MONODROMIES OF SINGULARITIES OF THE HADAMARD AND EÑE PRODUCT

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Abstract. We prove that singularities with holomorphic monodromies are preserved by Hadamard product, and we find an explicit formula for the monodromy of the singularities of their Hadamard product. We find similar formulas for the exponential eñe product whose monodromy is better behaved. With these formulas we give new direct proofs of classical results and prove the invariance of some rings of functions by Hadamard and eñe product.

"...la nature du point singulier $\alpha\beta$ ne dépend que de la nature des points singuliers α et β..."(Émile Borel on the Hadamard product, 1898)

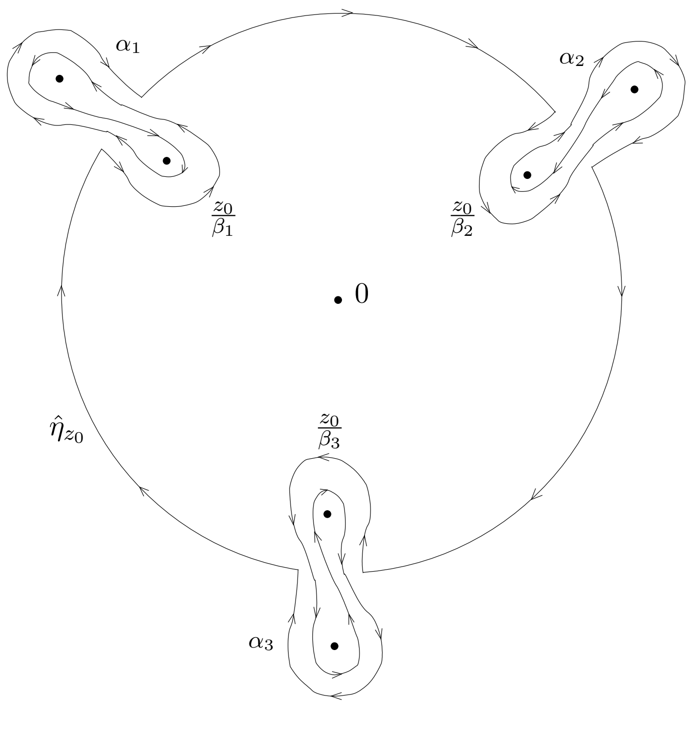

Figure 1. Choreographic monodromy integration contour.

2000 *Mathematics Subject Classification.* 08A02, 30D99, 30F99.

Key words and phrases. Eñe product, Hadamard product, singularities with monodromy, transalgebraic class.

1. Introduction and background.

1.1. **Hadamard and eñe product.** Given two power series

$$F(z) = A_0 + A_1 z + A_2 z^2 + \ldots = \sum_{n\geq 0} A_n\, z^n$$

$$G(z) = B_0 + B_1 z + B_2 z^2 + \ldots = \sum_{n\geq 0} B_n\, z^n$$

their classical Hadamard product is the power series

$$F \odot G(z) = A_0 B_0 + A_1 B_1 z + A_2 B_2 z^2 + \ldots = \sum_{n\geq 0} A_n B_n\, z^n$$

and their exponential eñe product is defined by

$$F \star_e G(z) = -A_1 B_1 z - 2 A_2 B_2 z^2 + \ldots = -\sum_{n\geq 0} n A_n B_n\, z^n$$

The Hadamard product and the exponential eñe product are commutative internal operations on the additive group of formal power series $\mathbb{C}[[z]]$ (or $A[[z]]$ for a commutative ring A), and $(\mathbb{C}[[z]], +, \odot)$ and $(\mathbb{C}[[z]], +, \star_e)$ are commutative rings. These products are also internal operations on the additive subgroup $\mathbb{C}\{\{z\}\}$ of power series with a positive radius of convergence, and $(\mathbb{C}\{\{z\}\}, +, \odot)$ and $(\mathbb{C}\{\{z\}\}, +, \star_e)$ are commutative subrings.

Hadamard (1899, [9]) proved the *Hadamard Multiplication Theorem* that locates the singularities of the principal branch of $F \odot G$ which are products of singularities of F and G. The exponential eñe product has also a beautiful interpretation in terms of divisors, i.e. zeros and poles (2019, [18]). More precisely, if $a_0 = b_0 = 0$ and we consider the exponential of the power series

$$f(z) = \exp(F(z)) = 1 + a_1 z + a_2 z^2 + \ldots = 1 + \sum_{n\geq 1} a_n\, z^n$$

$$g(z) = \exp(G(z)) = 1 + b_1 z + b_2 z^2 + \ldots = 1 + \sum_{n\geq 1} b_n\, z^n$$

and we define the eñe product as

$$f \star g(z) = \exp(F \star_e G(z)) = 1 + c_1 z + c_2 z^2 + \ldots = 1 + \sum_{n\geq 1} c_n\, z^n$$

then the coefficient c_n is a universal polynomial with integer coefficients on $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$. Thus the exponential eñe product is the bilinearization of the eñe product through the exponential. The universality of the expression of the coefficients (c_n) allows to define in general the eñe product over a commutative ring of coefficients A, more precisely, if $\mathcal{A} = 1 + A[[z]]$, then $\mathcal{A}$ is a group for the multiplication and when we adjoint the eñe product we obtain a commutative ring $(\mathcal{A}, ., \star)$.

If f and g are non-constant polynomials (note in particular that in such case F and G are not polynomials), with respective roots (α) and (β) (counted with multiplicity), so that

$$f(z) = \prod_{\alpha} \left(1 - \frac{z}{\alpha}\right)$$

$$g(z) = \prod_{\beta} \left(1 - \frac{z}{\beta}\right)$$

then we have the remarkable formula

$$f \star g(z) = \prod_{\alpha,\beta} \left(1 - \frac{z}{\alpha\beta}\right) \ .$$

This interpretation with zeros can be taken as the starting point of the theory of the eñe product as is done in [18] (2019). Then the divisor property can be extended to entire and meromorphic functions on the plane, and even to transcendental singularities (2019, [19]). The eñe product is both natural from the algebraic point of view, and from the analytic one (for example it is compatible with Hadamard-Weierstrass factorizations, see section 9 of [19]). This property of zeros is closely related to the Hadamard MultiplicationTheorem, although the Hadamard product lacks of a direct interpretation in terms of zeros. We have a neat formula relating the exponential eñe product and the Hadamard product, namely

$$F \star_e G = -K_0 \odot F \odot G$$

where K_0 is the Koebe function

$$K_0(z) = -\sum_{n\geq 1} n\, z^n = -\frac{z}{(1-z)^2}$$

For the definition of the eñe product and other algebraic and analytic properties and formulas we refer to [18] where we extend the divisor interpretation to meromorphic functions with zeros and poles. In [19] we extend further the eñe product to the transalgebraic class. For the Riemann sphere $\mathbb{P}^1\mathbb{C}$, this transalgebraic class is composed by functions with a finite number of exponential singularities. These are of the form $R_0 \exp R_1$ where R_0 and R_1 are rational functions. We prove in [19] that the divisor interpretation still holds for exponential singularities, which is natural when we view these singularities *à la Euler* as zeros or poles of "infinite order". From this point of view, we naturally introduce "eñe poles". Then functions with singularities with non-trivial monodromies arise naturally, in particular the hierarchy of polylogarithms.

