$ the images of the two access to the critical point in the inner boundary component, with $z_-^{(1)}$ being the first one encountered when going around the annulus counterclockwise from z_+. The pasting invariant is the unordered pair of angles defined modulo 1 (we measure angles modulo 1)

$$\tau(A) = \{\tau_1(A), \tau_2(A)\},$$
$$\tau_1(A) = \frac{1}{2\pi}(\arg z_+ - \arg z_-^{(1)}),$$
$$\tau_2(A) = \frac{1}{2\pi}(\arg z_+ - \arg z_-^{(2)}).$$

For the root A_0 we proceed as follows. We identify A_0 to the pointed unit disk $\mathbb{D} = \mathbb{D} - \{0\}$ and we consider the two points $(z_-^{(1)}, z_-^{(2)})$ corresponding to the two access of the critical on the inner boundary of A_0. We define modulo 1

$$\tau_1(A_0) = 1 - \tau_2(A_0) = \frac{1}{2\pi}(\arg z_-^{(1)} - \arg z_-^{(2)}).$$

The angular invariant of the root $\tau(A_0) = \{\tau_1(A_0), \tau_2(A_0)\}$ is only significative modulo the elements the diagonal, i.e. of the form (α, α) (only the difference of the arguments of $z_-^{(1)}$ and $z_-^{(2)}$ matters). For an end, we define $\tau(A) = 0$ (there is no significative angular invariant for an end).

Definition 2.11. ***(Analytic tree).*** *For a generic K we define the analytic tree of K or Ω to be the combinatorial tree with labeled vertices*

$$\mathcal{T}_\omega(K) = \mathcal{T}_\omega(\Omega) = ((A, \quad \text{mod }(A), \tau(A), <).$$

In general we call an analytic tree such a combinatorial object.

Remark 2.12. The angular invariants in the non-generic situation describe the relative position of critical points in the outer boundary and the position of critical accesses corresponding to critical points in the inner boundary. As in the generic situation all angles are only defined up to one rotation. More important, we have to add the information describing which sets of accesses do correspond to the same critical point.

Remark 2.13. Continuing this work, N. Emerson developped in his Thesis at UCLA the combinatorics and structure of general Analytic Trees and their combinatorics (see [9]).

Riemann surface associated to an analytic tree.

From the analytic tree of Ω we can reconstruct Ω by gluing together round annulus (that is, bounded by two concentric circles) of modulus $\text{mod }(A)$ in the prescribed way by the tree and the angular invariants. More precisely, we describe the gluing construction that associates to each analytic tree $\mathcal{T}_\omega$ a Riemann surface $\mathcal{S}(\mathcal{T}_\omega)$.

We start with $A_0 \cup \{\infty\}$ that we identify to the exterior of the closed unit disk $\overline{\mathbb{C}} - \overline{\mathbb{D}}$. We determine two points on the unit circle compatible with the angular invariant $\tau(A_0)$, i.e. two points whose difference of arguments modulo 1 is equal to $\tau_1(A_0) - \tau_2(A_0)$. We identify these two points of the boundary. Now there is a unique way of gluing a children of A_0 so that the harmonic parameterization of the identified boundaries do correspond. If not an end, the angular invariant of the children determines exactly the location on the inner boundary of two points. We identify these two points and we are again in the same situation for gluing the next two children. We go on and paste all children in the tree. Finally we paste the ends in such a way that the harmonic measures correspond on both sides of the pasted boundary. We denote $\mathcal{S}(\mathcal{T}_\omega)$ the abstract Riemann surface constructed from the analytic tree $\mathcal{T}_\omega$ in that way. Obviously there is a harmonic function on $\mathcal{S}(\mathcal{T}_\omega)$ that generates the analytic tree for which the traces of the pasted boundaries are equipotentials. Moreover we have that $\mathcal{S}(\mathcal{T}_\omega(\Omega))$ and Ω are biholomorphic

$$\mathcal{S}(\mathcal{T}_\omega(\Omega)) \approx \Omega,$$

because a conformal representation of $A_0 \cup \{\infty\}$ into its trace in $\mathcal{S}(\mathcal{T}_\omega(\Omega))$ extends into a holomorphic diffeomorphism from Ω into $\mathcal{S}(\mathcal{T}_\omega(\Omega))$ because the pastings are compatible (in a neighborhood of a non-singular glued point the harmonic functions on both sides have conjugate functions that do coincide on the boundary, so the harmonic functions and its conjugates do define analytic extensions of the same local holomorphic diffeomorphism). Observe that the pointed Riemann surfaces with the point ∞ for Ω and the corresponding point to ∞ for $\mathcal{S}(\mathcal{T}_\omega(\Omega))$ are biholomorphic also. From this construction we obtain:

Theorem 2.14. *The pointed planar Riemann surfaces* (Ω_1, ∞)*,* $\Omega_1 = \overline{\mathbb{C}} - K_1$*, and* (Ω_2, ∞)*,* $\Omega_2 = \overline{\mathbb{C}} - K_2$*, are biholomorphic (i.e. by a holomorphic diffeomorphism fixing* ∞*) if and only if their analytic trees coincide*

$$\mathcal{T}_\omega(\Omega_1) = \mathcal{T}_\omega(\Omega_2).$$

Proof. Obviously by construction $\mathcal{T}_\omega(\Omega)$ only depends on the conformal type of (Ω, ∞). Conversely, if the analytic trees agree then

$$\Omega_1 \approx \mathcal{S}(\mathcal{T}_\omega(\Omega_1)) \approx \mathcal{S}(\mathcal{T}_\omega(\Omega_2)) \approx \Omega_2 \ ,$$

are all biholomorphic as pointed Riemann surfaces. □

We can observe that there are analytic trees that can not be realized as the complement of a compact set with positive capacity. Just consider an analytic tree with asymptotic very large modulus invariants. The abstract pasted Riemann surface is then parabolic, i.e there are no Green functions, (use the criterium in [2] IV.15B p.229) and is not the complement of a compact set with positive capacity. Note that we could define analytic trees also in general for the complement of a compact set of zero capacity (or of compact sets with irregular points) using an Evans potential (see [20] V.3.13A p. 351 or [22] III.6 p.75). Such a tree depends then on the choice of the Evans potential which is not unique.

On the other hand we can observe that any tree corresponds to an analytic tree of the complement of a compact set (in the case of zero capacity the tree is constructed from the choice of an Evans-Selberg potential). The abstract Riemann surface $\mathcal{S}(\mathcal{T}_\omega)$ obtained from the pasting of abstract annulus according to the combinatorics of the tree $\mathcal{T}_\omega$ is planar, i.e. every cycle on $\mathcal{S}(\mathcal{T}_\omega)$ is dividing (see [Ah-Sa] I.30D p.66). Then $\mathcal{S}(\mathcal{T}_\omega)$ can be holomorphically embedded in $\mathbb{C}$ ([2] III.11A p.176), $\mathcal{S}(\mathcal{T}_\omega) \approx \Omega \subset \overline{\mathbb{C}}$, with ∞ corresponding to the non pasted end of A_0.

Observe that we can read on $\mathcal{T}_\omega(K)$ many properties of K or Ω. The Riemann surface Ω is finitely connected if and only if $\mathcal{T}_\omega(\Omega)$ is finite. An infinite branch of $\mathcal{T}_\omega(K)$ corresponds to a point in the ideal boundary of Ω and to a connected component of K. If the sum of the modulus invariants are infinite along this branch with the root removed, i.e. for $(A_1, A_2, \ldots)$,

$$\sum_{i\geq 1} \ \text{mod} \ (A_i) = +\infty,$$

then this component is a point. We observe that if $\mathcal{T}_\omega(K)$ has no finite branches and this condition holds for all infinite branches then K is a Cantor set. Observe also that this is not a necessary condition. But if we have a finite branch then K contains a non-trivial connected component, that is not a point because otherwise this isolated point will not be regular (contradicting one of our assumptions).

We study in more detail the Cantor set situation. More precisely we want to determine when the analytic tree (or equivalently the conformal type of Ω) determines $K \subset \overline{\mathbb{C}}$ up to a Moëbius transformation. Observe that this problem is hopeless when K has a non-trivial connected component (because of Riemann uniformization theorem), or when K has positive measure (because of Morrey-Ahlfors-Bers theorem), or for a "flexible" Cantor set as those constructed by R. Kaufman and C. Bishop [4]. In order to address this problem we review in the next section the classical theory of O_{AD} Riemann surfaces.

We observe also that all the information of the analytic tree can be recovered from φ_K and the skeleton $\Gamma'(K)$. In particular, the preimages of annulus of the analytic tree by φ_K are quadrilaterals bounded by circles and radial segments. The modulus of the rectangles are the modulus of the corresponding annulus.

The cylindrical model $\mathcal{S}$ is very convenient. The flat metric on the cylindrical model is the extremal metric for all annulus in the analytic tree (we have a simultaneous uniformization). Observe that if an annulus $A \subset \mathcal{S}$ corresponds to an annulus in the analytic tree then

$$\text{mod } A = \frac{1}{2\pi} \frac{g_+(A) - g_-(A)}{\mu_H(A)},$$

where $g_-(A) < g_+(A)$ are the potentials of the boundaries of A, and $\mu_H(A)$ is the harmonic measure of A, i.e. the linear measure of the angles of external rays intersecting A.

2.3. **The class of O_{AD} Riemann surfaces.** We review part of the classical classification theory of Riemann surfaces, and more precisely the results concerning the O_{AD} class. We remind the classical classification theory of Riemann surfaces. As a basic reference the reader can consult [2], and for a more encyclopedic treatment [20].

