A NATURAL EÑE PRODUCT CONSTRUCTION OF THE BIG WITT RING

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Abstract. We give a straighforward, self-contained, and natural construction of the Big Witt ring using the eñe product that is defined through the action on zeros of polynomials. This is in contrast with classical constructions of the Big Witt ring using formulas out of nowhere.

1. **Introduction.** The eñe product of a commutative ring A was defined by the second author through the action on divisors of polynomials [18] with the purpose of studying its analytical properties when extended to transcendental functions ([19], [20]). Although it is not obvious from its definition, it turns out that the eñe product is a twisted form of the multiplication in the Big Witt ring. In this way, the definition of the eñe product from [18] provides a novel and straightforward construction of the Big Witt ring that we present in this article. This construction shows the action as a twisted convolution of the multiplication of the Big Witt ring on divisors of power series representing meromorphic functions. This important fact has been surprisingly overlooked in the classical treatments. A notable exception is Bergman [2] Appendix B where the observation is made, but not used. It is also nearly missed in the exercises in the Commutative Algebra volumes of Bourbaki [3] Chapter IX (exercises 42 to 46). Manin considers the eñe product, which he names "tensor product" and denotes by $f \otimes g$, in algebraic groups in relation with the tensor product used in Étale Cohomology, but, inexplicably misses the relation with the Big Witt ring (see [16]). In relation with this, he presents Kurokawa's proposition for tensoring global zeta functions, which comes from an incomplete form of the eñe product [9]. The eñe product defines an eñe ring structure on the multiplicative group of split polynomials with constant coefficient 1 and is continuous for the Krull topology. We construct a continuous extension to $\mathcal{A}(A) = 1 + XA[[X]]$ that is a twist of the Big Witt ring. The split polynomials are dense in $\mathcal{A}(A)$ when for example A is an algebraically closed field, but this is not true in general. The usual constructions of the Big Witt ring use formulas out of nowhere. In this sense the construction presented here seem more natural. The construction is close to the philosophy of the elegant construction by

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Lenstra [15], with simplifications coming from the use of the eñe product in a ring of variables.

Structure of the article. In sections 2 we define the eñe product on finite divisors on a monoid or on a partial monoid that gives them a ring structure. In section 3 we define the eñe product for split polynomials over a commutative unitary ring A and its relation to the eñe product on divisors supported on the monoid A or on the partial monoid A^* (the two approaches are presented). In section 4 we construct the eñe ring structure $(\mathcal{A}(A), ., \star)$ where $\mathcal{A}(A)$ is the multiplicative group of formal power series $1+XA[[X]]$ for an arbitrary ring A. In section 5 we introduce Big Witt rings and state the main Theorem. In section 6 we identify the Big Witt ring with the twist of the eñe ring. In section 7 we single out the action of the Witt multiplication on divisors. Section 8 is devoted to the density problem and we provide counterexamples. In section 9 we discuss the functorial properties. In section 10 we give a short, less than two pages long, self-contained construction of the Big Witt ring using the previous ideas, but avoiding the divisor discussion. We conclude in section 11 with a brief historical discussion of the origins of the Big Witt ring.

2. Eñe ring on finite divisors.

Let $(G, .)$ be a magma, that is, just a set G with a binary operation. Recall that this is a semigroup if the binary operation is associative, and it is a monoid if the binary operation also has a neutral element. The space of divisors $\mathcal{D}(G)$ is the $\mathbb{Z}$-module of linear combinations $\delta = \sum_{g\in G} n_g.(g)$ where $n_g \in \mathbb{Z}$ is called the multiplicity (or coefficient) of $g \in G$. The multiplication of divisors is by definition the additive structure of this $\mathbb{Z}$-module and $(\mathcal{D}(G), .)$ is a group whose neutral element is the zero divisor ($n_g = 0$ for all $g \in G$), and [1]

$$\delta.\eta = \sum_{g\in G}(n_g + m_g).(g) \ .$$

The group of divisors $(\mathcal{D}(G), .)$ is an ordered group (Bourbaki Algèbre II, Chapter VI, §1, 1 in [4]) and the positive cone is the monoid of positive divisors $(\mathcal{D}^+(G), .)$ with non-negative multiplicities $n_g \geq 0$.

The convolution or eñe product of divisors is defined by

$$\delta \star_G \eta = \sum_{g\in G}\left(\sum_{g_1.g_2=g} n_{g_1} m_{g_2}\right).(g) \tag{1}$$

[1] We denote multiplicatively the sum of divisors thinking about the multiplication of the functions that generate the divisors. This convention is also compatible with the multiplicative notation for eñe and Witt rings.

where the multiplicity of $g \in G$ is 0 if there is no pair $(g_1, g_2) \in G^2$ such that $g_1.g_2 = g$. The convolution is a binary operation for finite divisors $\mathcal{D}_0(G)$, and for positive finite divisors $\mathcal{D}_0^+(G)$. The structure $(\mathcal{D}_0(G), \star_G)$ is a magma and $(\mathcal{D}_0^+(G), \star_G)$ is a submagma. The magmas $(\mathcal{D}_0(G), \star_G)$ and $(\mathcal{D}_0^+(G), \star_G)$ are associative, resp. commutative, if the binary operation of the magma $(G, .)$ is associative, resp. commutative. When $(G, .)$ is a monoid, then $(\mathcal{D}_0(G), \star_G)$ and $(\mathcal{D}_0^+(G), \star_G)$ are monoids with neutral element the divisor $\delta_e = (e)$ associated to the neutral element $e \in G$.

Proposition 1 (Eñe ring structure on finite divisors on a semigroup)**.** [2] *If $(G, .)$ is a semigroup, then we have a ring structure $(\mathcal{D}_0(G), ., \star_G)$. This is the eñe ring associated to the semigroup $(G, .)$. The eñe ring is commutative if the semigroup is commutative. If $(G, .)$ is a monoid, then the eñe ring $(\mathcal{D}_0(G), ., \star_G)$ is unitary and the unit element is the divisor $\delta_e = (e)$ associated to the neutral element $e \in G$.*

If $(G, .)$ is a semigroup then we have a semiring structure on positive divisors $(\mathcal{D}_0^+(G), ., \star_G)$. This is the eñe semiring associated to the semigroup $(G, .)$. Also $(\mathcal{D}_0(G), ., \star_G)$ is an ordered ring with positive cone $\mathcal{D}_0^+(G)$.