Therefore, it becomes natural to investigate the extension of the divisor interpretation of the eñe product to singularities with non-trivial monodromy, i.e. non-uniform singularities in the XIX-th century terminology. A uniform transcendental singularity is by definition an isolated singularity without monodromy.

Almost simultaneously to the discovery by Hadamard of his Theorem determining the location of the singularities of $F \odot G$, Émile Borel (1898, [6]) proved that if F and G have uniform isolated singularities, i.e. isolated without monodromy, then the singularities of $F \odot G$ are also isolated and uniform. This result is related to the action of the eñe product on exponential singularities. Borel also makes the vague, but on point, observation that the nature of the singularities of $F \odot G$ only depends on the nature of the singularities of F and G.

"...la nature du point singulier $\alpha\beta$ ne dépend que de la nature des points singuliers α et β..."[1]
(É. Borel, 1898)

The goal of this article is to make very precise this statement, by giving, in some natural situations, what seems to be a new explicit formula for the monodromy of the singularities of $F \odot G$ in terms of the monodromies of singularities of F and G.

[1] "... the nature of the singular point $\alpha\beta$ depends only on the nature of the singular points α and β..."

1.2. **Holomorphic monodromy formulas.** We need first some definitions and properties on monodromies.

Definition 1.1 (Monodromy of an isolated singularity)**.** *Let F be an holomorphic function with an isolated singularity $\alpha \in \mathbb{C}$. We denote F_+ the analytic continuation of F when winding around α once in the positive orientation (in a neighborhood without any other singularity). The monodromy of F at the point $\alpha \in \mathbb{C}$ is*

$$\Delta_\alpha F = F_+(z) - F(z) \ .$$

Example. The simplest and more basic example of non-trivial monodromy is given by the logarithmic function, that we normalize properly, at the isolated singularity $\alpha = 1$,

$$\Delta_1 \left(\frac{1}{2\pi i} \log(z-1) \right) = 1$$

We consider in this article only holomorphic monodromies:

Definition 1.2 (Holomorphic monodromy)**.** *A function F with an isolated singularity at $\alpha \in \mathbb{C}$ has a holomorphic monodromy at α when $\Delta_\alpha F$ is holomorphic in a neighborhood of α.*

Observe that if the monodromy is holomorphic then $\Delta_\alpha^2 F = 0$. When $\Delta_\alpha^2 F = 0$, if

$$F_0 = F - \frac{1}{2\pi i} \log(z-\alpha) \Delta_\alpha F$$

then $\Delta_\alpha F_0 = 0$, and F_0 has a uniform singularity at α (isolated without monodromy). We have proved:

Proposition 1.3. *When $\Delta_\alpha^2 F = 0$, we can decompose F uniquely as*

$$F = F_0 + \frac{1}{2\pi i} \log(z-\alpha) \Delta_\alpha F$$

where F_0 has a uniform singularity at 0.

Note the minor abuse of notation since F_0 depends on the singularity α. If needed we may use the notation $F_{\alpha,0}$.

Definition 1.4 (Totally holomorphic singularity)**.** *The singularity α is totally holomorphic when both $\Delta_\alpha F$ and the germ F_0 are holomorphic at α.*

Our main result computes the monodromy $\Delta_{\alpha\beta}(F \odot G)$ from the monodromies $\Delta_\alpha F$ and $\Delta_\beta G$ in the case of holomorphic singularities. We have a remarkable explicit formula:

Theorem 1.5 (Holomorphic monodromy formula for the Hadamard product)**.** *We consider F and G holomorphic germs at 0 with respective set of singularities (α) and (β) in $\mathbb{C}$. We assume that the singularities are isolated and holomorphic, that is, $\Delta_\alpha F$, resp. $\Delta_\beta G$, is holomorphic at α, resp. β. Then the set of singularities of the principal branch of $F \odot G$ is contained in the product set $(\gamma) = (\alpha\beta)$ and is composed by isolated singularities which are holomorphic, and we have the formula*

$$\Delta_\gamma(F \odot G)(z) = -\sum_{\substack{\alpha,\beta \\ \alpha\beta=\gamma}} \text{Res}_{u=\alpha} \left(\frac{F_0(u)\Delta_\beta G(z/u)}{u} \right) - \sum_{\substack{\alpha,\beta \\ \alpha\beta=\gamma}} \text{Res}_{u=\beta} \left(\frac{G_0(u)\Delta_\alpha F(z/u)}{u} \right)$$
$$- \frac{1}{2\pi i} \sum_{\substack{\alpha,\beta \\ \alpha\beta=\gamma}} \int_\alpha^{z/\beta} \Delta_\alpha F(u) \Delta_\beta G(z/u) \frac{du}{u} \tag{1}$$

When the singularities are totally holomorphic, we have the simpler formula

$$\Delta_\gamma(F \odot G)(z) = -\frac{1}{2\pi i} \sum_{\substack{\alpha,\beta \\ \alpha\beta=\gamma}} \int_\alpha^{z/\beta} \Delta_\alpha F(u) \Delta_\beta G(z/u) \, \frac{du}{u} \tag{2}$$

The convolution in our formula appears also in the work of J. Écalle in his study of algebras of resurgent functions (see for example formula 2.1.8 p.19 of [7] where he uses the convolution for germs modulo holomorphic functions). There is certainly a relation that we don't fully understand at this point with Écalle's theories.

Notice the exceptional situation at $z = 0$: it can happen that the monodromies of F and G are both holomorphic at $z = 0$, but the monodromy of the Hadamard product has a singularity at $z = 0$ with a non-trivial monodromy. This monodromy is generated by the term $1/u$ in the integrand of the monodromy convolution formula (1). This occurs for instance for polylogarithms (see section 6.1).

We observe that when $\Delta_\alpha F = \Delta_\beta G = 0$ for all singularities α and β, then the singularities are all holomorphic, $\Delta_\gamma(F \odot G) = 0$. Therefore, Borel's Theorem is a direct Corollary of our formula.