A Riemann surface $\mathcal{S}$ is in O_{AD} if there are no non-constant AD-functions, i.e. holomorphic functions F on $\mathcal{S}$ with finite Dirichlet integral

$$D_{\mathcal{S}}(F) = \int\int_{\mathcal{S}} |F'|^2 \; dxdy \; .$$

By Rado's theorem, any Riemann surface (defined to be connected) is σ-compact. Consider an exhaustion $(U_n)_{n\geq 0}$ of $\mathcal{S}$ by open sets U_n relatively compact in $\mathcal{S}$ and $\overline{U_n} \subset U_{n+1}$. Consider the connected components $(U_{nj})_j$ of $U_{n+1} - U_n$. We denote by

m_{nj} the extremal length of paths in U_{nj} joining $\partial U_n \cap \overline{U_{nj}}$ to $\partial U_{n+1} \cap \overline{U_{nj}}$. We have ([Ah-Sa] IV.13D p.225) that

$$m_{nj} = \frac{1}{D_{U_{nj}}(u)} \ ,$$

where

$$D_{U_{nj}}(u) = \int \int_{U_{nj}} |\operatorname{grad} u|^2 dx \ dy,$$

where u is the harmonic function in U_{nj} equal to 0 in $\partial U_n \cap \overline{U_{nj}}$ and equal to 1 in $\partial U_{n+1} \cap \overline{U_{nj}}$.

Define

$$m_n = \inf_j m_{nj} .$$

Then we have the following modular test due to L. Sario ([2] IV.16D p.233):

Theorem 2.15. *If there exists an exhaustion $(U_n)_{n\geq 0}$ such that*

$$\sum_{n=0}^{+\infty} m_n = +\infty$$

then $\mathcal{S}$ belongs to the class O_{AD}.

Observe that the strength of Sario's criterium relies on the choice of the exhaustion. Given a Riemann surface it is always possible to choose bad exhaustions for which the criterium fails. The most efficient choice of the exhaustion consists in those that give modulus m_{nj} that are comparable in magnitude for fixed n (the boundaries of the exhausting domains cut "evenly" the Riemann surface). Observe that Sario's criterium gives a sufficient condition only. There is a distinct class of Riemann surfaces for which Sario's criterium holds that is strictly distinct from the O_{AD} class. For these Riemann surfaces the O_{AD} property can be checked using Sario's geometric critetrium. Thus it is natural to denote this class by GO_{AD} for *geometric* O_{AD} class. Geometric O_{AD} Riemann surfaces enter in the effective explicit application of the rectification theorems presented in this article.

Planar O_{AD} Riemann surfaces.

We consider $\mathcal{S} = \overline{\mathbb{C}} - K$ where K is a compact set in $\mathbb{C}$. The following results can be traced back to L. Ahlfors, A. Beurling, R. Nevanlinna and L. Sario ([2] IV.2B p.199).

Theorem 2.16. *The following conditions are equivalent :*

(i) The Riemann surface $\mathcal{S} = \overline{\mathbb{C}} - K$ is in the class O_{AD}.

(ii) The compact set K has absolute measure zero: For any univalent map defined on the complement of K, the complement of the image has measure zero.

(iii) The compact set $K \subset \mathbb{C}$ is rigidly embedded in the complex plane: The only univalent functions on the complement of K are Moebius transformations.

(iv) The compact set $K \subset \mathbb{C}$ is neglectable for extremal length: The modulus of a path family Γ is the same than the modulus of the path family whose "paths" are the ones of Γ removing their intersection with K.

What makes the class O_{AD} particularly interesting for our purposes is the strong removability property (iii). The complement of a Cantor quadratic (or cubic, see [3]) Julia set is in O_{AD}. C. McMullen seems to have been the first to notice that. He formulated a useful variant of Sario's test for planar Riemann surfaces using modulus of annulus (see [14] p.20):

Theorem 2.17. *We assume that we have a tree whose vertex are annulus $A \subset \mathbb{C} - K$ such that if B is a children of A then B is contained in the bounded component $I(A)$ of $\mathbb{C} - A$. If for any $z \in K$ there is an infinite branch $(A_i(z))$ deprived from the root with $z \in I(A_i(z))$ for all index i and*

$$\sum_{i \geq 1} \mod A_i(z) = +\infty \ ,$$

then the Riemann surface $\mathcal{S}$ is in the class O_{AD}.

Remark 2.18. In the application to quadratic renormalization we use this geometric criterium in a weak form : All modulus of annuli will be bounded from below.

Definition 2.19. ***(Thin tree).*** *A modular tree is a tree of annulus nested inside each other in the prescribed way by the tree, and vertices labeled by the modulus of the corresponding annulus. Analytic trees have an underlying modular tree structure.*

A modular tree is thin if it has only infinite branches and along all branches the sum of the modulus of the annuli except the root is infinite.

McMullen's test can be applied to an Analytic Tree:

Theorem 2.20. *Let $\mathcal{T}_\omega$ be an analytic tree which is thin. Then the corresponding planar Riemann surface $\mathcal{S}(\mathcal{T}_\omega) \equiv \mathbb{C} - K(\mathcal{T}_\omega)$ is in the class O_{AD}, i.e. $K(\mathcal{T}_\omega)$ is a Cantor set of absolute measure zero and its embedding in the plane is uniquely determined up to a Moebius transformation.*

Notice that the condition can be checked in a purely combinatorial way.

One can prove the previous theorem using Sario's test choosing a suitable exhaustion (U_n) of $\mathcal{S}(\mathcal{T}_\omega)$ where the components of the boundary of U_n are composed by pieces of equipotentials (with different potentials) and all modulus $(m_{n_j})_j$ are the same.

Definition 2.21. ***(Thin and totally thin Cantor sets).*** *A tree of annulus $\mathcal{T}_\omega$ as in theorem 2.20 for a compact set K will be called a tree associated to K if $K = K(\mathcal{T}_\omega)$. In particular the analytic tree of K is a tree associated to K.*

A Cantor set K is a thin Cantor set if there is an associated tree which is thin. Then $\mathbb{C} - K$ is an O_{AD} Riemann surface and K has absolute measure zero.

A Cantor set K of positive capacity with all points regular is a potentially thin Cantor set if its analytic tree is thin.

Note that in the definition of thin Cantor set we include Cantor sets of zero capacity (but not for potentially thin Cantor sets).

2.4. **Potential and virtual conformal structures.** We define in this section *potential conformal structures* and the notions of *Green*, *equipotential* and *potential* equivalence between conformal structures.

Definition 2.22. ***(Equipotential, Green and potential equivalence).*** *Let $K \subset \mathbb{C}$ be a compact set with all points regular for the Dirichlet problem. A conformal structure ξ on $\mathbb{C} - K$ is equipotentially (resp. Green, potentially) equivalent to another conformal structure η on $\mathbb{C} - K$ if*

$$\xi' = (\varphi_K^{-1} \circ E)^* \ \xi$$
$$\eta' = (\varphi_K^{-1} \circ E)^* \ \eta,$$

are equi-potentially (resp. Green, potentially) equivalent in $\mathbb{H}/\mathbb{Z}$, that is, there exists

$$L : \mathbb{H} \to \mathbb{H}$$

of the form $(x, y) \mapsto (d(x), y)$ (resp. $(x, y) \mapsto (x, k(y))$; $(x, y) \mapsto (d(x), k(y))$), where $d : \mathbb{T} \to \mathbb{T}$ is an absolutely continuous homeomorphism (resp. and $k : \mathbb{R}_+ \to \mathbb{R}_+$ is an absolutely continuous increasing homeomorphism) and

$$\eta' = L_* \xi'.$$

Definition 2.23. ***(Potential conformal structure).*** *A potential conformal structure is a conformal structure potentially equivalent to σ_0.*

Remark 2.24. The ellipses in the tangent spaces defining a potential conformal structure have their principal axes tangent or orthogonal to equipotentials. Observe that L is differentiable almost everywhere thus a potential conformal structure has a well defined Beltrami form.

A potential conformal structure is locally quasi-conformal on $\overline{\mathbb{C}} - K$ if and only if both d and k are locally lipchitz.

In other words, a potential conformal structure ξ is compatible with the potential of K. For the potential theory with respect to the new conformal structure equipotentials and Green lines are the same than for the potential theory with respect to the standard conformal structure. More precisely we have the following theorem:

Theorem 2.25. *Assume that ξ is a quasi-conformal potential conformal structure with the above notations. Let $h_\xi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rectification of ξ such that $h_\xi(\infty) = \infty$. Then all points of $h_\xi(K)$ are regular and if $K' = h_\xi(K)$ we can choose a normalization of h_ξ such that the map*

$$\varphi = h_\xi^{-1} \circ \varphi_K \circ E^{-1} \circ L \circ E$$

is tangent to the identity at ∞ and then we have

$$\varphi_{K'} = \varphi.$$

In particular $h_\xi(\Gamma(K))$ is the skeleton of K' and for $z \in \overline{\mathbb{C}} - K$,

$$G_{K'}(h_\xi(z)) = k(G_K(z)) \tag{1}$$

$$\tau_{K'}(h_\xi(z)) = d(\tau_K(z)). \tag{2}$$

(3)

Proof. The proof is straightforward. The mapping

$$\varphi = h_\xi^{-1} \circ \varphi_K \circ E^{-1} \circ L \circ E$$

is holomorphic and defines a holomorphic diffeomorphism in a neighborhood of ∞. With the appropriate normalization of h_ξ (composing with a linear dilatation) we can make φ tangent to the identity at ∞. Then $\log|\varphi|$ is the Green function of K'. □

We observe that a potential conformal structure is uniquely determined by the compact K and by the mapping L, or more precisely by the mappings d and k. Now if d is a uniform limit of orientation homeomorphisms of the circle d_n we can think to (d, k) as defining a singular conformal structure ξ even in the case when d is not a homeomorphism (for example when it maps intervals into single points). The main result of next section proves that under natural restrictions on d and k the rectifications of ξ_n defined by (d_n, k) do converge to a "rectification" of ξ.

Virtual conformal structures.