We recall that a semiring (or hemiring) satisfies the same axioms of a ring except for the additive structure that is has a semigroup structure instead of a group structure (see [12]). For the definition of ordered ring see Bourbaki Algèbre II, Chapter VI, 2, 1, Definition 1 in [4].

Divisors on a partial magma.

A partial magma $(G, .)$ is a set with a partial binary operation defined in a subset $U \subset G \times G$, that is, a map $U \subset G \times G \to G$. The structure $(G, .)$ is a partial semigroup if we have associativity: for $x, y, z \in G$ it is equivalent that $x.y$ and $(x.y).z$ both exist and that $y.z$ and $x.(y.z)$ both exist, and it that case we have

$$(x.y).z = x.(y.z)$$

and we denote by $x.y.z$ these products. The partial magma $(G, .)$ has a neutral element $e \in G$ if for any $x \in G$ we have that both $e.x$ and $x.e$ exist and

$$e.x = x.e = x \ .$$

Example. If A is a ring, then $A^* = A - \{0\}$ is a partial magma for the multiplication in A defined on those pairs $(a, b) \in A^*$ such that $a.b \neq 0$. The partial magma $(A^*, .)$ is a magma if and only if A has no divisors of 0.

The construction of the eñe ring of divisors of a magma extends to the case when $(G, .)$ is a partial magma. The definition is the same except for the convolution

[2] It turns out that this is the same as $\mathbb{Z}G$, the monoid ring of G over $\mathbb{Z}$.

formula (1) where the summation only extends to those $(g_1, g_2) \in G^2$ such that $g_1.g_2$ exists (and the empty sum is 0). We have that Proposition 1 extends to partial semigroups.

Proposition 2 (Eñe ring structure on finite divisors on a partial semigroup)**.** *If $(G,.)$ is a partial semigroup, then we have a ring structure $(\mathcal{D}_0(G),.,\star_G)$. This is the eñe ring associated to the partial semigroup $(G,.)$. The eñe ring is commutative if the partial semigroup is commutative. If $(G,.)$ is a partial monoid, then the eñe ring $(\mathcal{D}_0(G),.,\star_G)$ is unitary and the unit element is the divisor $\delta_e = (e)$ associated to the neutral element $e \in G$.*

If $(G,.)$ is a partial semigroup then we have a semiring structure on positive divisors $(\mathcal{D}_0^+(G),.,\star_G)$. This is the eñe semiring associated to the semigroup $(G,.)$. Also $(\mathcal{D}_0(G),.,\star_G)$ is an ordered ring with positive cone $\mathcal{D}_0^+(G)$.

3. **Eñe ring structure associated to a commutative ring.** We consider an unitary commutative ring A and the multiplicative group of formal power series $\mathcal{A}(A) = 1 + XA[[X]]$. We consider also the commutative multiplicative semigroup $P_A \subset 1 + XA[X] \subset \mathcal{A}(A)$ of *split polynomials* with constant coefficient 1. So for $f \in P_A$ we can write

$$f(X) = \prod_{a \in A} (1 - aX)^{n_a}$$

with $n_a \in \mathbb{N}$, and almost all $n_a = 0$. Note that the split factorization is never unique as n_0 can be arbitrary and we can always multiply by a positive integer power of the constant polynomial equal to 1. Of course, the non-uniqueness can be worse when A has zero divisors. Let S_A be the multiplicative subgroup $S_A \subset \mathcal{A}(A)$ generated by P_A. We name S_A the space of *split "rational" functions*. We can write any element $f(X) \in S_A$ as a finite product

$$f(X) = \prod_{a \in A} (1 - aX)^{n_a}$$

with $n_a \in \mathbb{Z}$, and almost all $n_a = 0$. Note that

$$(1 - aX)^{-1} = 1 + \sum_{k \geq 1} a^k X^k \in \mathcal{A}(A) \ .$$

We can consider divisors with support on A^* considering the partial monoid structure $(A^*,.)$ (the convention is that $a.b$ does not exist in A^* when $a.b = 0$ in A), or divisors with support on A considering the monoid structure $(A,.)$. We start considering the first case that geometrically is more natural, but maybe algebraically less orthodox.

Divisors with support on the partial monoid $(A^*,.)$.

The following Proposition is obvious from the definition of P_A.

Proposition 3 (Positive divisor map)**.** *The positive divisor map* $\varphi^+_{A^*} : \mathcal{D}^+_0(A^*) \to P_A \subset A[X]$ *defined by*

$$\delta = \sum_{a\in A^*} n_a.(a) \mapsto f(X) = \prod_{a\in A^*} (1-aX)^{n_a}$$

is a surjective semigroup morphism.

Lemma 4. *If A has no zero divisors then $\varphi^+_{A^*}$ is an isomorphism of semigroups,* $\operatorname{Ker}\varphi^+_{A^*} = \{0\}$.

Proof. Consider $\delta \in \operatorname{Ker}\varphi^+_{A^*}$, with $\delta = \sum_{k=1}^m n_{a_k}.(a_k)$ given by a finite formal sum of $m \geq 0$ non-zero terms, $a_k \neq 0$. If $m \geq 1$, looking at the coefficient of highest degree, we have $a_1^{n_1} \dots a_m^{n_m} = 0$. Since A has no zero divisors, we have for some $1 \leq k_0 \leq m$, $a_{k_0} = 0$. Therefore $m = 0$ and $\operatorname{Ker}\varphi^+_{A^*} = \{0\}$. □

In general when A has divisors of 0, $\varphi^+_{A^*}$ is not an isomorphism but we can define the eñe product on P_A by the following procedure:

Theorem 5 (Eñe product on P_A)**.** *Given $f, g \in P_A$, we choose $\delta_f, \delta_g \in \mathcal{D}^+_0(A^*)$, then we define*

$$f \star g = \varphi^+_A(\delta_f \star_{A^*} \delta_g)$$

that is,

$$f \star g = \prod_{\exists a.b\in A^*} (1-abX)^{n_a m_b} = \prod_{c=a.b\in A^*} (1-cX)^{\left(\sum_{a.b=c} n_a.m_b\right)} \tag{2}$$

and the result does not depend on the choices of δ_f and δ_g.