Borel also observes that when F is a rational function, then one can construct a differential operator D_F such that the the singularities of $\alpha\beta \; F \odot G$ are "of the same nature" as those of $D_F G$ at β. The holomorphic monodromy formula makes this explicit when G has holomorphic singularities in the more general situation of a meromorphic function F.

Corollary 1.6. *Let F be a meromorphic function in $\mathbb{C}$ (for example a rational function), holomorphic at 0, with set of poles (α), and G with totally holomorphic singularities (β). The monodromies of $F \odot G$ are in the differential ring generated by the $\Delta_b G(z/\alpha)$, with field of constants generated by the coefficients of the polar parts of F. More precisely, consider the polar part of F at each pole α*

$$\sum_{k=1}^{d} \frac{a_{k,\alpha}}{(u-\alpha)^k}$$

then we have

$$\Delta_\gamma(F \odot G) = -\sum_{\substack{\alpha,\beta,k \\ \alpha\beta=\gamma}} \frac{a_{k,\alpha}}{(k-1)!} \left[\frac{d^{k-1}}{du^k} \left(\frac{\Delta_\beta G(z/u)}{u} \right) \right]_{u=\alpha} .$$

Example. A simple example occurs when we take $F = -K_0$, where $K_0(z) = z/(1-z)^2$ is the Koebe function which has a simple pole of order 2 at $z = 1$ and polar part

$$-K_0(z) = -\frac{1}{z-1} - \frac{1}{(z-1)^2} .$$

Then we compute

$$\begin{aligned}
\Delta_\beta(-K_0 \odot G) &= -\operatorname{Res}_{u=1} \left(\left(-\frac{1}{u-1} - \frac{1}{(u-1)^2} \right) \frac{\Delta_\beta G(z/u)}{u} \right) \\
&= [\Delta_\beta G(z/u)/u]_{u=1} + \left[\frac{\Delta_\beta G'(z/u)}{u} - \frac{\Delta_\beta G(z/u)}{u^2} \right]_{u=1} \\
&= \Delta_\beta G(z) + \Delta_\beta G'(z) - \Delta_\beta G(z) \\
&= \Delta_\beta G'(z) \\
&= (\Delta_\beta G)'(z)
\end{aligned}$$

Proposition 1.7. *The monodromies of the singularities of the Hadamard product with the negative of the Koebe function are the derivatives of the monodromies.*

We have a similar formula for the monodromies of the exponential eñe product.

Theorem 1.8 (Holomorphic monodromy formula for the exponential eñe product)**.** *We consider F and G holomorphic germs at 0 with respective sets of singularities (α) and (β). We assume that the singularities are isolated and holomorphic. Then the set of singularities of the principal branch of $F \star_e G$ is contained in the product set $(\gamma) = (\alpha\beta)$ and is composed by isolated singularities with holomorphic monodromies, and we have*

$$\Delta_\gamma(F \star_e G)(z) = \sum_{\substack{\alpha,\beta \\ \alpha\beta=\gamma}} \mathrm{Res}_{u=\alpha} \left(F_0'(u)\Delta_\beta G(z/u)\right) + \sum_{\substack{\alpha,\beta \\ \alpha\beta=\gamma}} \mathrm{Res}_{u=\beta} \left(G_0(u)\Delta_\alpha F'(z/u)\right)$$
$$+ \frac{1}{2\pi i} \sum_{\substack{\alpha,\beta \\ \alpha\beta=\gamma}} \Delta_\alpha F(\alpha)\Delta_\beta G(z/\alpha) + \frac{1}{2\pi i} \sum_{\substack{\alpha,\beta \\ \alpha\beta=\gamma}} \int_\alpha^{z/\beta} \Delta_\alpha F'(u)\Delta_\beta G(z/u)\, du \tag{3}$$

When the singularities are totally holomorphic, we have the simpler formula

$$\Delta_\gamma(F \star_e G)(z) = \frac{1}{2\pi i} \sum_{\substack{\alpha,\beta \\ \alpha\beta=\gamma}} \Delta_\alpha F(\alpha)\Delta_\beta G(z/\alpha) + \frac{1}{2\pi i} \sum_{\substack{\alpha,\beta \\ \alpha\beta=\gamma}} \int_\alpha^{z/\beta} \Delta_\alpha F'(u)\Delta_\beta G(z/u)\, du \tag{4}$$

Notice this time the absence of the factor $1/u$ in the integral of the monodromy convolution formula (3) for the exponential eñe product generates no extra singularities at $z = 0$ for non-principal branches of $F \star_e G$. This explains why the monodromies of singularities of the exponential eñe product have a better analytic behavior than those for the Hadamard product. The symmetry of the formula on F and G is clear in the first line, and for the second line it follows by integration by parts using basic properties of the monodromy operator Δ_α (see Proposition 5.3).

For the exponential eñe product we have the same Borel's type of Theorem. It follows from formula (3) that if $\Delta_\alpha F = \Delta_\beta G = 0$ then $\Delta_\gamma(F \star_e G) = 0$, hence we have:

Corollary 1.9. *If F and G have only uniform singularities then $F \star_e G$ has only uniform singularities, i.e. $\Delta_\gamma(F \star_e G) = 0$.*

We have also an analogue of Corollary 1.6. The result is stronger because of the better analytic properties of the eñe product.

Corollary 1.10. *Let F be a function in $\mathbb{C}$ with a discrete set (α) of singularities with constant monodromies, such that the germs F_0 are meromorphic, F is holomorphic at 0, and G with totally holomorphic singularities (β). The monodromies of $F \star_e G$ are in the differential ring generated by the $\Delta_\beta G(z/\alpha)$, with field of constants generated by the coefficients of the polar parts of F and the constants $\left(\frac{\Delta_\alpha F}{2\pi i}\right)$. More precisely, consider the polar part of each F_0 at each pole α*

$$\sum_{k=1}^{d} \frac{a_{k,\alpha}}{(u-\alpha)^k} ,$$

then we have

$$\Delta_\gamma(F \odot G) = -\sum_{\substack{\alpha,\beta,k \\ \alpha\beta=\gamma}} \frac{a_{k,\alpha}}{(k-1)!} \left[\frac{d^k}{du^k}\left(\Delta_\beta G(z/u)\right)\right]_{u=\alpha} + \sum_{\substack{\alpha,\beta \\ \alpha\beta=\gamma}} \frac{\Delta_\alpha F(\alpha)}{2\pi i} \Delta_\beta G(z/\alpha) .$$

A Corollary of the algebraic nature of these monodromy formulas is the invariance by the Hadamard and eñe product of some natural rings of functions. Consider a field $K \subset \mathbb{C}$ with $2\pi i \in K$, and consider the ring, for the usual addition and multiplication, $PML(K)$ (Polynomial Logarithmic Monodromy ring with coefficients in K) of germs holomorphic at 0 with only isolated singularities located at points in K, and with monodromy in $K[z, \log z]$.