Consider a metric space X. Let $\mathcal{C}(X)$ be the space of closed continuous correspondences on X, this is the set of closed connected subsets of $X \times X$ endowed with the Hausdorff topology associated to the product distance in $X \times X$. Recall that correspondences can be composed: If $d_1, d_2 \in \mathcal{C}(X)$,

$$d_2 \circ d_1 = \{(x,z) \in X^2; \text{there exists } y \in X, \ (x,y) \in d_1 \text{and } (y,z) \in d_2\}.$$

Consider now Homeo(X) the space of homeomorphisms of X endowed with the topology of uniform convergence.

The graph map that associates to an element $\varphi \in \text{Homeo}(X)$ its graph $\mathcal{G}$: Homeo$(X) \to \mathcal{C}(X)$,

$$\varphi \mapsto \mathcal{G}(\varphi) = \{(x,y) \in X^2; y = \varphi(x)\}$$

gives a continuous embedding of Homeo(X) into $\mathcal{C}(X)$.

Definition 2.26. ***(Map correspondences).***

We define

$$\mathcal{C}_0(X) = \overline{\mathcal{G}(\text{Homeo}_+(X))} \subset \mathcal{C}(X)$$

to be the space of limit homeomorphisms, and we define

$$\hat{\mathcal{C}}_0(X) = \mathcal{C}_0(X) \cap C^0(X,X)$$

to be the space of map correspondences (where $C^0(X,X)$ denotes the spaces of continuous maps from X into X).

We are interested in the case $X = \mathbb{T}$. The following proposition is immediate.

Proposition 2.27. *The space $\mathcal{C}_0(\mathbb{T})$ is homeomorphic to the space $\mathcal{M}(\mathbb{T})$ of probability measures on $\mathbb{T}$ endowed with the weak topology. We have a homeomorphism $\int : \mathcal{M}(\mathbb{T}) \to \mathcal{C}_0(\mathbb{T})$*

$$\mu \mapsto d = \int d\mu$$

where $d = \int d\mu$ is defined by the fact that $d \cap (\{x\} \times \mathbb{T})$ is always a point or an interval and

$$|d \cap (\{x\} \times \mathbb{T})| = \mu(\{x\}),$$

$$\sup_{\substack{(x,y)\in d \\ 0 \leq x \leq x_0}} y = \mu([0,x_0]) = \int_0^{x_0} d\mu.$$

The same map induces a homeomorphism from the space $\mathcal{M}_{na}(\mathbb{T})$ *of non atomic probability measures into the space of map correspondences* $\hat{\mathcal{C}}_0(\mathbb{T})$.

To each homeomorphism $d \in \text{Homeo}_+(\mathbb{T})$ we have associated before a homeomorphism $L : \mathbb{H} \to \mathbb{H}$. Now to $d \in \hat{\mathcal{C}}_0(\mathbb{T})$ and $k \in \text{Homeo}_+(\mathbb{R}_+)$ we can also associate in the same way a map correspondence of $\mathbb{H}/\mathbb{Z}$, $L \in \hat{\mathcal{C}}_0(\mathbb{H})$, $L : \mathbb{H}/\mathbb{Z}\mathbb{H}/\mathbb{Z}$,

$$(x, y) \mapsto (d(x), k(y)).$$

From a classical point of view, it doesn't make sense to talk about the "complex structure" $L_*\sigma_0$, but all the information is contained in L or more precisely on the pair $(d, k) \in \hat{\mathcal{C}}_0(\mathbb{T}) \times \text{Homeo}_+(\mathbb{R}_+)$.

Definition 2.28. ***(Virtual conformal structure).*** *A virtual conformal structure on* $\mathbb{H}/\mathbb{Z}$ *or* $\overline{\mathbb{C}}$ *is identified with a map correspondence* $d \in \hat{\mathcal{C}}_0(\mathbb{T})$ *(or with its corresponding non-atomic probability measure* $\mu_d \in \mathcal{M}_{na}(\mathbb{T})$*) and an orientation preserving homeomorphism* $k \in \text{Homeo}_+(\mathbb{R}_+)$.

Observation.

We can give a more general definition allowing $k \in \hat{\mathcal{C}}_0(\mathbb{R}_+)$ and prove similar rectification theorems but this is unnecessary for the applications.

2.5. **Rickman's theorem.** We use at different places Rickman's removability theorem for quasi-conformal mappings (see [19]). Rickman's theorem has been used before in the theory of polynomial-like mappings (see [8]). Rickman's theorem holds in higher dimension.

Theorem 2.29. *(S. Rickman) Let* $U \subset \mathbb{C}$ *be an open set,* $K \subset U$ *be a closed in* U *(for the relative topology in* U*) ,* f *and* g *be two mappings* $U \to \mathbb{C}$ *which are homeomorphisms onto their images. Suppose that* g *is quasi-conformal, that* f *is quasi-conformal on* $U - K$ *and that* $f = g$ *on* K*. Then* f *is quasi-conformal, and* $Df = Dg$ *almost everywhere on* K*.*

We use in the applications to renormalization a version of Rickman's theorem for quasi-regular mappings. First we recall the definition of a topological branched covering.

Definition 2.30. *A map* f *is a topological branched covering if* f *is locally the composition of a holomorphic map and a homeomorphism.*

The map f *is quasi-regular if* f *is locally the composition of a holomorphic map and a quasi-conformal homeomorphism.*

Theorem 2.31. *Let $U \subset \overline{\mathbb{C}}$ be an open set and $K \subset U$ be a closed set in U. Let f and g be two topological branched coverings from U into their images.*

We assume that g is quasi-regular, f is quasi-regular on $U - K$, and that $f = g$ on K. Then f is quasi-regular and almost everywhere in K

$$Df = Dg.$$

Proof. At each point z which is not a critical point of f or g we use Rickman's theorem for homeomorphisms in a neighborhood V of z to get $Df = Dg$ almost everywhere in $K \cap V$ ($K \cap V$ is closed in V for the relative topology in V). The remaining points form a discrete set in U of zero Lebesgue measure. □

3. Rectification of degenerate conformal structures.

3.1. **Introduction.** Morrey-Ahlfors-Bers theorem proves the existence of solutions of the Beltrami equation

$$\bar{\partial} f = \mu \, \partial f$$

when μ is measurable and bounded $||\mu||_{L^\infty} < 1$. This is a very powerful theorem of rectification of conformal structures quasi-conformal with respect to the standard structure. Numerous efforts have been devoted to weak the boundeness assumption, i.e. to remove the quasi-conformality assumption. O. Lehto [10] [11] and G. David [6] proved existence theorems for degenerate Beltrami forms with $||\mu||_{L^\infty} = 1$ with assumptions on the measure of the set where μ degenerates (i.e. where μ gets close to 1 in absolute value). Typically David's condition requires that the measure of the set where μ degenerates decreases exponentially fast:

$$|\{z \in \mathbb{C}; 1 - \epsilon < |\mu(z)|\}| \leq C_0 e^{-\frac{1}{\epsilon}}.$$

Lehto's condition is more general, allows cancellations, and is expressed in an integral form (we refer to Lehto's articles cited before for a precise definition). This type of conditions is necessary in order to preserve the topology (i.e. to have a rectification that is indeed a homeomorphism). It is easy to construct conterexamples where the topology is destroyed (for example when there are annulus with infinite modulus have finite modulus with respect to the new conformal structure). The solutions that these authors obtain are always almost everywhere differentiable homeomorphisms. Unfortunately these theorems don't seem to be adapted to the type of wild "collapsing"

We present in this section new general theorems of existence of solutions of "the Beltrami equation" for Beltrami forms μ such that $||\mu||_{L^\infty} = 1$.

In the first theorem (section 3.2) the Beltrami form μ will be bounded away from 1 on compact subsets of $\mathbb{C} - K$ where K is a thin Cantor set. The main requirement is that there is a tree associated to K that is thin with respect to the *new* conformal structure. Then the solution will be a unique almost everywhere differentiable homeomorphism. The same idea will prove the existence of solutions of the Beltrami equation in a unique way for a class of extremely degenerate Beltrami forms not even defining quasi-conformal structures in compact subsets of the complement of K, which this time is assumed to be potentially thin (section 3.2) The conformal structures defined by such degenerate Beltrami coefficients are of a very special nature. They are *potential* conformal structures which in general terms it means that they are compatible with the potential of K. The solutions will still be homeomorphisms. The next step (section 3.4) generalizes the previous rectification theorems to *virtual* conformal structures that are not conformal structures in a classical sense. They are defined by sequences of degenerating potential conformal structures. These rectifications do have a uniquely determined limit that can be though of as the "rectification" of the virtual conformal structure. In general the solutions this time will not even be homeomorphisms. They will be continuous mappings from the Riemann sphere into itself. For example, we will show examples where a full measure dense open set is collapsed into a set of measure zero.

The idea at the base of these new rectification theorems is to consider Beltrami coefficients that do degenerate into a thin or potentially thin Cantor set. The removability of this thin Cantor set is used to construct the unique rectification. More precisely the image of the thin Cantor set is uniquely determined and is the Hausdorff limit of its image by rectifications of approximating quasi-conformal structures. When considering conformal structures that do respect the potential theory in the exterior of the potentially thin Cantor set, from the convergence of the images of the Cantor sets we get the convergence of the potential and therefore the convergence of the rectifications to a unique limit mapping.

3.2. **First rectification theorems.**

Theorem 3.1. *We consider a compact set $K \subset \mathbb{C}$. Let ξ be a conformal structure on $\mathbb{C}-K$. We assume that ξ is quasi-conformal on compact subsets of $\mathbb{C}-K$. We denote by μ its Beltrami coefficient. The locally rectifiable conformal structure ξ defines a new Riemann surface structure on $\mathbb{C}-K$. We assume that $(\mathbb{C}-K, \xi)$ is a Riemann surface in the class O_{AD}.*

There exists a continuous mapping $h : (\overline{\mathbb{C}}, \xi) \to (\overline{\mathbb{C}}, \sigma_0)$ rectifying the conformal structure ξ, i.e. the restriction $h_{/\mathbb{C}-K}$ is a homeomorphism that is quasi-conformal on compact subsets of $\mathbb{C}-K$ and is a solution to the Beltrami equation

$$\bar{\partial} h = \mu \; \partial h.$$

Moreover h is unique up to composition to the left by a Moebius transformation. Thus if z_0 and z_1 are two points in $\mathbb{C}-K$, h is uniquely determined by the normalization

$h(z_0) = z_0$, $h(z_1) = z - 1$ and $h(\infty) = \infty$. If K is totally disconnected then h is a homeomorphism of the Riemann sphere. Otherwise h collapses components of K into points.