Proof. Observe that the coefficients $(A_k)_{k\geq 1}$ of the expansion of the product

$$f(X) = \prod_{a\in A}(1-aX)^{n_a} = 1 + A_1X + A_2X^2 + \dots$$

are the elementary symmetric functions evaluated in the a's repeated with multiplicities, that we also name elementary symmetric functions of the associated divisor. If we choose another divisors δ'_f that incarnates f then the elementary symmetric functions of δ_f and δ'_f give the same coefficients (A_k) that coincide. From Lemma 10 in Section 3 we have that the coefficients of $f \star g$ are universal polynomial functions with integer coefficients of the elementary symmetric functions of δ_f and δ_g, hence they are universal polynomial formulas with integer coefficients on the coefficients of f and g. Therefore, the result is independent of the choices of δ_f and δ_g. □

Divisors with support on the monoid $(A,.)$.

The following Proposition is immediate from the definition of S_A.

Proposition 6 (Divisor map)**.** *The divisor map $\varphi_A : \mathcal{D}_0(A) \to S_A$ defined by*

$$\delta = \sum_{a\in A} n_a.(a) \mapsto f(X) = \prod_{a\in A}(1-aX)^{n_a}$$

is a surjective group morphism.

Its restriction to positive divisors $\varphi_A^+ : \mathcal{D}_0^+(A) \to P_A$ is a surjective morphism of semigroups.

Observe that, contrary to the precedent situation, even when A has no divisors of 0, we have that φ_A^+, resp. φ_A, is never an isomorphism since $\operatorname{Ker}\varphi_A^+$, resp. $\operatorname{Ker}\varphi_A$, always contains $\mathbb{N}.(0)$, resp. $\mathbb{Z}.(0)$. Note that $\mathbb{Z}.(0) = \big((0)\big)$ is a principal ideal of the eñe ring $(\mathcal{D}_0(A), ., \star_A)$.

Again, we desire to define the eñe product $f \star g$ for $f, g \in P_A$, or for $f, g \in S_A$, by

$$(f\star g)(X) = \prod_{a,b\in A}(1-abX)^{n_a.m_b} = \prod_{c=a.b\in A^*}(1-cX)^{\left(\sum_{a.b=c} n_a.m_b\right)} \tag{3}$$

but we need to justify that the result is independent of the choice of the presentation of $f(X)$ and $g(X)$ as split factorizations. When A has no divisors of 0 the same argument as before proves that $\operatorname{Ker}\varphi_A^+ = \mathbb{N}.(0)$ and $\operatorname{Ker}\varphi_A = \mathbb{Z}.(0)$. Therefore, when A has no divisors of 0 the split factorizations of f is unique up to the freedom of choosing n_0 that does not change the product.

For a general ring A, the independence of the definition of the eñe product of the choice of the divisors incarnating the functions in P_A (Theorem 5) works as well, but not for S_A. Instead we can use the following alternative argument due to Hendrik Lenstra if we insist in defining the eñe product for the full group S_A (something that is not necessary for the construction in the next section, but is certainly satisfactory to have). It is noteworthy to observe that the argument avoids the use of elementary symmetric functions.

Proposition 7 (H. Lenstra)**.** *The kernel $\operatorname{Ker}\varphi_A$ is an ideal of the eñe ring $(\mathcal{D}_0(A), ., \star)$, and the group isomorphism*

$$\mathcal{D}_0(A)/\operatorname{Ker}\varphi_A \approx S_A$$

induces on S_A a ring structure and formula (3) for the eñe product is well defined independently of the split factorizations.

Proof. In order to prove that $\operatorname{Ker}\varphi_A$ is an ideal of $\mathcal{D}_0(A)$, it is enough to prove that for $c \in A$ we have $(c)\star\operatorname{Ker}\varphi_A \subset \operatorname{Ker}\varphi_A$. The ring isomorphism $\psi_c : A[[X]] \to A[[X]]$

(for the usual ring structure of $A[[X]]$) defined by

$$\psi_c\left(\sum_{n\geq 0} a_n X^n\right) = \sum_{n\geq 0} a_n (cX)^n$$

induces on the multiplicative group S_A a group morphism $\psi_c : S_A \to S_A$ (we keep the same notation for this restriction). For a divisor $\delta \in \mathcal{D}_0(A)$,

$$\delta = \sum_{a\in A} n_a.(a)$$

we have

$$\prod_{a\in A}(1-caX)^{n_a} = \prod_{a\in A}(1-a(cX))^{n_a}$$

which means

$$\varphi_A((c)\star\delta) = \psi_c(\varphi_A(\delta)) \ .$$

Therefore, if $\delta \in \operatorname{Ker}\varphi_A$ we get $(c)\star\delta \in \operatorname{Ker}\varphi_A$.

Finally, observe that the choice of a split factorization corresponds to the choice of δ modulo $\operatorname{Ker}\varphi_A$. □

Remark. As observed before we always have $\mathbb{Z}.(0) \subset \operatorname{Ker}\varphi_A$. When A has no divisors of 0 then we have $\mathbb{Z}.(0) = \operatorname{Ker}\varphi_A$, and this gives the isomorphism of eñe rings,

$$\mathcal{D}_0(A)/\mathbb{Z}.(0) \approx \mathcal{D}_0(A^*) \ .$$

In this case, the map $\varphi_{A^*} : \mathcal{D}_0(A^*) \to S_A$ is an isomorphism of eñe rings, and it induces an isomorphism of eñe semirings $\varphi^+_{A^*} : \mathcal{D}^+_0(A^*) \to P_A$.

Definition 8. *We define the twisted divisor of $f \in S_A$,*

$$f(X) = \prod_{a\in A^*}(1-aX)^{n_a}$$

as

$$\#\operatorname{Div}(f(X)) = \sum_{a\in A} n_a.(a) \in \mathcal{D}_0(A)/\operatorname{Ker}\varphi_A \ .$$

This is just $\operatorname{Div} f(1/X)$ when this makes sense.