Corollary 1.11. *The $PLM(K)$ ring is closed under Hadamard, resp. eñe, product, and is the minimal Hadamard ring, resp. eñe ring, containing the class of functions with polynomial monodromies in $K[z]$ with isolated singularities located at points of K.*

2. Convolution formulas.

The main tool in the proof of Hadamard Theorem is Pincherle (or Hadamard) convolution formula (1885, see [16], [12] and [9]):

Proposition 2.1 (Pincherle convolution formula)**.** *The Hadamard product has the integral form*

$$F \odot G(z) = \frac{1}{2\pi i} \int_\eta F(u) G(z/u) \, \frac{du}{u} \tag{5}$$

where η is a positively oriented circle centered at 0 of radius $r > 0$ with $|z|/R_G < r < R_F$, where R_F and R_G are the respective radii of convergence of F and G, so that $F(u)$ and $G(z/u)$ are well defined.

We can take for η any Jordan curve located in the annulus bounded by the circles of radii R_F and $|z|/R_G$ and with winding number $+1$ with respect to 0.

Proof. The convolution formula immediately follows from the integration term by term of the series

$$\frac{F(u)G(z/u)}{u} = \sum_{n,m \geq 0} F_n G_m u^{n-m-1} z^m$$

and the application of Cauchy formula

$$\frac{1}{2\pi i} \int_\eta u^{n-m-1} \, du = \delta_{n,m}$$

where $\delta_{n,m}$ denotes the Kronecker symbol. □

For the exponential eñe product we have a similar convolution formula:

Proposition 2.2 (Exponential eñe product convolution formula)**.** *The exponential eñe product has the integral form*

$$F \star_e G(z) = -\frac{1}{2\pi i} \int_\eta F'(u) G(z/u) \, du \tag{6}$$

where η is a circle centered at 0 with the same conditions as before.

Proof. The proof is similar observing that $R_{F'} = R_F$ and integrating term by term

$$F'(u)G(z/u) = \sum_{n,m \geq 0} n F_n G_m u^{n-m-1} z^m$$

and using Cauchy formula as before. □

The convolution formula (5) gives the analytic continuation of the Hadamard product $F \odot G$ using the analytic continuation of F and G. We only need to deform homotopically the contour η when we move z around. The only obstruction to this continuation follows from a close inspection of the convolution formula: $F \odot G(z)$ does extend analytically unless $z = 0$ (except for the principal branch) or when we hit a point $z \in \mathbb{C}$ such that we have $u \in \mathbb{C}$ fulfilling with both conditions

$$\begin{cases} u & = \alpha \\ z/u & = \beta \end{cases}$$

with α and β singularities of F and G respectively. This happens if and only if $z = \alpha\beta$. Thus we have proved Hadamard Theorem:

Theorem 2.3 (Hadamard Multiplication Theorem, 1899, [9])**.** *The singularities of the principal branch of $F \odot G$ are of the form $\gamma = \alpha\beta$ where α and β are singularities of F and G respectively.*

The origin 0 is not a singularity of the principal branch by assumption. But the convolution formula shows that the origin 0 can become a singularity of other branches of the analytic continuation of $F \odot G$ because of the $1/u$ factor in the integrand. Moreover, the convolution formula also proves that if F and G are fluent in the sense of Liouville and Ritt, which, roughly speaking, means that the functions have an analytic extension around singularities, then their Hadamard product $F \odot G$ is fluent. The definition of fluency given by Ritt in his book[2] (1948,[20]) on Liouville theory of integration on finite terms (1833, [14], [15]) is not precise and indeed there exists more or less stronger versions of fluency (see section 5.4.1 of [11], and in particular [13]). Fluency is a key property of functions in the old Liouville classification of transcendental functions. In simpler terms, we can state the following Corollary of Hadamard Multiplication Theorem (or the Convolution Formula),

Corollary 2.4. *If both F and G have only isolated singularities with monodromy, then $F \odot G$ has only isolated singularities with monodromy.*

These considerations give the following geometric improvement of Hadamard Theorem (the reader, if not familiar, can skip this Theorem where we use the language of log-Riemann surfaces, see [2] for general background, [3] for more general definitions, and [4] and [5] for further properties).

Theorem 2.5. *Let $\mathcal{S}_F$ and $\mathcal{S}_G$ be the log-Riemann surfaces of the germs at 0 defined by F and G respectively. Then the log-Riemann surface $\mathcal{S}_{F\odot G}$ has a ramification set $\mathcal{R}_{F\odot G}$ such that*

$$\pi_{F\odot G}(\mathcal{R}_{F\odot G}) \subset \{0\} \cup (\pi_F(\mathcal{R}_F).\pi_G(\mathcal{R}_G))$$

where $\pi_{F\odot G} : \mathcal{S}_{F\odot G} \to \mathbb{C}$, $\pi_F : \mathcal{S}_F \to \mathbb{C}$ and $\pi_G : \mathcal{S}_G \to \mathbb{C}$ are the canonical projections, and $A.B$ denotes the set of all products ab with $a \in A$ and $b \in B$.

In particular, if the ramification sets $\mathcal{R}_F$ and $\mathcal{R}_G$ are discrete, then $\mathcal{R}_{F\odot G}$ is discrete.

For the exponential eñe product we a similar Theorem as Hadamard Multiplication Theorem using the formula $F \star_e G = -K_0 \odot F \odot G$ (or using the convolution formula for the same proof observing that that the singularities of F and F' are the same). Hence we obtain:

Theorem 2.6 (Singularities of the exponential eñe product)**.** *The singularities of the principal branch of $F \star_e G$ are of the form $\gamma = \alpha\beta$ where α and β are singularities of F and G respectively.*

We have a corresponding Theorem for the log-Riemann surface $\mathcal{S}_{F\star_e G}$ which is simpler since we don't need to add the origin in the locus of the projection.

[2]Ritt's book is from the pre-differential algebra era.

Theorem 2.7. *Let $\mathcal{S}_F$ and $\mathcal{S}_G$ be the log-Riemann surfaces of the germs at 0 defined by F and G respectively. Then the log-Riemann surface $\mathcal{S}_{F\star_e G}$ has a ramification set $\mathcal{R}_{F\star_e G}$ such that*

$$\pi_{F\star_e G}(\mathcal{R}_{F\star_e G}) \subset \pi_F(\mathcal{R}_F).\pi_G(\mathcal{R}_G)$$

where $\pi_{F\star_e G} : \mathcal{S}_{F\star_e G} \to \mathbb{C}$, $\pi_F : \mathcal{S}_F \to \mathbb{C}$ and $\pi_G : \mathcal{S}_G \to \mathbb{C}$ are the canonical projections, and $A.B$ denotes the set of all products ab with $a \in A$ and $b \in B$.