Also if we truncate μ in an ϵ_n-neighborhood of $K \cup \{\infty\}$, $\mu_n = \mu.\mathbf{1}_{\mathbb{C}-V_{\epsilon_n}(K\cup\{\infty\})}$, and we consider the classical rectifications (h_n) of (μ_n) normalized as h, $h_n(z_0) = z_0$, $h_n(z_1) = z_1$ and $h_n(\infty) = \infty$. Then $h_n \to h$ uniformly on the Riemann sphere.

Proof. Consider the Riemann surface $\mathcal{S} = \mathbb{C} - K$ endowed with the new conformal structure ξ. Local rectifications of ξ provide charts for the new complex structure on $\mathcal{S}$. Obviously for this new complex structure $\mathcal{S}$ is still a planar Riemann surface (it is a topological property). Thus there is an embedding of $\mathcal{S}$ in $\mathbb{C}$ (with the end corresponding to ∞ corresponding to ∞). We obtain in this way a rectification mapping $h : \mathbb{C} - K \to \mathbb{C} - K_\infty$. The compact set K_∞ is a totally disconnected set set of absolute area zero. So h extends continuously into a map $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ mapping each connected component of K into the corresponding point of K_∞.

We prove the uniqueness. Let h and h' be two solutions satisfying the theorem. Then $h' \circ h^{-1} : \mathbb{C} - h(K) \to \mathbb{C} - h'(K)$ is a holomorphic diffeomorphism. Since $\mathbb{C} - h(K)$ is a O_{AD} Riemann surface, using theorem 2.16 condition (iii) we have that $h' \circ h^{-1}$ is a Moëbius transformation as claimed.

It remains to prove the second part of the theorem about the approximation of h by classical rectifications. We obtain also another proof of the existence of h. Assume that z_0 and z_1 are two distinct points in $\mathbb{C} - K$. Consider a sequence (ϵ_n), $\epsilon_n > 0$, $\epsilon_n \to 0$, and consider the ϵ_n-neighborhood (for the chordal metric) of $K \cup \{\infty\}$, $V_{\epsilon_n}(K \cup \{\infty\})$. We truncate μ near $K \cup \infty$. Let $\mu_n = \mu \,.\mathbf{1}_{\mathbb{C}-V_{\epsilon_n}(K\cup\{\infty\})}$. The Beltrami form μ_n satisfies $||\mu_n||_{L^\infty} < 1$ and defines a classical quasi-conformal structure. Let ξ_n be its associated complex structure. Consider a sequence of classical Morrey-Ahlfors-Bers rectifications $h_n : (\overline{\mathbb{C}}, \xi_n) \to (\overline{\mathbb{C}}, \sigma_0)$ normalized such that $h_n(z_0) = z_0$, $h_n(z_1) = z_1$ and $h_n(\infty) = \infty$. We prove that the sequence (h_n) converges uniformly to the desired h.

Observe that the sequence (h_n) is equicontinuous in $\mathbb{C} - K$ because the h_n are uniformly Hölder on compact subsets of $\mathbb{C} - K$ because they are quasi-conformal (see [13] for classical results on quasi-conformal theory). Thus extracting a subsequence we can also assume that

$$\lim_{k\to+\infty} h_{n_k} = h$$

uniformly on compact subsets of $\mathbb{C} - K$. Observe that by the same classical proof in quasi-conformal theory (see [13]), the limit h is a homeomorphism from $\mathbb{C} - K$ into its image and is a quasi-conformal homeomorphism on compact subsets of $\mathbb{C} - K$.

Moreover $h(\mathbb{C}-K)=\mathbb{C}-K_\infty$ is biholomorphic to $(\mathbb{C}-K,\xi)$, thus it is a Riemann surface in the class O_{AD} and K_∞ is totally disconnected of absolute area 0. Thus the limit h is unique and we have on $\mathbb{C}-K$

$$h=\lim_{n\to+\infty} h_n.$$

Also if C is a component of K then $h_n(C)$ converges to a point of K_∞ (the point corresponding to the corresponding end of $\mathbb{C}-K$.) Thus on K the mappings h_n converge pointwise to an extension of h still denoted by $h:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$. The convergence is uniform on the Riemann sphere. Consider an exhaustion (U_i) of $\mathbb{C}-K$ with $\overline{U_i}$ a compact set in $\mathbb{C}-K$. Let $\epsilon>0$ and choose i large enough so that the diameter of the components of the complement of $h(\overline{U_i})$ have diameter less than $\epsilon/2$. The sequence (h_n) converges uniformly on U_i to h. There is $N\geq 1$ so that for $n\geq N$, $||h-h_n||_{C^0(U_i)}\leq\epsilon/2$. Thus, for $n\geq N$, the diameter of the image by h_n of a component of the complement of $\overline{U_i}$ is less than ϵ and $||h-h_n||_{C^0(\overline{\mathbb{C}})}\leq\epsilon$. Q.E.D. □

3.2.1. *Rectification theorem for thin Cantor sets.* In the applications we need an effective version of this theorem. First we have to be able to check that the Riemann surface $(\mathbb{C}-K,\xi)$ is in the class O_{AD}. For this we can use Sario's criterium or McMullen's version of it. In practice K is not an arbitrary compact set in the plane but a thin Cantor set with an associated tree of annulus.

Let ξ be a conformal structure on $\Omega=\overline{\mathbb{C}}-K$ such that ξ is quasi-conformal on compact subsets of $\mathbb{C}-K$ (observe that we allow ξ to degenerate in a neighborhood of ∞). Observe that we can consider modulus of annulus relatively compact in $\mathbb{C}-K$ with respect to ξ: The modulus

$$\text{mod}\,_\xi A$$

is the modulus of $h(A)$ where h is a quasi-conformal rectification of ξ in a neighborhood of A. Observe that this quantity is independent of the rectification chosen. We denote μ be the Beltrami form associated to ξ (defined to be 0 on K and at the point ∞, i.e. we consider there $\xi=\sigma_0$).

We assume the following fundamental hypothesis (we say that ξ is *admissible*) :

(H) *There exists a thin tree $\mathcal{T}$ associated to K such that $\mathcal{T}$ is also thin with respect to the conformal structure ξ, i.e. for any infinite branch (A_i) of $\mathcal{T}$ deprived from the root we have*

$$\sum_i \text{mod}\,_\xi A_i=+\infty,$$

and

$$\text{mod}\,_\xi A_0=+\infty.$$

Since A_0 is a pointed disk at ∞ and so it is not relatively compact in $\mathbb{C} - K$, $\mathrm{mod}\ _\xi A_0 = +\infty$ requires an explanation. It means that

$$\lim_{\epsilon \to 0} \quad \mathrm{mod}\ _\xi A_0 - V_\epsilon(\infty) = +\infty,$$

where $V_\epsilon(\infty)$ denotes an ϵ neighborhood of ∞ for the chordal metric.

Note that by McMullen's criterium the condition (H) implies that $(\mathbb{C} - K, \xi)$ is a Riemann surface in the class O_{AD}. Moreover since K is a Cantor set, the rectification given by the previous theorem is a global homeomorphism of the Riemann sphere. Also, transporting the tree of annulus by h and using McMullen's criterium, we have that $h(K)$ is a thin Cantor set. In conclusion we have:

Theorem 3.2. *Let ξ be an admissible conformal structure on $\mathbb{C} - K$. We denote by μ its Beltrami form.*

There exists a unique absolutely continuous almost everywhere differentiable homeomorphism rectifying the conformal structure ξ, $h : (\overline{\mathbb{C}}, \xi) \to (\overline{\mathbb{C}}, \sigma_0)$, i.e. solving the Beltrami equation

$$\bar{\partial} h = \mu\ \partial h,$$

with h normalized such that $h(0) = 0$, $h(1) = 1$ and $h(\infty) = \infty$.

Moreover, h is a quasi-conformal homeomorphism on compact subsets of $\mathbb{C} - K$ and $h(K)$ is also a thin Cantor set.

Also if we truncate μ in an ϵ_n neighborhood of $K \cup \{\infty\}$, $\mu_n = \mu.\mathbf{1}_{\mathbb{C} - V_{\epsilon_n}(K \cup \{\infty\})}$, and we consider the classical rectifications (h_n) of (μ_n), then $h_n \to h$ uniformly on the Riemann sphere.

Definition 3.3. ***(Generalized rectifications).*** *The solutions of the Beltrami equation provided by theorem 3.1 are called generalized rectifications.*

A similar proof as for theorem 3.1 gives

Theorem 3.4. *Let (μ_n) be a sequence of Beltrami forms with $||\mu_n||_{L^\infty(C)} \leq k(C) < 1$ for compact subsets $C \in \mathbb{C} - K$ such that*

$$\lim_{n \to +\infty} \mu_n(z) = \mu(z)$$

for almost every $z \in \overline{\mathbb{C}}$, where μ is a Beltrami form as in theorem 3.2. Let (h_n) be the classical Morrey-Ahlfors-Bers rectifications normalized in the usual way, and h the generalized rectification for μ. Then we have uniformly on the Riemann sphere

$$\lim_{n \to +\infty} h_n = h.$$

3.3. **Rectification of potential conformal structures.** The "effective" version of the rectification theorem in the previous section are more explicit for potential conformal structures that are allowed to be even not locally quasi-conformal.