Corollary 9. *(Eñe product) For $f, g \in S_A$, the eñe product $f\star g \in S_A$ is uniquely defined by*

$$\#\operatorname{Div}(f\star g) = \#\operatorname{Div}(f) \star_A \#\operatorname{Div}(g)$$

4. **Continuous extension of the eñe product to $\mathcal{A}(A)$.** We consider an arbitrary unitary commutative ring A. Let $\mathcal{A}(A) = 1 + XA[[X]]$ the multiplicative group of formal power series with coefficients in A endowed with the Krull topology where a bases of neighborhoods of 0 is given by $(1 + X^n A[[X]])_{n\geq 1}$. For the next Lemma we consider two polynomials given in split form.

Lemma 10. *Let $P(x), Q(x) \in P_A \cap A[X]$ be two split polynomials and write*

$$P(X) = 1 + a_1 X + a_2 X^2 + \ldots = \prod_{a\in A} (1 - aX)^{n_a}$$
$$Q(X) = 1 + b_1 X + b_2 X^2 + \ldots = \prod_{b\in A} (1 - bX)^{m_b}$$

with all but a finite number $(n_a), (m_b), (a_n), (b_n)$ being 0. We have that $P \star Q \in P_A$ is a split polynomial and

$$(P \star Q)(X) = 1 + c_1 X + c_2 X^2 + \ldots = \prod_{c\in A} (1 - cX)^{\left(\sum_{a.b=c} n_a.m_b\right)}$$

with

$$c_n = C_n(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

where the $C_n \in \mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$, for $n \geq 1$, are universal polynomial with integer coefficients.

Proof. In the ring $\mathbb{Z}[(X_k)_{1\leq k\leq n}, (Y_k)_{1\leq k\leq n}, T]$ of polynomials in $2n + 1$ variables consider the expansions

$$\prod_{k=1}^{n} (1 - X_k T) = \sum_{k=1}^{n} \Sigma_k(X) T^k \quad \text{and} \quad \prod_{l=1}^{n} (1 - Y_l T) = \sum_{l=1}^{n} \Sigma_l(Y) T^l$$

where the $\Sigma_k(X)$, resp. $\Sigma_l(Y)$, are the elementary symmetric functions on the variables $(X_k)_{1\leq k\leq n}$, resp. $(Y_k)_{1\leq k\leq n}$. Then we have

$$\prod_{k,l=1}^{n} (1 - X_k Y_l T) = \sum_{m=1}^{n^2} \Sigma_{m,n}(X, Y) T^m$$

The polynomial $\Sigma_{m,n}(X, Y) \in \mathbb{Z}[(X_k)_{1\leq k\leq n}, (Y_k)_{1\leq k\leq n}]$ is symmetric individually on the two groups of variables $(X_k)_{1\leq k\leq n}$ and $(Y_k)_{1\leq k\leq n}$. Applying the Fundamental Theorem on Symmetric Functions with the base ring $\mathbb{Z}[(X_k)_{1\leq k\leq n}]$ (see Bourbaki Algebra Chapter 4 [3]), we have that

$$\Sigma_{m,n}(X, Y) \in \mathbb{Z}[(X_k)_{1\leq k\leq n}][(\Sigma_l((Y_k)_{1\leq k\leq n})_{1\leq k\leq n}]$$

Applying a second time the same Theorem to each coefficient in $\mathbb{Z}[(X_k)_{1\leq k\leq n}]$ that are symmetric in the variables $(X_k)_{1\leq k\leq n}$, we obtain that

$$\Sigma_{m,n}(X, Y) \in \mathbb{Z}[(\Sigma_k(X))_{1\leq k\leq n}, (\Sigma_k(Y))_{1\leq k\leq n}]$$

For $1 \leq m \leq n$ the polynomials $\Sigma_{m,n}(X,Y) = \Sigma_m(X,Y)$ stabilize and are independent of $n \geq m$ because the m-th germ of $P \star Q$ depends only on the m-th germ of P and Q. Replacing the variables by the values in A, we have $c_n = \Sigma_n((a_l)_{1\leq l\leq n}, (b_l)_{1\leq l\leq n})$ and the result follows.

□

Corollary 11 (Universality)**.** *For an arbitrary ring A we define the eñe ring structure $(\mathcal{A}(A), ., \star)$ using the coefficient formulas given by the universal polynomials $(C_n)_{n\geq 1}$. This defines a commutative topological ring for the Krull topology.*

The eñe ring structure $(\mathcal{A}(A), ., \star)$ is a continuous extension of the eñe semi-ring structure on $(P_A, ., \star)$.

Proof. Note that it makes sense to define coefficientwise the eñe product $f \star g$ for any $f, g \in \mathcal{A}(A)$ because the polynomials $(C_n)_{n\geq 1}$ have integer coefficients. This binary operation is continuous for the Krull topology since if $f_1(X) - f_2(X) \in X^n A[[X]]$ and $g_1(X) - g_2(X) \in X^n A[[X]]$, then from the universal formulas we have $(f_1 \star g_1)(X) - (f_2 \star g_2)(X) \in X^n A[[X]]$. The extension of the eñe product defined in this way extends the commutative semi-ring structure $(P_A, ., \star)$ to a commutative ring structure since the properties as associativity or commutativity are encoded in universal polynomial relations on the coefficients that continue to hold, and also the coefficients of the inverse of an element f in $\mathcal{A}$ are universal polynomials with integer coefficients on the coefficients of f. □

Corollary 12. *When P_A is dense in $\mathcal{A}(A)$ the extension of the eñe product to $\mathcal{A}(A)$ is unique.*

For example, this happens when A is an algebraic closed field or when A is a ring where every $a \in A$ has a n-th root and the polynomial $1 - X^n$ splits for every $n \geq 2$. In general this is not true. We prove in section 8 that neither $P_{\mathbb{R}}$ is dense in $S_{\mathbb{R}}$, nor $S_{\mathbb{R}}$ is dense in $\mathcal{A}(\mathbb{R})$.

The next Corollary follows from the universality of the formulas but not from the density.