In particular, if the ramification sets $\mathcal{R}_F$ and $\mathcal{R}_G$ are discrete, then $\mathcal{R}_{F\star_e G}$ is discrete.

3. Monodromy of singularities and monodromy operator.

As already observed, the inspection of the convolution formula shows also that if both F and G have isolated singularities with monodromy then the singularities of $F \odot G$ and of $F \star_e G$ are also isolated with monodromy (including the case of trivial monodromy). More precisely, we recall again the definition of the monodromy and we define the monodromy operator:

Definition 3.1 (Monodromy and operator monodromy of an isolated singularity)**.** *Let F be an holomorphic function with an isolated singularity $\alpha \in \mathbb{C}$. We denote F_+ the analytic continuation of F when turning around α once in the positive orientation (in a neighborhood without any other singularity). The monodromy of F at the point $\alpha \in \mathbb{C}$ is*

$$\Delta_\alpha F = F_+(z) - F(z) \ .$$

This definition can be extended also to regular points α of F. The map Δ_α defines a linear operator, the monodromy operator at α, on the vector space V_α of holomorphic functions having a regular point or an isolated singularity at α. We also define the operator Σ_α such that

$$\Sigma_\alpha F = F_+$$

and we have

$$\Sigma_\alpha = I + \Delta_\alpha$$

In general, we define that monodromy operator along a path γ, $\sigma_\gamma : (\mathbb{C}, \gamma(0)) \to (\mathbb{C}, \gamma(1))$, which associates to a holomorphic germ at $\gamma(0)$ its Weierstrass holomorphic continuation along γ at $\gamma(1)$. In a domain where the germ is holomorphic, we have $\sigma_\gamma = \sigma_{[\gamma]}$ only depends on the homotopy class $[\gamma]$ of γ, and for a loop γ with winding number 1 with respect to α, we have $\Sigma_\alpha = \sigma_{[\gamma]}$.

Note that $\Delta_\alpha F$ can develop non-local singularities elsewhere at other points distinct from α. The structure of the monodromy at α is important, and justifies the following definitions.

Definition 3.2. *A function F with an isolated singularity at $\alpha \in \mathbb{C}$ has a holomorphic, resp. meromorphic, uniform, monodromy at α when $\Delta_\alpha F$ is holomorphic, resp. meromorphic, with a uniform isolated singularity at α, in a neighborhood of α. The singularity α is totally holomorphic if both $\Delta_\alpha F$ and $F_0 = F - \frac{1}{2\pi i}\log(z-\alpha)\Delta_\alpha F$ are holomorphic.*

Recall (see Proposition 1.3) that when $\Delta_\alpha^2 F = 0$ we can write uniquely in a neighborhood of α

$$F = F_0 + \frac{1}{2\pi i}\log(z-\alpha)\Delta_\alpha F$$

where F_0 has a uniform isolated singularity at α. The growth behavior of F near the singularity α is important.

Definition 3.3. *Let F be function with a regular point or an isolated singularity at $\alpha \in \mathbb{C}$. Then F is integrable at α, or α is an integrable singularity, if we have*

$$\text{S-lim}_{z\to\alpha}(z-\alpha)\,F(z) \to 0$$

where this is a Stolz limit, that is, $z \to \alpha$ with $\arg(z-\alpha)$ bounded. The singularity α has order $\rho > 0$, if $F(z) = \mathcal{O}(|z-\alpha|^{-\rho})$ in any Stolz angle[3].

The logarithmic function branched at α, $\log(z-\alpha)$, is integrable at α. The space of functions that are integrable, of order ρ or finite order functions at α form a $\mathbb{C}$-vector space. These vector spaces are $\mathcal{O}_\alpha$-modules, where $\mathcal{O}_\alpha$ is the local ring of holomorphic germs at α. For an integrable singularity we have

$$\int_{[\alpha,\alpha+\epsilon]} F(u)\,du \to 0$$

and

$$\int_\eta F(u)\,du \to 0$$

when $\epsilon \in \mathbb{C}$ is small and $\epsilon \to 0$, and the loop η is a small circle around α with radius $\to 0$. Adapting the arguments given in [10] we can prove that the Hadamard product of integrable singularities is integrable, but we don't need this result in this article.

Proposition 3.4. *If F is holomorphic in a pointed neighborhood of α, i.e. $\Delta_\alpha F = 0$, and integrable, then F is holomorphic at α.*

Proof. We can write a converging Laurent expansion in a pointed neighborhood of α,

$$F(z) = \sum_{n\in\mathbb{Z}} a_n (z-\alpha)^n$$

and we have the Cauchy formula for the coefficients

$$a_n = \frac{1}{2\pi i}\int_\eta \frac{f(u)}{(z-u)^n}\,du\ .$$

When we shrink η to α, the integrability estimates show that $a_n = 0$ for all $n < 0$, hence F is holomorphic. □

Proposition 3.5. *We have*

$$\Delta_\alpha(F.G) = \Delta_\alpha(F).G + F.\Delta_\alpha(G) + \Delta_\alpha(F).\Delta_\alpha(G)$$

Proof. It follows by analytic continuation that

$$\Sigma_\alpha(F.G) = \Sigma_\alpha(F).\Sigma_\alpha(G)$$

and using $\Sigma_\alpha = I + \Delta_\alpha$ gives the result. □

Corollary 3.6. *If $\Delta_\alpha^2(F) = 0$, in particular when the monodromy of F at α is holomorphic or meromorphic, then*

$$\Delta_\alpha\Big(F - \frac{1}{2\pi i}\log(z-\alpha)\,\Delta_\alpha(F)\Big) = 0$$

In that case we can write,

$$F(z) = F_0(z) + \frac{1}{2\pi i}\log(z-\alpha)\,\Delta_\alpha(F)$$

[3]These are also called "singularities with moderate growth in sectors", see [21], section 9.2, or "regular singularities" in the context of solutions of differential systems, see [1], p.8.

where $\Delta_\alpha F_0 = 0$.