From now on we fix K to be a totally thin Cantor set with all points regular for the Dirichlet problem.

Let ξ be a potential conformal. To each annulus A in the analytic tree of K, there corresponds by $E^{-1} \circ \varphi_K$ a finite number of rectangles $(R_j(A))$ with horizontal and vertical sides in the log-Böttcher upper half plane (the vertical sides are contained in the preimage of the skeleton $E^{-1}(\Gamma'(K))$). There are natural identification of the lateral sides of the preimage of the skeleton $E^{-1}(\Gamma'(K))$ defined by φ_K. Pasting in the prescribed way the vertical boundaries of the rectangles $(R_j(A))$ we recover the annulus A, and we observe that t is easy to check that i

$$\text{mod}\ (A) = \sum_j R_j(A).$$

We define the modulus of A with respect to ξ by

$$\text{mod}\,_\xi A = \sum_j \quad \text{mod}\,_{\xi'} R_j(A) = \sum_j \quad \text{mod}\ L(R_j(A)).$$

t is easy to check that i Observe that when ξ is quasi-conformal with respect to σ_0 in a neighborhood of A this definition coincides with the one given before (observe that L defines a local rectification of each $R_j(A)$ that respects the vertical gluing).

Definition 3.5. ***(Admissible potential conformal structures).*** *A potential conformal structure ξ is admissible if the analytic tree $\mathcal{T}_\omega(K)$ is thin for the conformal structure ξ. This means that for any infinite branch (A_i) deprived from the root*

$$\sum_i \quad \text{mod}\,_\xi A_i = +\infty$$

(note that $\text{mod}\,_\xi A_0 = +\infty$ *is automatic).*

Observation.

We remind that an admissible potential conformal structure is not necessarily admissible in the sense of the definition in theorem 3.2 because it can be non-quasi-conformal on compact subsets of $\mathbb{C} - K$. Thus we allow even more degenerate conformal structures. On the other hand the ellipse field defining ξ is of a very special nature: All ellipses have principal axes tangent or orthogonal to equipotentials.

Note that a potential conformal structure ξ has a well defined Beltrami form μ because L is almost everywhere differentiable.

Theorem 3.6. *Let ξ be an admissible potential conformal structure with Beltrami form μ.*

There exists a unique absolutely continuous almost everywhere differentiable homeomorphism rectifying the conformal structure ξ, $h : (\overline{\mathbb{C}}, \xi) \to (\overline{\mathbb{C}}, \sigma_0)$, i.e. solving the Beltrami equation

$$\bar{\partial} h = \mu \ \partial h,$$

with h normalized such that $h(0) = 0$, $h(1) = 1$ and $h(\infty) = \infty$.

Moreover, $h(K)$ is a totally thin Cantor set. Also if ξ_n are potential conformal structures with corresponding (d_n) and (k_n) lipschitz maps (so that ξ_n is quasi-conformal) such that $d_n \to d$ and $k_n \to k$ uniformly, then uniformly on $\overline{\mathbb{C}}$,

$$\lim_{n \to +\infty} h_n = h$$

where the (h_n) are the classical rectifications normalized as usual.

The rectification h is called the generalized rectification of ξ.

Proof. The proof follows the same lines as the one of theorem 3.1. We consider sequences (d_n) and (k_n) of Lipschitz homeomorphisms $d_n : \mathbb{T} \to \mathbb{T}$ and $k_n : \mathbb{R}_+ \to \mathbb{R}_+$, that converge uniformly to d and k respectively. We consider the corresponding conformal structures

$$\xi_n = (\varphi_K^{-1} \circ E)_* \ L_n^* \ \sigma_0 \ ,$$

where $L_n(x, y) = (d_n(x), k_n(y))$.

The conformal structures ξ_n are quasi-conformal with respect to σ_0. As in theorem 3.1 we prove that the sequence of normalized rectifications $h_{\xi_n} : (\overline{\mathbb{C}}, \xi_n) \to (\overline{\mathbb{C}}, \sigma_0)$ converges to a unique mapping h that satisfies the conditions of the theorem. We have a similar proof. First the sequence of $(h_{\xi_n}(K))$ is uniformly bounded.

This sequence has a limit K_∞ in Hausdorff topology because any limit compact is a totally thin Cantor set and its analytic tree is determined by ξ, (more precisely d and k determine the modular invariants and d alone determines the angular invariants) and the rectifications are normalized.

Now, on the exterior of K every point is determined by its potential $G_K(z)$ and its set of external angles $\tau_K(z)$ (in general there is only one external angle but it can have more than one when it belongs to a critical external ray). We observe that the Green map of $h_{\xi_n}(K)$ converges uniformly on compact sets to the Green map of K_∞. The rectifications h_{ξ_n} transport equipotentials (resp. external rays) of K into equipotentials (resp. external rays) of $h_{\xi_n}(K)$ according to the map k_n (resp. d_n). Moreover the Green function of $h_{\xi_n}(K)$ converges uniformly on compact sets of $\mathbb{C} - K$ to the Green function of K_∞. It follows that h_{ξ_n} converges uniformly on compact sets of $\mathbb{C} - K$ to a map h that maps a point $z \in \mathbb{C} - K$ to the point $h(z) \in \mathbb{C} - K_\infty$ determined by

$$G_{K_\infty}(h(z)) = k(G_K(z)),$$
$$\tau_{K_\infty}(h(z)) = d(\tau_K(z)),$$

The map h extends continuously to the Cantor set K and $h(K) = K_\infty$.

We finally observe that on the complement of K and $h(K)$, the map h is an absolutely continuous almost everywhere differentiable homeomorphism (this map is just L expressed in the Böttcher coordinates of the domain and the range).

Again the uniform convergence on K is obtained from the convergence of finite parts of the analytic tree to the analytic tree of the limit Cantor set. □

We observe that in the precedent proof we didn't make essential use of the fact that the limit mapping on external angles d was a homeomorphism. It was only used to define the modulus of annulus with respect to ξ in order to identify the analytic tree of the limit. This is the key point observation before going to the most general rectification theorem.

3.4. **Rectification of virtual conformal structures.** We consider a totally thin Cantor set K and a virtual conformal structure ξ on $\overline{\mathbb{C}}$ given by L (or (d, k)). We denote by ξ' the corresponding conformal structure on $\mathbb{H}/\mathbb{Z}$ defined by the same L. The Cantor set K is the ideal boundary of the Riemann surface $\overline{\mathbb{C}} - K$. The measure μ_d being non-atomic and the critical external rays being countable, there is a natural way of projecting μ_d into a non-atomic probability measure of the ideal boundary. The support of this projected measure is a Cantor set $K_0 \subset K$.

Let R be a rectangle in the log-Böttcher upper half cylinder $\mathbb{H}/\mathbb{Z}$ having vertical and horizontal sides. It is then natural to define by analogy with a classical complex structure

$$\mathrm{mod}_{\xi'} R = \mathrm{mod}\, L(R),$$

and

$$\mathrm{mod}_{\xi} A = \sum_j \mathrm{mod}_{\xi'} R_j(A),$$

for an annulus A in the analytic tree of K. Note that this definition is compatible with the previous ones when the virtual conformal structure is a classical conformal structure. Observe also that if $J_0(A) \subset \mathbb{T}$ denotes the set of external angles whose corresponding external ray intersects A and if $J_1(A) \subset \mathbb{R}_+$ is the interval of equipotential values intersecting A, then

$$\mod{}_{\xi} A = \frac{|J_0(A)|}{\mu_d(J_0(A))} \frac{|k(J_1(A))|}{|J_1(A)|} \quad \mod A,$$

where μ_d is the non-atomic probability measure corresponding to $d \in \hat{\mathcal{C}}_0(\mathbb{T})$, and $|J(A)|$ denotes the Lebesgue (linear) measure of $J(A)$. Thus we observe that if $\operatorname{supp} \mu_d \neq \mathbb{T}$ (i.e. d is not a homeomorphism) then we may have for some annulus $\mod{}_{\xi} A = +\infty$. This is a new feature that did not happen for non virtual conformal structures. Also from the above formula it follows that we cannot have $\mod{}_{\xi} A = +\infty$ for all annulus A at a given depth. Otherwise by additivity of μ_d it would follow $\mu_d(\mathbb{T}) = 0$ contradicting that μ_d is a probability measure.

Definition 3.7. *A virtual conformal structure ξ is admissible if for each infinite branch (A_i) deprived from the root of the analytic tree $\mathcal{T}_\omega(K)$ we have*

$$\sum_i \mod{}_{\xi} A_i = +\infty.$$

We have the following rectification theorem.

Theorem 3.8. *Let ξ be a virtual conformal structure in $\overline{\mathbb{C}} - K$ (where K is a totally thin Cantor set as above) defined by $d \in \hat{\mathcal{C}}_0(\mathbb{T})$ and $k \in \mathrm{Homeo}_+(\mathbb{R}_+)$. We assume that ξ is admissible and that $0, 1 \in K_0$ where K_0 is the support of the projection of μ_d on K.*

There is a unique map correspondence $h \in \hat{\mathcal{C}}_0(\overline{\mathbb{C}}$ which rectifies ξ such that $h(0) = 0$, $h(1) = 1$ and $h(\infty) = \infty$.

More precisely, the subset K_∞ of regular (or non-isolated) points of $h(K)$ is a totally thin Cantor set, $K_\infty = h(K_0)$ and on the complement of $E^{-1}(\Gamma'(K))$,

$$h \circ \varphi_K^{-1} \circ E = \varphi_{h(K_0)}^{-1} \circ E \circ L.$$

If there is no annulus A in the analytic tree that degenerates (that is $\mod{}_{\xi} A < +\infty$, i.e. the support of μ_d is $\mathbb{T}$ so $K_0 = K$), then h is a homeomorphism and $h(K) = K_\infty$.