Corollary 13. *The eñe product defined in $\mathcal{A}(A)$ in Corollary 11 coincides with the eñe product in S_A defined by equation (3).*

Proof. By construction it coincides in P_A. When $A = \mathbb{C}$, simply looking at divisors, we have

$$f^{-1} \star g^{-1} = f \star g \ ,$$
$$f^{-1} \star g = (f \star g)^{-1} \ .$$

By universality, these equations remain true for any ring A. More precisely, the polynomial equations that give these identities coefficientwise are the same as those for $A = \mathbb{C}$. When we replace in this formulas f and g by $1 + aX$ and $1 + bX$, we can see that the extension of the eñe product in P_A to $\mathcal{A}(A)$ coincides on the generators of S_A with the one defined by equation (3) (or Corollary 9), hence it coincides in all S_A. □

5. **The Big Witt ring.** We consider the multiplicative group of formal power series $\mathcal{A}(A) = 1 + XA[[X]]$ and the Big Witt ring structure $(\mathcal{A}(A), ., \star_w)$ defined by G. Bergman ([2], 1966), following Witt work on p-adic rings ([25], 1937). A very direct construction was given by P. Cartier ([5], 1967), and another very enlightening construction by H. Lenstra ([15], 2002). We endow $\mathcal{A}(A)$ with the Krull topology generated by the ideal (X) and the Big Witt ring is a topological ring.

Theorem 14. *The twisted eñe ring structure* $(\mathcal{A}(A), ., \check{\star})$ *defined by*

$$f \check{\star} g = (f \star g)^{-1}$$

coincides with the Big Witt ring.

We prove this Main Theorem in the next section.

6. **Identification with the twisted Big Witt ring.** We work in the ring of formal power series in a countable set of variables with rational coefficients $\mathbb{Q}[[(X_k)_{k\geq 1}, (Y_k)_{k\geq 1}, \ldots]]$. For background, we refer the reader to Bourbaki, Algèbre, Chapter IV, section 4, [3]. This ring is a $\mathbb{Q}$-algebra.

Lemma 15 (Formal Exponential form)**.** *We have*

$$f(T) = \prod_{k=1}^{+\infty}(1 - X_k T) = \exp\left(-\sum_{n=1}^{+\infty}\frac{1}{n}N_n(X)T^n\right) \quad \textit{with} \quad N_n(f) = N_n(X) = \sum_{k=1}^{+\infty} X_k^n$$

and

$$f(T) = \prod_{k=1}^{+\infty}(1 - X_k T^k)^{-1} = \exp\left(\sum_{n=1}^{+\infty}\frac{1}{n}W_n(X)T^n\right) \quad \textit{with} \quad W_n(f) = W_n(X) = \sum_{d|n} d X_d^{n/d}\ .$$

Proof. Develop $\log(1 - X_k T)$ or $-\log(1 - X_k T^k)$ in the exponential and regroup terms with the same exponent of T. The first case is clear. For the second case we have

$$-\sum_{k\geq 1}\log(1 - X_k T^k) = \sum_{k,l\geq 1}\frac{1}{l}X_k^l\, T^{kl} = \sum_{n\geq 1}\ \frac{1}{n}\left(\sum_{k,l\geq 1;kl=n} k X_k^l\right)T^n = \sum_{n\geq 1}\frac{1}{n}W_n(X)T^n.$$

□

The first left formula in the Lemma is the exponential form of the eñe product that appears in [18] Section 4. The second left formula is the exponential form for the Big Witt product that appears in [26] formula (1.3).

The power series $N_n(X) \in \mathbb{Z}[[(X_k)_{k\geq 1}]]$ are the classical Newton sums of the variables (X). These are power series in an infinite number of variables. The polynomials $W_n(X) = \sum_{d|n} dX_d^{n/d} \in \mathbb{Z}[X_1, \ldots, X_n]$, for all $n \geq 1$, also called "ghost components", appear first in print independently in Bergman [2] p.180 and in Lang's Algebra book as an exercise [13] (Exercise in Chapter VIII). Lang gives credit to an oral communication by Witt, and Witt left a manuscript note from a seminar he gave in Hamburg in June 1964 (see [27] p.164, and also the essay by Harder [10]). It seems historically justified to name them the Bergman-Witt polynomials. Traditionally the pre-1940 theory of Witt rings is derived from the subsequence W_{p^n} (see [2], [3], [5], [13], [23]). The Big Witt multiplication is defined by

$$W_n(f \star_w g) = W_n(f).W_n(g) \ .$$

So the Witt multiplication corresponds to the simple multiplication of ghost components. They are introduced in the construction of the Big Witt ring without proper motivation. A notable exception is Lenstra's elegant construction [15] that does not use them. In our construction, the Bergman-Witt polynomials appear naturally through the exponential form of the eñe product, as shown in the above Lemma 15, and the next Lemma gives the exponential form of the eñe product.

Lemma 16 (Formal exponential form of the eñe product)**.**

$$\left(\prod_{k=1}^{+\infty}(1-X_kT)\right) \star \left(\prod_{l=1}^{+\infty}(1-Y_lT)\right) = \exp\left(-\sum_{n=1}^{+\infty}\frac{1}{n}(N_n(X).N_n(Y))T^n\right) \ .$$

Hence, the eñe product corresponds to the simple multiplication of the Newton sums.

Proof. We have

$$\left(\sum_{k=1}^{+\infty}X_k^n\right).\left(\sum_{l=1}^{+\infty}Y_l^n\right) = \left(\sum_{k,l=1}^{+\infty}(X_kY_l)^n\right) \ .$$

□

Corollary 17 (Relation with the Hadamard product)**.** *If*

$$\mathcal{D}f(T) = -T\,\frac{f'(T)}{f(T)}$$

and if we denote by $\odot$ the Hadamard product, then we have

$$\mathcal{D}(f \star g) = \mathcal{D}f(T) \odot \mathcal{D}g(T)$$
$$\mathcal{D}(f \star_w g) = -\mathcal{D}f(T) \odot \mathcal{D}g(T)$$

The first identity appears in [18], Theorem 10.5, the second one in [15], page 1241.