Proof. We observe that

$$\Delta_\alpha \left(\frac{1}{2\pi i} \log(z-\alpha) \right) = 1$$

and then from the previous Proposition we get

$$\begin{aligned} \Delta_\alpha \Big(F - \frac{1}{2\pi i} \log(z-\alpha)\, \Delta_\alpha(F) \Big) &= \Delta_\alpha(F) - 1.\Delta_\alpha(F) - \frac{1}{2\pi i} \log(z-\alpha).\Delta_\alpha^2(F) - 1.\Delta_\alpha^2(F) \\ &= \Delta_\alpha(F) - \Delta_\alpha(F) - 0 - 0 \\ &= 0 \end{aligned}$$

□

Corollary 3.7. *We assume that* $\Delta_\alpha^2 F = 0$. *The singularity* α *of* F *is integrable if and only if the singularity is totally holomorphic, that is,* $\Delta_\alpha F$ *and* F_0 *are both holomorphic at* α.

Proof. The condition is necessary. If F is integrable at α, directly from the definition we get that F_+ is integrable, hence $\Delta_\alpha F = F_+ - F$ is integrable and holomorphic in a pointed neighborhood of α. Proposition 3.4 implies that $\Delta_\alpha F$ is a holomorphic function at α. Then also $\frac{1}{2\pi i} \log(z-\alpha)\, \Delta_\alpha(F)$ is integrable at α, hence $F_0(z) = F(z) - \frac{1}{2\pi i} \log(z-\alpha)\, \Delta_\alpha(F)$ is also integrable. Again, by Proposition 3.4, F_0 being holomorphic in a pointed neighborhood and integrable, it is holomorphic.

The condition is sufficient. We assume that F_0 and $\Delta_\alpha F$ are both holomorphic at α. We have that $\frac{1}{2\pi i} \log(z-\alpha)\, \Delta_\alpha(F)$ is integrable because $r \log r$ is integrable on $\mathbb{R}_+$ at $r = 0$. Also, F_0 is integrable since is it holomorphic at α. Now, adding these two integrable functions it follows that F is integrable. □

Now, observe that we have $F_+ = F + \Delta_\alpha F$, hence

$$(F_+)' = F' + (\Delta_\alpha F)'$$

and since $(F')_+ = (F_+)'$, which is obvious by analytic continuation, then we conclude that the monodromy operator Δ_α commutes with the derivation.

Proposition 3.8. *We have*

$$\Delta_\alpha(F') = (\Delta_\alpha F)'$$

Corollary 3.9. *Let* $\vartheta = P(D) = \sum_{n=0}^{N} a_n D^n$ *with* $D = d/dz$ *be a differential operator with coefficients* (a_n) *with isolated singularities without monodromy. We have*

$$\Delta_\alpha(\vartheta F) = \vartheta(\Delta_\alpha F)\ .$$

Therefore, if F *satisfies the differential equation* $\vartheta F = 0$ *then the monodromy* $\Delta_\alpha F$ *satisfies the same differential equation*

$$\vartheta(\Delta_\alpha F) = 0$$

and $\Delta_\alpha F$ *belongs to the finite dimensional vector space of solutions. We have the same result for a system of differential equations.*

Proof. Using Proposition 3.5 we have that

$$\Delta_\alpha(a_n F^{(n)}) = a_n \Delta_\alpha(F^{(n)}) = a_n \Delta_\alpha(F)^{(n)}$$

and the result follows. □

The monodromy $\Delta_\alpha F$ can have an isolated singularity at α with non-trivial monodromy. For example this happens at singular points of differential equations, and of course for algebraic functions. For these we have the simple, but important, observation:

Proposition 3.10. *We assume that F is an algebraic function satisfying the algebraic equation*

$$P(z,F)=0$$

with $P\in\mathbb{C}[x,y]$. Then $\Delta_\alpha F$ is an algebraic function satisfying an algebraic equation of lower degree

$$Q(z,\Delta_\alpha F)=0$$

with $Q\in\mathbb{C}[x,y]$ and $\deg_y Q<\deg_y P$

Proof. By analytic continuation around α we have

$$P(z,F+\Delta_\alpha F)=0$$

developing the powers of $F+\Delta_\alpha F$, using $P(z,F)=0$, and factoring out $\Delta_\alpha F$ we get an explicit expression for Q that has a lower degree in the second variable. □

We will use the following change of variables formula.

Proposition 3.11 (Change of variables formula)**.** *Let $\varphi:(\mathbb{C},\beta)\to(\mathbb{C},\alpha)$ a local holomorphic diffeomorphism, $\alpha=\varphi(\beta)$. We have*

$$\Delta_{\varphi^{-1}(\alpha)}(F\circ\varphi)=\Delta_\alpha(F)\circ\varphi\ .$$

Proof. We consider a local loop γ enclosing α with winding number 1 with respect to α. Its pre-image $\varphi^{-1}(\gamma)$ has winding number 1 with respect to β. When we continue analytically along $\varphi^{-1}(\gamma)$, the return map for $F\circ\varphi$ is $F\circ\varphi+\Delta_\beta(F\circ\varphi)$ by definition of the monodromy. The return analytic continuation of F along γ is $F+\Delta_\alpha F$, hence we have

$$F\circ\varphi+\Delta_\beta(F\circ\varphi)=(F+\Delta_\alpha(F))\circ\varphi=F\circ\varphi+\Delta_\alpha(F)\circ\varphi$$

and we get the stated formula. □

Corollary 3.12. *Let $z_0\in\mathbb{C}^*$. We have*

$$\Delta_\beta(F(z_0/z))=\Delta_{z_0/\beta}(F)(z_0/z)\ .$$

Proof. With the change of variables $\varphi(z)=z_0/z$, thus $\varphi^{-1}(z)=z_0/z$, we get the result using Proposition 3.11. □

The formula in this Corollary remains valid for the monodromy at ∞ in the Riemann sphere, $\alpha=\infty\in\overline{\mathbb{C}}$. Thus, the formula holds in general for a Moëbius transformation φ.

This result is a particular case for $n=1$ (local diffeomorphism) of the local degree $n\geq 1$ case (we will not use this more general result).