Moreover, if (ξ_n) is a sequence of potential conformal structures with associate homeomorphisms (d_n) and $k_n = k$ with $d_n \to d$ uniformly on $\mathbb{T}$, then uniformly on the Riemann sphere

$$\lim_{n \to +\infty} h_{\xi_n} = h,$$

where the rectifications have been properly normalized composing by a suitable Moëbius transformation.

Before going into the proof, we have to study the limit Cantor set K_∞ and more precisely how its analytic tree is related with the one of K.

3.4.1. *Combinatorial collapsing.* We consider the above situation. The virtual conformal structure ξ is determined by (d,k). We construct the analytic tree $\tau_\omega^\xi(K)$ of K with respect to ξ from $\tau_\omega(K)$ as follows.

We first delete from $\tau_\omega(K)$ all branches after a vertex A such that $\mod_\xi A = +\infty$. Given a vertex B, we denote by $\alpha_+^{(1)}(B), \alpha_+^{(2)}(B) \in \mathbb{T}$ the two external angles of the critical point in the outer boundary of B, and $\alpha_-^{(1)}(B), \alpha_-^{(2)}(B) \in \mathbb{T}$ the two angles of the critical point in the inner boundary. We have that

$$\left\{ \frac{\alpha_+^{(1)} - \alpha_-^{(1)}}{|\alpha_+^{(1)} - \alpha_+^{(2)}|}, \frac{\alpha_+^{(1)} - \alpha_-^{(2)}}{|\alpha_+^{(1)} - \alpha_+^{(2)}|} \right\}$$

is the angular invariant of B (the expression $|\alpha_+^{(1)} - \alpha_+^{(2)}|$ is the harmonic measure of the annulus B). ic Observe that a vertex A satisfies $\mod_\xi A = +\infty$ if and only if

$$d(\alpha_+^{(1)}(A)) = d(\alpha_+^{(2)}(A)),$$

or equivalently, if and only if for its parent B we have

$$d(\alpha_-^{(1)}(B)) = d(\alpha_-^{(2)}(B)).$$

Next we remove all vertices in the remaining tree that have only one children. The new branches of the tree can be composed by several old branches. We change the modular invariant of the father vertex (denote it by A_1) by the sum of its old modular invariant with the modular invariants of the deleted old children (denote them by $A_2, \dots, A_n$). So

$$M^\xi(A_1) = \sum_{i=1}^{n} \mod A_i.$$

Its new angular invariants will be

$$
\begin{aligned}
\tau_1^\xi(A_1) &= \frac{1}{|d(\alpha_+^{(1)}(A_1)) - d(\alpha_+^{(2)}(A_1))|} \sum_{i=1}^{n} d(\alpha_+^{(1)}(A_i)) - d(\alpha_-^{(1)}(A_i)) \\
&= \frac{d(\alpha_+^{(1)}(A_1)) - d(\alpha_+^{(1)}(A_n))}{|d(\alpha_+^{(1)}(A_1)) - d(\alpha_+^{(2)}(A_1))|} \\
\tau_2^\xi(A_1) &= \frac{1}{|d(\alpha_+^{(1)}(A_1)) - d(\alpha_+^{(2)}(A_1))|} \sum_{i=1}^{n} d(\alpha_+^{(1)}(A_i)) - d(\alpha_-^{(2)}(A_i)) \\
&= \frac{d(\alpha_+^{(1)}(A_1)) - d(\alpha_+^{(1)}(A_n))}{|d(\alpha_+^{(1)}(A_1)) - d(\alpha_+^{(2)}(A_1))|} \\
\tau^\xi(A_1) &= \{\tau_1^\xi(A_1), \tau_2^\xi(A_2)\}
\end{aligned}
$$

We denote by $(\hat{A}_i)$ the final tree. It is a binary tree (we cannot remove all vertices of an infinite branch of the original tree because the measure μ_d has no atoms). The new analytic tree is

$$\tau_\omega^\xi(K) = \left(\hat{A}_i, M^\xi(\hat{A}_i), \tau^\xi(\hat{A}_i)\right).$$

Proof of Theorem 3.8.

Consider a sequence of classical conformal structures (ξ_n) quasi-conformal with respect to σ_0 (just take d_n and k_n to be lipschitz) as in the Theorem.

Observe that on $E^{-1}(\Gamma'(K))$ we have,

$$h_{\xi_n} \circ \varphi_K^{-1} \circ E = \varphi_{h_{\xi_n}(K)}^{-1} \circ E \circ L_n.$$

As before, the mappings h_{ξ_n} being normalized, the sequence of compacts sets (K_n), $K_n = h_{\xi_n}(K)$, is uniformly bounded. We can extract converging sub-sequences

$$
\begin{aligned}
\lim_{k\to+\infty} K_{n_k} &= K'_\infty, \\
\lim_{k\to+\infty} \varphi_{K_{n_k}} &= \varphi_\infty, \\
\lim_{k\to+\infty} \Gamma(K_{n_k}) &= \Gamma_\infty(K).
\end{aligned}
$$

The map φ_∞^{-1} is holomorphic near ∞ and is defined in the kernel of the domain of definition of the $(\varphi_{K_n}^{-1})$,

$$\varphi_\infty^{-1} : \overline{\mathbb{C}} - E \circ L \circ E^{-1}(\Gamma'(K)) \to \overline{\mathbb{C}} - (K'_\infty \cup \Gamma_\infty(K)).$$

When there is collapsing (i.e. when d is not a homeomorphism) the limit K'_∞ contains non-regular points. More precisely, to each collapsing annulus A there corresponds an isolated point of K'_∞ which the limit of all points of K enclosed by A. This point if the image of a tip of a segment of $E \circ L \circ E^{-1}(\Gamma'(K))$. The map φ_∞^{-1} extends holomorphically at this point in a univalent way. We denote by K_∞ the subset of regular points of K'_∞ (i.e. K'_∞ deprived of its isolated points). The extension of φ_∞ constructed in this way is the Green map of K_∞. It is not difficult to see that the analytic tree of K_∞ coincides with the analytic tree $\tau_\omega^\xi(K)$ of K with respect to the virtual conformal structure ξ constructed above. Thus K_∞ is a totally thin Cantor set uniquely determined from K and ξ. Thus we don't need to extract converging sub-sequences: The limit K'_∞ is uniquely determined. The limit of the mappings (h_{ξ_n}) is uniquely determined on K, but also outside K by the potential outside of K_∞ (as in the proof of theorem 3.1). Passing to the limit in the above relation we obtain on $E^{-1}(\Gamma'(K))$,

$$h \circ \varphi_K^{-1} \circ E = \varphi_{K_\infty}^{-1} \circ E \circ L.$$

The other properties of the theorem are obtained as in theorem 3.2.

3.5. **Generalized rectifications for continua.** In this section we consider a *connected* compact set K as above (so $\Gamma_K = \emptyset$ and φ_K is a conformal representation).

Definition 3.9. ***(Green equivalence in*** $\mathbb{H}/\mathbb{Z}$***).*** *Two conformal structures ξ and η on $\mathbb{H}/\mathbb{Z}$ are Green equivalent if there exists an absolutely continuous almost everywhere differentiable homeomorphism $L : \mathbb{H}/\mathbb{Z} \to \mathbb{H}/\mathbb{Z}$ of the form*

$$L(x, y) = (x, h(y)),$$

such that

$$\eta = L_*\xi.$$

In that case L extends into a homeomorphism of $\overline{\mathbb{H}}$ by the identity on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The map L is an absolutely continuous almost everywhere differentiable homeomorphism if and only if $h : \mathbb{R}_+ \to \mathbb{R}_+$ is an absolutely continuous homeomorphism.

This is a well defined equivalence relation since the class of homeomorphisms L considered is a composition subgroup. Observe that L leaves globally invariant vertical lines, i.e. Green lines for the potential $z \mapsto \Im z$ in $\mathbb{H}/\mathbb{Z}$.

We recall that we denote $E(z) = e^{-2\pi i z}$.

Definition 3.10. ***(Green equivalence outside a compact connected set*** K***).*** *Let $K \subset \mathbb{C}$ be a compact connected set as above. Two conformal structures ξ and η*

on $\Omega = \overline{\mathbb{C}} - K$ *are Green equivalent outside* K *if*

$$\xi' = (\varphi_K^{-1} \circ E)^* \xi,$$
$$\eta' = (\varphi_K^{-1} \circ E)^* \eta,$$

are Green equivalent on $\mathbb{H}/\mathbb{Z}$.

Then there exists an absolutely continuous almost everywhere differentiable homeomorphism $l : \overline{\mathbb{C}} - K \to \overline{\mathbb{C}} - K$, *such that*

$$\eta = l_* \xi,$$

and l *leaves invariant Green lines, maps equipotentials into equipotentials, and is defined by*

$$l \circ \varphi_K^{-1} \circ E = \varphi_K^{-1} \circ E \circ L,$$

where $L : \mathbb{H} \to \mathbb{H}$ *is the mapping realizing the equipotential equivalence of* ξ' *and* η'.

The following result shows how are related the Morrey-Ahlfors-Bers rectification of two Green equivalent conformal structures that are quasi-conformal (with respect to the standard complex structure σ_0).