Corollary 18. *We have $\check{\star} = \star_w$*

Proof. Observe that $\mathcal{D}(f(T)^{-1}) = -\mathcal{D}f(T)$. □

Corollary 19. *The result holds for an arbitrary commutative ring A.*

Proof. We have universal polynomials (Q_n^w) with $Q_n^w \in \mathbb{Z}[Z_1, \ldots, Z_n]$ such that

$$f \star_w g = 1 + \sum_{k\geq 1} Q_k^w\, T^k = \left(1 + \sum_{k\geq 1} Q_k\, T^k\right)^{-1} = (f \star g)^{-1}\ .$$

We can replace variables $(X_k)_{k\geq 1}$ and $(Y_k)_{k\geq 1}$ with arbitrary values in the ring A, since the polynomials have integer coefficients. The identity remains true for all commutative rings. □

7. **Action on divisors of the Big Witt multiplication.** The following Corollary is obvious from our construction. We single it out because it is generally overlooked in the standard references (except in [2] Appendix 8). We define the split "rational" functions S_A as those elements in $\mathcal{A}(A)$ that can be written as finite products

$$f(X) = \prod_{a\in A} (1 - aX)^{n_a}$$

with $n_a \in \mathbb{Z}$. Note that $(1 - aX)^{-1} = 1 + aX + a^2X^2 + \ldots \in \mathcal{A}(A)$.

Corollary 20. *The multiplicative sub-group $S_A \subset \mathcal{A}(A)$ is invariant by the Big Witt ring multiplication.*

This is to be compared with the invariance of $P_A \subset A[X]$ by the eñe product.

8. **Density of P_A and S_A and counterexamples.**

Proposition 21. *Let A be a commutative ring such that for every $n \geq 2$, every $a \in A$ has an n-th root, and $1 - X^n \in P_A$. Then, P_A and S_A are dense in $\mathcal{A}(A)$.*

Proof. From [5] (or [13], exercise in chapter VIII) we have that any element $f \in \mathcal{A}(A)$ can be written as an infinite product

$$f(X) = \prod_{n=1}^{+\infty}(1 - a_n X^n)^{-1}$$

Also, considering the inverse of f, any $f \in \mathcal{A}(A)$, can be written as

$$f(X) = \prod_{n=1}^{+\infty}(1 - a_n X^n)$$

If A satisfies the conditions, then for every $n \geq 1$, each factor $1 - aT^n$ splits as

$$1 - aX^n = 1 - (bX)^n = \prod_{\omega^n=1}(1 - b\omega X)$$

where $b^n = a$. This proves that finite products that build P_A and S_A are dense in $\mathcal{A}(A)$. □

Corollary 22. *When A is an algebraically closed field then P_A and S_A are dense in $\mathcal{A}(A)$.*

We have counter-examples for non-algebraically closed fields.

Proposition 23. *The group $S_{\mathbb{R}}$ is not dense in $\mathcal{A}(\mathbb{R})$.*

Proof. Consider $A = \mathbb{R}$. We prove that $1 + X^2 \in \mathcal{A}(\mathbb{R})$ is not in the closure of $S_{\mathbb{R}}$. Otherwise there are finite sequences $(a_k)_{1\leq k\leq n}$, $a_k \in \mathbb{R}$, and $(\epsilon_k)_{1\leq k\leq n}$, $\epsilon_k = \pm 1$, such that

$$\prod_{k=1}^{n}(1 + a_k X)^{\epsilon_k} = 1 + X^2 + \mathcal{O}(X^3)$$

then from the coefficients of X and X^2 we have

$$\sum_k \epsilon_k a_k = 0 \quad \text{and} \quad \sum_{k\neq l} \epsilon_k \epsilon_l a_k a_l + \sum_{\epsilon_k=-1} a_k^2 = 1 \ .$$

The second Newton relation gives a contradiction

$$0 = \left(\sum_k \epsilon_k a_k\right)^2 = \sum_k a_k^2 + \sum_{k\neq l} \epsilon_k \epsilon_l a_k a_l = \sum_k a_k^2 + \left(1 - \sum_{\epsilon_k=-1} a_k^2\right) = 1 + \sum_{\epsilon_k=1} a_k^2 \geq 1 \ .$$

□

Also, in general we don't have that P_A is dense in S_A.

Proposition 24. *The semigroup $P_{\mathbb{R}}$ is not dense in the group $S_{\mathbb{R}}$.*

Proof. Let $a \in \mathbb{R}$ with $a \neq 0$. We prove that $(1+aX)^{-1}$ is not in the closure of $P_{\mathbb{R}}$. We have

$$(1+aX)^{-1} = 1 - aX + a^2X^2 + \dots$$

By contradiction there would be a sequence $(a_k)_{1\leq k\leq n}$, $a_k \in \mathbb{R}$, such that

$$\prod_{k=1}^{n}(1+a_kX) = 1 - aX + a^2X^2 + \mathcal{O}(X^3).$$

Then we have

$$\sum_{k=1}^{n} a_k = -a \quad \text{and} \quad \sum_{k\neq l} a_k a_l = a^2 \ .$$

so

$$a^2 = \left(\sum_{k=1}^{n} a_k\right)^2 = \sum_{k=1}^{n} a_k^2 + \sum_{k\neq l} a_k a_l = \sum_{k=1}^{n} a_k^2 + a^2$$

therefore

$$\sum_{k=1}^{n} a_k^2 = 0$$

which implies $a_k = 0$ for all $k \geq 1$ and $a = 0$. Contradiction.

□

9. **Functoriality of the eñe product.** We prove the functoriality property given by Lenstra [15] for the Big Witt ring structure and the eñe ring structure (we give the statement in this last situation). As noted by Lenstra, $\mathcal{A}$ is a functor from the category of rings into the category of abelian groups. We prove that it is also a functor of rings with $\mathcal{A}(A)$ endowed with the eñe ring structure.