Proposition 3.13. *Let $\varphi:(\mathbb{C},\beta)\to(\mathbb{C},\alpha)$ a local holomorphic map with β a critical point of degree $n\geq 1$ (hence φ is of local degree n). We have*

$$\Delta_\alpha(F)\circ\varphi=\sum_{k=1}^{n}\binom{n}{k}\Delta_\beta^k(F\circ\varphi)$$

Lemma 3.14. *Let γ be a local loop with winding number $n \geq 1$ with respect to $\beta \in \mathbb{C}$ in a neighborhood where β is the only singularity of F in this neighborhood. We denote $\Delta_\beta^{(n)} F$ the monodromy of F along γ that only depends on the homotopy class $[\gamma]$, i.e. on its winding number $n \geq 1$. We have*

$$\Delta_\beta^{(n)} F = \sum_{k=1}^{n} \binom{n}{k} \Delta_\beta^k F \ .$$

Proof. We observe that

$$\Delta_\beta^{(n)} = \sigma_{n.[\gamma]}^n - I = \Sigma_\beta^n - I \ .$$

Since $\Sigma_\beta = I + \Delta_\beta$, the result follows from Newton binomial formula. □

Proof of Proposition 3.13. We carry out the same proof as before, but this time the pre-image $\varphi^{-1}(\gamma)$ has winding number $n \geq 1$ with respect to $\beta = \varphi^{-1}(\alpha)$. When we continue analytically along γ the return map of F is $F + \Delta_\alpha(F)$ by definition of the monodromy. The return analytic continuation of $F \circ \varphi$ along $\varphi^{-1}(\gamma)$ is $F \circ \varphi + \Delta_\beta^{(n)}(F \circ \varphi)$ hence we have

$$F \circ \varphi + \Delta_\alpha(F) \circ \varphi = F \circ \varphi + \Delta_\beta^{(n)}(F \circ \varphi)$$

so $\Delta_\alpha(F) \circ \varphi = \Delta_\beta^{(n)}(F \circ \varphi)$ and the formula follows from Lemma 3.14. □

We leave the general study of monodromies $\Delta_\alpha F$ having an isolated singularity at α with non-trivial monodromy to future articles.

4. Totally holomorphic monodromy formula.

As a preparation for the general case, we first prove the monodromy formula (2) in the totally holomorphic or, equivalently, the integrable case. Thanks to the integrability conditions, there is no local contributions in the integration path argument that is central in the proof. In the next section we treat the general case.

4.1. **Proof of the totally holomorphic monodromy formula: Single singularity case.** We prove in this section Theorems 1.5 and 1.8 in the totally holomorphic case. The proof of Theorem 1.8 is similar and indications will be given at the end, so we concentrate in the proof of 1.5.

The first observation is that the result is purely local near the singularities. We may have some multiplicity when there exists distinct pairs $(\alpha, \beta) \neq (\alpha', \beta')$ such that $\alpha\beta = \alpha'\beta'$. Then we have a linear superposition of the different contributions. So, we can be more precise in Theorems 1.5 since there is no need to assume in the Theorem that all singularities have the same holomorphic structure but only those α's and β's that contribute to the location γ.

We consider first the case where there is no multiplicity, that is $\gamma = \alpha\beta$ for a unique pair (α, β), and we can assume that $\alpha = \beta = \gamma = 1$, and the radii of convergence are $R_F = R_G = 1$. The general case is a superposition of this case. We consider a current value z close to 1, and we follow the analytic extension of $F \odot G$ when z, starting at z_0, turns around 1 once in the positive direction. We can start at z_0 with $|z_0| < 1$, and z_0 close to 1.

We start by integrating Hadamard convolution formula on a circle η_{z_0} of radius r with $|z_0| < r < 1$, so that both series giving $F(u)$ and $G(z_0/u)$ are converging. When z, starting at z_0, moves around the point 1, we deform homotopically the integration path into η_z so that z never crosses the integration path. With such condition, the convolution formula yields the analytic continuation of

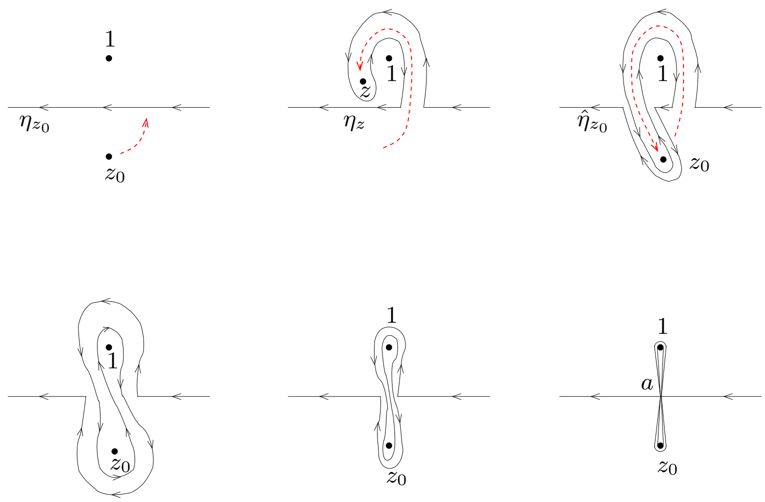

FIGURE 2. Homotopical deformation of the integration path when z_0 turns around 1.

$F \odot G$. When we return to z_0, the path is deformed into $\hat{\eta}_{z_0}$ (see Figure 2) the Hadamard product takes the value $(F \odot G)_+(z_0)$. The difference

$$\Delta_1(F \odot G)(z_0) = (F \odot G)_+(z_0) - (F \odot G)(z_0)$$

is the monodromy at 1. According to the convolution formula, this difference can be computed by integrating on the homotopical difference of the two paths $\hat{\eta}_{z_0} - \eta_{z_0}$ (Figure 2). Considering the intersection point $a = [z_0, 1] \cap \eta_{z_0}$, and shrinking the path as shown in Figure 2, this difference is composed by four (indeed only two, repeated twice) vertical segments (it is indeed a "train track") $\eta_1 = [a, 1]$, $\eta_2 = [a, z_0]$, $\eta_3 = [a, 1]$ and $\eta_4 = [a, z_0]$ where we integrate in both directions different functions. Note also that the small turning loops around 1 and z_0 give no contribution when they shrink because of the integrability condition (in the proof of the general case there is a non-trivial residual contribution here). We decompose further each path η_j into two consecutive paths $\eta_j = \eta_j^- \cup \eta_j^+$ so that the difference $\hat{\eta}_{z_0} - \eta_{z_0}$ decomposes as

$$\hat{\eta}_{z_0} - \eta_{z_0} = \eta_1^- \cup \eta_1^+ \cup \eta_2^- \cup \eta_2^+ \cup \eta_3^- \cup \eta_3^+ \cup \eta_4^- \cup \eta_4^+$$

where the paths $\eta_j^{\pm}$ follow each other in the order this union is written. We first list the functions that are integrated against the differential du/u in each path. To compute these functions, notice that the monodromy around $u = 1$ of $F(u)$ is $\Delta_1 F(u)$, and, according to Corollary 3.12, the monodromy around z_0 of $G(z_0/u)$ is $\Delta_1 G(z_0/u)$, i.e.

$$\Delta_{z_0}(G(z_0/u)) = (\Delta_1 G)(z_0/u) \ .$$

We need to take into account the sign corresponding to the orientation of the loop around each singularity. Note that we are assuming that the monodromies $\Delta_1 F$ and $\Delta_1 G$ are holomorphic and, in particular, there are no second monodromies at 1.