Theorem 3.11. *Let* $K \subset \overline{\mathbb{C}}$ *a compact connected set as above. Assume that* $0 \in K$ *and* $1 \in K$. *Let* ξ *and* η *two conformal structures on* $\Omega = \overline{\mathbb{C}} - K$ *which are quasi-conformal with respect to* σ_0 *and Green equivalent outside* K. *Then the mapping* l *is quasi-conformal (i.e. uniformly quasi-conformal on* $\overline{\mathbb{C}} - K$*) and we can extend* ξ *and* η *into quasi-conformal structures on* $\overline{\mathbb{C}}$ *defining*

$$\xi_{/K} = \eta_{/K} = \sigma_{0\ /K}.$$

Let $h_\xi : (\overline{\mathbb{C}}, \xi) \to (\overline{\mathbb{C}}, \sigma_0)$ *and* $h_\eta : (\overline{\mathbb{C}}, \eta) \to (\overline{\mathbb{C}}, \sigma_0)$ *be the unique Morrey-Ahlfors-Bers rectifications normalized so that*

$$h_\xi(0) = 0,\ h_\xi(1) = 1,\ h_\xi(\infty) = \infty,$$
$$h_\eta(0) = 0,\ h_\eta(1) = 1,\ h_\eta(\infty) = \infty,$$

Then we have

$$h_{\xi/K} = h_{\eta\ /K},$$
$$h_{\xi/\overline{\mathbb{C}}-K} = h_{\eta/\overline{\mathbb{C}}-K} \circ l.$$

In particular we have

$$h_\xi(K) = h_\eta(K).$$

Lemma 3.12. *If* $L : \mathbb{H}/\mathbb{Z} \to \mathbb{H}/\mathbb{Z}$, $(x, y) \mapsto (x, h(y))$ *is a* C*-quasi-conformal homeomorphism then* h *(and also* L*) is* C*-bilipschitz, i.e. for* $y, y' \geq 0$,

$$C^{-1}|y - y'| \leq |h(y) - h(y')| \leq C|y - y'|.$$

In particular, if l *is related to* L *as in definition 3.9, if* d_P *is the Poincaré distance in* $\mathbb{C} - K$*, then* l *is bounded away from the identity for the uniform norm for* d_P*, more precisely,for* $z \in \mathbb{C} - K$,

$$d_P(l(z), z) \leq \log C.$$

Proof. **(Lemma 3.12).**The map L is differentiable almost everywhere and we compute

$$DL = \begin{pmatrix} 1 & 0 \\ 0 & h'(y) \end{pmatrix}.$$

Thus for almost every $y > 0$, $|h'(y)| \leq C$ and the first statement follows. For the last statement we observe that it is enough to check that $L - \mathrm{id}$ is bounded for the Poincaré distance $d_{\mathbb{H}/\mathbb{Z}}$ of $\mathbb{H}/\mathbb{Z}$. But

$$d_{\mathbb{H}/\mathbb{Z}}(L(z), z) = \left|\log\left(\frac{h(y)}{y}\right)\right| \leq \log C.$$

□

Proof. **(Theorem 3.11).** Given h_η we can define $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by

$$h_{/K} = h_{\eta\ /K},$$
$$h_{/\overline{\mathbb{C}}-K} = h_{\eta\ /\overline{\mathbb{C}}-K} \circ l.$$

If $z_n \to z_0 \in K$ with $z_n \in \overline{\mathbb{C}} - K$ then $l(z_n) - z_n \to 0$ because from lemma 3.12 we know that $d_P(l(z_n), z_n)$ is bounded. So $l(z_n) \to z_0$ and l extends continuously to K by the identity. Thus the map h is a homeomorphism. Moreover h is quasi-conformal on $\overline{\mathbb{C}} - K$ because l is quasi-conformal there, and coincides with the global quasi-conformal homeomorphism h_η in K. Now Rickman's theorem (see Theorem 2.29) implies that h is quasi-conformal and on K,

$$\bar{\partial} h = \bar{\partial} h_\eta = 0.$$

So h rectifies ξ as is normalized as h_ξ. By uniqueness of Morrey-Ahlfors-Bers theorem we have $h_\xi = h$. □

A corollary of this result justifies the terminology used:

Corollary 3.13. *If ξ and η are Green equivalent as above, if η happens to be the standard complex structure σ_0 then $h_\xi(K) = K$, and h_ξ maps equipotentials of K into equipotentials of K, and leaves globally invariant external rays of K.*

Also if h_η maps equipotentials of K into equipotentials of $h_\eta(K)$, and external rays of K into external rays of $h_\eta(K)$ then h_ξ has the same property.

Proof. In the case that $\eta = \sigma_0$, we have $h_\eta = \mathrm{id}$ and $h_{\xi/\overline{\mathbb{C}}-K} = l$. The statement follows from the fact that $\varphi_K \circ l \circ \varphi_K^{-1}$ maps circles centered at 0 into concentric circles, and radial lines into radial lines.

The second statement follows from $h_{\xi/\overline{\mathbb{C}}-K} = h_{\eta/\overline{\mathbb{C}}-K} \circ l$. □

The last theorem suggest that when ξ is Green equivalent to a quasi-conformal structure η we can define a generalized rectification map using the rectification of η.

Definition 3.14. ***(Generalized rectification).*** *Let ξ and η be Green equivalent conformal structures on $\overline{\mathbb{C}} - K$ with η quasi-conformal with respect to σ_0. We assume $0, 1 \in K$. The generalized rectification of ξ is $h_\xi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined by*

$$h_{\xi/K} = h_{\eta/K},$$
$$h_{\xi/\overline{\mathbb{C}}-K} = h_{\eta/\overline{\mathbb{C}}-K} \circ l.$$

Observe that h_ξ is absolutely continuous but not necessarily almost everywhere differentiable on K and not even continuous (!) because there is no reason that l extends continuously by the identity on K when K is not locally connected. It is almost everywhere differentiable on $\overline{\mathbb{C}} - K$ and

$$(h_\xi)_* \xi = \sigma_0.$$

If ∂K has measure 0 then h_ξ is almost everywhere differentiable, $(h_\xi)_* \xi = \sigma_0$, although h_ξ may be discontinuous. When all external rays land at K (in particular when K is locally connected), the map h_ξ is a homeomorphism of the Riemann sphere.

We prove that the generalized rectification does not depend on the choice of η.

Remark 3.15. It is not difficult to construct non continuous generalized rectifications. Consider a continuum K with a non landing external ray γ. Choose a sequence of points $z_n \in \gamma$ with decreasing potential such that $|z_n - z_{n+1}| \geq \epsilon_0 > 0$ for some $\epsilon_0 > 0$. Then construct by interpolation an absolutely continuous homeomorphism $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h(G_K(z_{n+1})) = G_K(z_n)$. The corresponding map l is the generalized rectification for $\xi = l^* \sigma_0$ in $\mathbb{C} - K$ and does not extend continuously to the identity on K. But h_ξ is the identity on K.

The precedent corollary also holds for generalized rectifications (i.e. when ξ is not assumed to be quasi-conformal) with the same proof.

The unsatisfactory part of the generalized rectification is that in general it is not even a homeomorphism and also it is not almost everywhere differentiable. We study this problem below and give some sufficient conditions on the equipotential equivalence that implie the almost everywhere differentiability of the generalized rectification.

On the other hand, the generalized rectification can be regarded as a (highly singular) solution to a Beltrami equation with unbounded Beltrami form (i.e. $||\mu||_{L^\infty} = 1$). Therefore it is satisfactory to have the following uniqueness result.

Theorem 3.16. ***(Uniqueness).****If ξ has a generalized rectification, then it is unique (i.e. it does not depend on the choice of η).*

Proof. If h and h' are two generalized rectifications corresponding to two quasi-conformal structures η and η', then $h' \circ h^{-1}$ is the identity on K. Moreover on $\Omega = \overline{\mathbb{C}} - K$ we have

$$h' \circ h^{-1} = h_{\eta'} \circ (l' \circ l^{-1}) \circ h_\eta^{-1}.$$

This is a composition of quasi-conformal mappings on Ω because $l' \circ l^{-1}$ is quasi-conformal on Ω since it transports η into η'. Also from lemma 3.12 we get that the map $l' \circ l^{-1}$ is bounded from the identity in Ω for the Poincaré metric. Thus $h' \circ h^{-1}$ is continuous and it is a homeomorphism of the Riemann sphere. Now, $h' \circ h^{-1}$ is a homeomorphism, quasi-conformal in $\overline{\mathbb{C}} - K$, and coinciding with the identity on K. We can use Rickman's theorem to conclude that $h' \circ h^{-1}$ is conformal, and then necessarily $h = h'$. □

Now we have the following approximation theorem:

Theorem 3.17. *Let ξ be a conformal structure on $\overline{\mathbb{C}} - K$ Green equivalent to a quasi-conformal structure η.*

We assume that ξ is quasi-conformal on compact subsets of $\mathbb{C} - K$ (i.e. l is quasi-conformal on compact subsets of $\mathbb{C} - K$). Let μ be its associated Beltrami form extended by 0 on K.

Let $(\mu_n)_{n\geq 0}$ be a sequence of quasi-conformal (i.e. $||\mu_n||_{L^\infty} < 1$) Beltrami forms converging to μ almost everywhere and such that $|\mu_n| \leq |\mu|$. Let ξ_n be the associated complex structure to μ_n and $h_n : (\overline{\mathbb{C}}, \xi_n) \to (\overline{\mathbb{C}}, \sigma_0)$ be the Morrey-Ahlfors-Bers rectification homeomorphism, normalized such that $h_n(0) = 0$, $h_n(1) = 1$ and $h_n(\infty) = \infty$. We assume that the conformal structures ξ_n are Green equivalent to ξ. We made also the assumption that the modulus of an annulus bounded by K and an equipotential *of K for the quasi-conformal structure ξ_n is uniformly bounded on n from above and away from 0.*

Then

$$\lim_{n\to+\infty} h_n = h_\xi$$

uniformly on compact subsets of $\mathbb{C} - K$.

Moreover, if the complex structures ξ_n are Green equivalent to ξ (such a sequence always exists from Proposition 3.19) then $h_n = h_\xi$ on K.