Theorem 25. *A morphism $\varphi : A \to B$ of commutative rings induces a commutative diagram*

$$\begin{array}{ccc} \mathcal{A}(A)\times\mathcal{A}(A) & \xrightarrow{\star} & \mathcal{A}(A) \\ {\scriptstyle (\mathcal{A}(\varphi),\mathcal{A}(\varphi))}\downarrow & & \downarrow{\scriptstyle \mathcal{A}(\varphi)} \\ \mathcal{A}(B)\times\mathcal{A}(B) & \xrightarrow{\star} & \mathcal{A}(B) \end{array}$$

Proof. We consider the enveloping ring of polynomials $\mathbb{Z}[(X_a)_{a\in A}]$ with an infinite, eventually uncountable, number of variables. We refer to Bourbaki's Algebra Chapter IV section 1 [3] for calculus on formal power series in an infinite number of variables. There is a surjective quotient

$$\pi_A : \mathbb{Z}[(X_a)_{a\in A}] \to A$$

such that $\pi_A(X_a) = a$ where the kernel encodes all the ring axioms and the ring relations. The map $\varphi : A \to B$ induces a map $(X_a)_{a\in A} \to (X_b)_{b\in B}$ (defined by $X_a \mapsto X_{\varphi(a)}$) and a ring morphism $\mathbb{Z}[(X_a)_{a\in A}] \to \mathbb{Z}[(X_b)_{b\in B}]$. We have a commuting diagram

$$\begin{array}{ccc} \mathbb{Z}[(X_a)_{a\in A}] & \longrightarrow & \mathbb{Z}[(X_b)_{b\in B}] \\ \downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_B} \\ A & \xrightarrow{\quad\varphi\quad} & B \end{array}$$

The quotient also defines a ring morphism between eñe rings $(\mathcal{A}(\mathbb{Z}[(X_a)_{a\in A}]), ., \star)$ and $(\mathcal{A}(A), ., \star)$. The result follows.

□

Remark. In the proof of the properties of the eñe product in section 6, we use the ring of formal power series in a countable set of variables $\mathbb{Q}[[(X_k)_{k\geq 1}, (Y_k)_{k\geq 1}, \ldots]]$. In this section, for different purposes, we consider the enveloping ring of polynomials $\mathbb{Z}[(X_a)_{a\in A}]$ which is generated a priori by an uncountable number of variables. In the next section we consider a ring of formal power series with an a priori uncountable number of variables. One should not be confused with the instrumental countable variables that serve to construct the eñe product and to prove its properties, and those other variables that are used to construct the enveloping ring that serve for establishing functorial properties. Rings of variables are useful in proving the properties of the eñe product when they are $\mathbb{Q}$-algebras (hence, for example, we can write directly exponential forms without needing a yoga with logarithmic derivatives). Note that if the ring A has non-zero characteristic it cannot be embedded into a larger ring of characteristic zero. This is the reason why is natural to consider enveloping rings. Their usefulness can be seen, for example, at the begining of section 9.10 in [11] where there is some confusion and the "larger ring" considered should instead be an enveloping ring.

10. **Short self-contained proof.** Consider the ring

$$A = \mathbb{Q}[[(X_k)_{k\geq 1}, (Y_l)_{l\geq 1}, (Z_m)_{m\geq 1}, \ldots]]$$

of commuting variables[3], the multiplicative group $\mathcal{A}(A) = 1 + TA[[T]]$ and the subgroup $S[T] \subset A[T]$ generated by $(1 - XT)_X$, where the X run over monomials of A, i.e. $f \in S[T]$ if there is a finite or countable sequence $(X_k)_{k\geq 1}$ of monomials in A such that

$$f(T) = \prod_{k=1}^{+\infty}(1 - X_k T) = \sum_{k\geq 0} \Sigma_k(X) T^k$$

[3]Maybe this time an uncountable number, but we write them in finite or countable groups.

where $\Sigma_k(X)$ is the k-th elementary symmetric function on the monomials $(X_k)_{k\geq 1}$ ($\Sigma_0(X)=1$).

Definition 26. *Let $f,g\in S[T]$, we define the eñe product $f\star g\in S[T]$ by*

$$f\star g=\left(\prod_{k=1}^{+\infty}(1-X_kT)\right)\star\left(\prod_{l=1}^{+\infty}(1-Y_lT)\right)=\prod_{k,l\geq 1}(1-X_kY_lT)=\sum_{k\geq 0}\Sigma_k(X,Y)T^k$$

Lemma 27. *$(S(T),.,\star)$ is a commutative ring.*

Proof. Obvious. If X_k and Y_l are monomials, then X_kY_l is a monomial. □

Lemma 28. *For $n\geq 0$,*

$$\Sigma_n(X,Y)=Q_n(\Sigma_1(X),\Sigma_2(X),\dots,\Sigma_n(X),\Sigma_1(Y),\Sigma_2(Y),\dots,\Sigma_n(Y))$$

for a universal polynomial $Q_n\in\mathbb{Z}[Z_1,\dots,Z_{2n}]$.

Proof. The polynomial $\Sigma_n(X,Y)$ is symmetric individually on each group of variables $(X_k)_{1\leq k\leq n}$ and $(Y_k)_{1\leq k\leq n}$. Using the Fundamental Theorem on symmetric functions (FTSF) over the coefficient ring $\mathbb{Z}[(X_k)_{1\leq k\leq n}]$ we have that $\Sigma_n(X,Y)$ is a polynomial with coefficients in $\mathbb{Z}[(X_k)_{1\leq k\leq n}]$ of $\Sigma_1(Y),\Sigma_2(Y),\dots,\Sigma_n(Y)$. Applying a second time to each coefficient the FTSF over the coefficient ring $\mathbb{Z}$ we prove the result. □

Corollary 29. *The eñe product extends as a binary operation to $\mathcal{A}(A)$ using the universal formulas and the commutative ring structure $(S(T),.,\star)$ extends to $(\mathcal{A}(A),.,\star)$.*

Proof. The condition of associativity and commutativity are polynomial universal relations with integer coefficients that remain true. □

Lemma 30 (Exponential form of the eñe product.)**.** *We have*

$$f(T)=\prod_{k=1}^{+\infty}(1-X_kT)=\exp\left(-\sum_{n=1}^{+\infty}\frac{1}{n}N_n(f)T^n\right)\quad\textit{with}\quad N_n(f)=\sum_{k=1}^{+\infty}X_k^n$$

and $N_n(f\star g)=N_n(f).N_n(g)$.