Lemma 3.18. *The family $(h_n)_{n\geq 0}$ is equicontinuous in $\mathbb{C} - K$.*

Proof. The proof runs as the classical proof of equicontinuity of families of uniformly quasi-conformal homeomorphisms. We prove that the family $(h_n)_{n\geq 0}$ is uniformly Hölder on compact subsets of $\mathbb{C} - K$. Let $z_0 \in \mathbb{C} - K$. Let $r_0 > 0$ small enough such that $\overline{B(z_0, r_0)} \subset \mathbb{C} - K$ and $B(z_0, r_0)$ does not contain one of the points 0 or 1, say 1 for example (furthermore this disk does not contain ∞). Let $0 < r < r_0$. The quasi-conformal mappings h_n are M-quasi-conformal for a uniform M on $\overline{B(z_0, r_0)}$. We consider the annulus

$$A = B(z_0, r_0) - \overline{B(z_0, r)}.$$

We have

$$\text{mod } h_n(A) \geq M^{-1} \quad \text{mod } A = M^{-1} \frac{1}{2\pi} \log\left(\frac{r_0}{r}\right).$$

The component of the complement of A not containing $h_n(z_0)$ contains 1 and ∞. It follows from Teichmuller estimate ([Le-Vi]) that there exists a universal constant $C > 0$ such that in the spherical metric

$$\begin{aligned} \operatorname{diam} h_n(\overline{B(z_0, r)}) &\leq C e^{-2\pi \quad \text{mod } (h_n(A))} \\ &\leq C e^{M^{-1} \log\left(\frac{r_0}{r}\right)} = C\left(\frac{r_0}{r}\right)^{M^{-1}}. \end{aligned}$$

This proves that the family $(h_n)_{n\geq 0}$ is uniformly Hölder on compact subsets of $\mathbb{C} - K$. □

Proof. **(Theorem 3.17)** Let h be an accumulation point of the family $(h_n)_{n\geq 0}$ in $\mathbb{C} - K$,

$$\lim_{k\to+\infty} h_{n_k} = h.$$

The assumption on uniformity of the modulus of annulus bounded by K and an equipotential implies that the limit h is not the infinite constant and that $h(\mathbb{C} - K) = \mathbb{C} - K$. As in the classical situation, the uniform quasi-conformality on compact subsets of $\mathbb{C} - K$ proves that such a limit h is a homeomorphism of $\mathbb{C} - K$ (see [Le-Vi]). Here the Green equivalence of ξ_n to ξ proves directly this fact. Let $l_n = h_\eta^{-1} \circ h_n$. Then $\lim_{k\to+\infty} l_{n_k} = \hat{l} = h_\eta^{-1} \circ h$. By construction $\hat{l}_* \xi = \eta$, so $(l \circ \hat{l}^{-1})_* \eta = \eta$. Now $l \circ \hat{l}^{-1}(\infty) = \infty$ and $l \circ \hat{l}^{-1}$ is a conformal automorphism of $(\overline{\mathbb{C}} - K, \eta)$ whose angular action at ∞ is the identity. So $l = \hat{l}$ and $h = h_\eta \circ \hat{l} = h_\eta \circ l = h_\xi$. □

Proposition 3.19. *Given a conformal structure ξ as in theorem 3.17, there exists a sequence of Green equivalent quasi-conformal conformal structures $(\xi_n)_{n\geq 0}$ such that the corresponding sequence of Beltrami forms $(\mu_n)_{n\geq 0}$ converge to μ_ξ almost everywhere and for $z \in \mathbb{C} - K$,*

$$|\mu_n(z)| \leq |\mu(z)|.$$

Proof. The conformal structure ξ is Green equivalent to a quasi-conformal structure η. Let $L(x, y) = (x, h(y))$ be the corresponding mapping in the upper half plane. The continuous strictly increasing map h is almost everywhere differentiable. The differential of L for almost all points (x, y) is

$$DL(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & h'(y) \end{pmatrix} .$$

So the Beltrami form of μ_ξ at the point $z \in \mathbb{C} - K$ corresponding to the point $(x, y) \in \mathbb{H}$ is

$$\mu_\xi(z) = \frac{\bar{\partial} L}{\partial L} = \frac{1 - h'(y)}{1 + h'(y)}.$$

We define $\varphi_n = \min(n, h')$. The sequence of functions $h_n(x) = \int_0^x \varphi_n(u)du$ converges pointwise to h, and the mappings $L_n(x, y) = (x, h_n(y))$ define complex structures ξ_n in $\overline{\mathbb{C}} - K$ that satisfy the required properties. □

Theorem 3.17 together with Proposition 3.19 show that if the generalized rectification is quasi-conformal on compact subsets of $\mathbb{C} - K$ then it can be obtained as uniform limit on compact subsets of $\mathbb{C} - K$ of quasi-conformal homeomorphisms of the Riemann sphere. Note that this result provides an alternative way to define the generalized rectification.

The problem of almost everywhere differentiability of the generalized rectification h_ξ is equivalent to the problem of extension of l to $\overline{\mathbb{C}}$ into an almost everywhere differentiable homeomorphism. The map l extends radially to the identity for almost all $z \in K$ for the harmonic measure. If K is locally connected l extend continuously to the identity on K into a global homeomorphism of the Riemann sphere. But we are looking for results that are independent of the structure of K.

In the following theorems we show that just a C^0 control on $l - \text{id}$ on Ω for the Poincaré metric implies a differentiability result.

Theorem 3.20. *Let K be a full compact connected set in $\overline{\mathbb{C}}$ with at least two points. If $l : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a homeomorphism such that:*

- *$l_{/K} = \text{id}_K$ (so $l(K) = K$),*
- *The map $z \mapsto d_P(l(z), z)$ converges to 0 when $z \to K$, $z \in \overline{\mathbb{C}} - K$ (we denote by d_P the Poincaré distance of $\overline{\mathbb{C}} - K$).*

Then l is differentiable on all points of K and

$$Dl_{/K} = \text{id}\,.$$

Proof. Let $z_0 \in K$ and $h \in \mathbb{C}$, $h \neq 0$. If $z_0 + h \in K$ then

$$l(z_0 + h) - l(z_0) = (z_0 + h) - z_0 = h.$$

We consider the case $z_0 + h \notin K$.

Let $\epsilon > 0$. There exists $\delta > 0$ such that for any $h' \in \mathbb{C}$, $|h'| < \delta$, we have $d_P(l(z_0 + h'), z_0 + h') \leq \epsilon$. Now the Poincaré infinitesimal length is related to the Euclidean infinitesimal length by

$$\frac{1}{2}\frac{|dz|}{\delta(z)} \leq ds_P(z) \leq 2\frac{dz}{\delta(z)},$$

where $\delta(z) = d_E(z, K) = \min_{w \in K} |z - w|$. For ϵ small, if $|h| \leq \delta$ then $[z_0 + h, l(z_0 + h)] \subset \overline{\mathbb{C}} - K$ and

$$\delta_0 = \max_{w \in [z_0 + h, l(z_0 + h)]} d_E(w, K) \leq C|h|,$$

because $d_E(z_0 + h, K) \leq |h|$ and $|l(z_0 + h) - (z_0 + h)| \leq 2\delta_0 \epsilon$ (we can put $C = (1 - 2\epsilon)^{-1}$.)

So we obtain,

$$\frac{1}{2}\frac{1}{\delta_0}|l(z_0 + h) - (z_0 + h)| \leq d_P(l(z_0 + h), z_0 + h) \leq \epsilon.$$

And for $|h| \leq \delta$,

$$|l(z_0 + h) - z_0 - h| \leq 2\epsilon\delta_0 \leq 2C\epsilon|h|,$$

and

$$\left|\frac{l(z_0 + h) - z_0}{h} - 1\right| \leq 2C\epsilon.$$

That means that $Dl(z_0) = \text{id}$. □

We can get a better result for almost everywhere differentiability. The next theorem is such a result. The proof of the next theorem gives more: We can balance how far is l from the identity in the Poincaré metric to the porosity of K. We leave other more precise statement for a future version.

Theorem 3.21. *Let K be a full compact connected set in $\overline{\mathbb{C}}$. If $l : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a homeomorphism such that:*

- $l_{/K} = \mathrm{id}_K$ *(so $l(K) = K$),*
- *For $z \in \overline{\mathbb{C}}$ the map $z \mapsto d_P(l(z), z)$ is bounded (d_P is the Poincaré distance). Let $M > 0$ be an upper bound.*

Then l is differentiable almost everywhere on K and, more precisely, for all Lebesgue density points $z \in K$,

$$D_z l = \mathrm{id}\,.$$

Proof. By Lebesgue density theorem, for a.e. $z \in K$,

$$\lim_{r\to 0} \frac{\lambda(B(z,r)\cap K)}{\lambda(B(z,r))} = 1.$$

Thus for a set $K_0 \subset K$ of full measure, for any $z \in K_0$ there exists an increasing function $\epsilon_z(r) > 0$, such that

$$\lim_{r\to 0} \epsilon_z(r) = 0,$$

and

$$\frac{\lambda(B(z,r)\cap K)}{\lambda(B(z,r))} \geq 1 - \epsilon_z(r).$$

We carry out the same proof as before for $z_0 \in K_0$. Given $\epsilon > 0$ we choose $r_0 > 0$ such that

$$2\epsilon_z(r_0)^{1/2} M \leq \epsilon/2.$$

Now if $r \leq r_0$, for any $w \in B(z, r/2)$, we have

$$\delta(w) \leq r\epsilon_z(r)^{1/2},$$

because $B(w, r\epsilon_z(r)^{1/2})$ must intersect K or we would have a too big hole in $B(z,r)$ incompatible with the Lebesgue density condition at this scale. Thus if $|h| = r/2 < r_0/2$,

$$\delta(z_0 + h) \leq 2|h|\epsilon_z(r)^{1/2} \leq 2|h|\epsilon_z(r_0)^{1/2}.$$

So

$$\frac{1}{2}\frac{1}{\delta(z_0+h)}|l(z_0+h)-(z_0+h)| \leq d_P(l(z_0+h), z_0+h),$$
$$|l(z_0+h)-(z_0+h)| \leq 4|h|\epsilon_z(r_0)^{1/2}M,$$
$$\left|\frac{l(z_0+h)-z_0}{h}-1\right| \leq 4\epsilon_z(r_0)^{1/2}M,$$
$$\left|\frac{l(z_0+h)-z_0}{h}-1\right| \leq \epsilon.$$

□

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