Proof. Develop $-\log(1-X_kT)$ in the exponential and regroup terms for the first statement. For the last one we use

$$\left(\sum_{k=1}^{+\infty}X_k^n\right).\left(\sum_{l=1}^{+\infty}Y_l^n\right)=\left(\sum_{k,l=1}^{+\infty}(X_kY_l)^n\right)\ .$$

□

Lemma 31 (Ghost exponential form of the Big Witt product.)**.** *We have*

$$f(T)=\prod_{k=1}^{+\infty}(1-X_kT^k)^{-1}=\exp\left(\sum_{n=1}^{+\infty}\frac{1}{n}W_n(f)T^n\right)\quad \textit{with}\quad W_n(f)=\sum_{d|n}dX_d^{n/d}\ .$$

and $W_n(f\star g)=-W_n(f).W_n(g)$.

Proof. Same as before for the first statement. Use the previous Lemma for the second. □

The Ghost components (W_n) are used to define the classical Big Witt product by

$$W_n(f\star_w g)=W_n(f).W_n(g)\ .$$

Therefore, we have constructed $\star_w$ from the eñe product:

Theorem 32. *We have* $f\star_w g=(f\star g)^{-1}$.

Corollary 33. *The result holds for an arbitrary commutative ring A.*

Proof. We have universal polynomials (Q_n^w) with $Q_n^w\in\mathbb{Z}[Z_1,\dots,Z_n]$ such that

$$f\star_w g=1+\sum_{k\geq 1}Q_k^w\,T^k=\left(1+\sum_{k\geq 1}Q_k\,T^k\right)^{-1}=(f\star g)^{-1}\ .$$

When the monomials are variables, we can replace $(X_k)_{k\geq 1}$ and $(Y_k)_{k\geq 1}$ with arbitrary elements of the ring A, since the polynomials have integer coefficients. The identity remains true for all commutative rings. □

11. **Historical origin of the Big Witt ring.** The original motivation of Witt was the study of cyclic field extensions of degree a power of a prime number p^n, which was part of the problems of interest to Hasse's Number Theory school around Class Field theory (see the historical survey by Roquette [21] and the biography of E. Witt by Kersten [8]). In this context, Witt introduced ring structures and Witt polynomials associated to a prime number p, $(W_{p^n})_{n\geq 1}$ in his work from 1937 [25].

Only many years later the full sequence of Bergman-Witt polynomials $(W_n)_{n\geq 1}$ and the Big Witt ring appear, almost simultaneously in different places. First, in 1965 in the first edition of Lang's algebra book [13] as an exercise in section VIII which gives credit to Witt for an oral communication. In Lang's exercise, the formula for the Bergman-Witt polynomial appears first in print. In Witt's Collected papers [27] there is an uncirculated manuscript (dated June 23rd 1964) by Witt from a seminar he gave in Hamburg with the construction of the Big Witt ring, but not the explicit formula of the Bergman-Witt polynomial. Then, G.M. Bergman, in chapter 26 of Mumford's book published in 1966 [2], gives a full construction of the Big Witt ring and the

Bergman-Witt polynomials appear in page 180. Bergman was a graduate student at the time. Later, P. Cartier in 1967 [5] gives a very economical construction of the Big Witt ring based on Bergman-Witt polynomials (although he leaves all details to the reader). Cartier cites Bergman indirectly by citing Mumford's book. Cartier's construction runs along the same lines as the one in Lang's book. He observes that, for each $n \geq 1$, the map from the product ring $W(A) = A^{\mathbb{N}^*}$ into A

$$W_n(\mathbf{a}) = \sum_{d|n} d a_d^{n/d}$$

is a ring morphism. Also he observes that the map $\mathbf{E} : W(A) \to \mathcal{A}(A)$ given by

$$\mathbf{E}(\mathbf{a}) = \prod_{n\geq 1} (1 - a_n T^n)^{-1} = f(T)$$

is a bijection, that satisfies $\mathbf{E}(\mathbf{a} + \mathbf{b}) = \mathbf{E}(\mathbf{a}).\mathbf{E}(\mathbf{b})$ and he defines

$$f \star_w g = \mathbf{E}(\mathbf{E}^{-1}(f).\mathbf{E}^{-1}(g)) \ .$$

So the Witt multiplication is just the multiplication of ghost components.

To add more confusion, years before, in 1958, A. Grothendieck, in his work in the theory of Chern classes [7], introduces the notion of λ-ring structure which have a Big Witt ring structure. Grothendieck refers to explicit universal formulas but he does not provide any, nor the Bergman-Witt polynomials. Also, in a letter to Mumford years later (31st August 1964 [17]), he praises Bergman "I liked also Bergman's Chapter 26–27 [2], and especially his universal Witt scheme, realized as a formal power series functor" and "...since Gabriel's seminar on formal groups I had the feeling that the Witt rings must also have a λ-ring structure" (he is talking here about the classical Witt rings from 1937 Witt's article). Cartier seems to have been the first one to clarify the relation between the Big Witt ring structure and the λ-ring structure [5].

Since then, the Big Witt structure has appeared in different branches of mathematics, many times it went unnoticed by a collective hallucination. We have seen the example of Manin and Kurokawa tensor product. Another apparently not well known example is the relation with the theory of Symmetric Functions (see the article of Scharf and Thibon [22], and the work of A. Lascoux [14]). Another example is the subject of Free Probability iniciated by D.-V. Voiculescu in the 80's, see [24], and only in 2012 the relation was noticed by Friedrich and McKay [6]. Also, recently, there is some renewed interest in the rational Witt vectors, that is, the extension of the Witt product to general rational functions (not normalized to be 1 at 0). This is very natural from our point of view and was developed for the eñe product by the second author in [18]. This seems to have noticed first in 1978 by Almkvist [1].

4

The diverse appearance in different contexts of the Big Witt ring is a clear sign of its universal and rich structure. For other examples and a rich background information, the reader is invited to go through Hazewinkel's survey of this vast subject [11].

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[4]Another example is the relation to the eñe product, that the second author experienced first hand in a curious episode. Around the year 2010, the second author spend one morning at the IHES with Pierre Cartier explaining to him the eñe product and its analytic properties. The presentation was similar to the one given in [18], but over the field $\mathbb{C}$ and stressing the analytic properties on finite order meromorphic functions. But Cartier didn't realize the link with the Big Witt ring! Sometimes the analytic context hides the algebraic essence of the subject and conversely.

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