$$\begin{aligned}
\eta_1^- &\to F(u)G(z_0/u)\\
\eta_1^+ &\to F(u)G(z_0/u) + \Delta_1 F(u)G(z_0/u)\\
\eta_2^- &\to F(u)G(z_0/u) + \Delta_1 F(u)G(z_0/u)\\
\eta_2^+ &\to F(u)G(z_0/u) + F(u)\Delta_1 G(z_0/u) + \Delta_1 F(u)G(z_0/u) + \Delta_1 F(u)\Delta_1 G(z_0/u)\\
\eta_3^- &\to F(u)G(z_0/u) + F(u)\Delta_1 G(z_0/u) + \Delta_1 F(u)G(z_0/u) + \Delta_1 F(u)\Delta_1 G(z_0/u)\\
\eta_3^+ &\to F(u)G(z_0/u) - \Delta_1 F(u)G(z_0/u) + F(u)\Delta_1 G(z_0/u) - \Delta_1 F(u)\Delta_1 G(z_0/u)+\\
&\qquad + \Delta_1 F(u)G(z_0/u) + \Delta_1 F(u)\Delta_1 G(z_0/u)\\
&\quad = F(u)G(z_0/u) + F(u)\Delta_1 G(z_0/u)\\
\eta_4^- &\to F(u)G(z_0/u) + F(u)\Delta_1 G(z_0/u)\\
\eta_4^+ &\to F(u)G(z_0/u) - F(u)\Delta_1 G(z_0/u) + F(u)\Delta_1 G(z_0/u) = F(u)G(z_0/u)
\end{aligned}$$

Now, for the contributions of each integral we have

$$\int_{\eta_j} = \int_{\eta_j^-} - \int_{\eta_j^+}$$

and these contributions for each integral η_j are, respectively,

$$\int_{\eta_1} -\Delta_1 F(u)G(z_0/u)\,\frac{du}{u} = -\int_a^1 \Delta_1 F(u)G(z_0/u)\,\frac{du}{u}$$

$$\int_{\eta_2} \left(-F(u)\Delta_1 G(z_0/u) - \Delta_1 F(u)\Delta_1 G(z_0/u)\right)\,\frac{du}{u} = -\int_a^{z_0} \left(F(u)\Delta_1 G(z_0/u) + \Delta_1 F(u)\Delta_1 G(z_0/u)\right)\,\frac{du}{u}$$

$$\int_{\eta_3} \left(\Delta_1 F(u)G(z_0/u) + \Delta_1 F(u)\Delta_1 G(z_0/u)\right)\,\frac{du}{u} = \int_a^1 \left(\Delta_1 F(u)G(z_0/u) + \Delta_1 F(u)\Delta_1 G(z_0/u)\right)\,\frac{du}{u}$$

$$\int_{\eta_4} F(u)\Delta_1 G(z_0/u)\,\frac{du}{u} = \int_a^1 F(u)\Delta_1 G(z_0/u)\,\frac{du}{u}$$

Adding up, and after some cancellations when pairing η_1 and η_3, and pairing of η_2 and η_4, we get

$$\begin{aligned}
\int_{\eta_1} + \int_{\eta_3} &= \int_a^1 \Delta_1 F(u)\Delta_1 G(z_0/u)\,\frac{du}{u}\\
\int_{\eta_2} + \int_{\eta_4} &= -\int_a^{z_0} \Delta_1 F(u)\Delta_1 G(z_0/u)\,\frac{du}{u}
\end{aligned}$$

and finally

$$\int_{\eta_{z_0} - \hat{\eta}_{z_0}} = \int_{\eta_1} + \int_{\eta_2} + \int_{\eta_3} + \int_{\eta_4} = -\int_1^{z_0} \Delta_1 F(u)\Delta_1 G(z_0/u)\,\frac{du}{u}$$

which gives the formula for the case of a single singularity.

In the general case when $\alpha\beta = \gamma$, we get the contribution

$$-\int_\alpha^{z_0/\beta} \Delta_1 F(u)\Delta_1 G(z_0/u)\,\frac{du}{u}$$

In this path deformation argument, there is no residual contribution contribution at 1 because the functions and their monodromies are all integrable at 1.

4.2. **Proof for higher multiplicity singularities.** The initial Jordan loop of integration η_{z_0} separates the singularities (α) of F, which are in the outside unbounded region, from the singularities (z_0/β) of $G(z_0/u)$, which are in the inside bounded region. As before, when the point z starting at z_0 circles once around $\alpha\beta = \gamma$, then all points z/β circles once around the corresponding α in a synchronized choreography. We end-up with a path $\hat{\eta}_{z_0}$ as shown in Figure 1 (reversing the orientation). Then the difference of paths $\eta_{z_0} - \hat{\eta}_{z_0}$ is decomposed into a finite number of quadruple loops as the one considered before, one for each pair (α, β) such that $\gamma = \alpha\beta$. The total contribution adds the contribution of each quadruple loop and the formula follows.

5. General holomorphic monodromy formula.

5.1. **Formula for the Hadamard product.** We prove the general result by following the ideas from the previous section. We are reduced to consider the case of a singularity $\alpha\beta$ without multiplicitym and again we can assume $\alpha = \beta = 1$. We consider the same integration path.

The only difference appears when we shrink loops at $u = 1$ and $u = z_0$. For example, for the first path η_1, when we turn around $u = 1$, we get an extra contribution of

$$\lim_{\epsilon_1 \to 1} \frac{1}{2\pi i} \int_{\epsilon_1} F(u)G(z_0/u)\, \frac{du}{u}$$

where ϵ_1 is a local positive circle loop around $u = 1$, and we take the limit when this loop converges to $u = 1$. We have

$$\lim_{\epsilon_1 \to 1} \frac{1}{2\pi i} \int_{\epsilon_1} F(u)G(z_0/u)\, \frac{du}{u} = \lim_{\epsilon_1 \to 1} \frac{1}{2\pi i} \int_{\epsilon_1} F_0(u)G(z_0/u)\, \frac{du}{u} + \lim_{\epsilon_1 \to 1} \frac{1}{2\pi i} \int_{\epsilon_1} \frac{1}{2\pi i} \log(u-1)\Delta_1 F(u)G(z_0/u)\, \frac{du}{u}$$

and the last limit is zero because of the integrability condition, and more precisely because $\Delta_1 F(u)G(z_0/u)/u$ is holomorphic and $\log(u-1)$ integrable at $u = 1$.

Therefore, for each path η_j we get the following extra residue contributions to the integral (the orientations give the proper signs):

$$\begin{aligned}
\eta_1 &\to \operatorname{Res}_{u=1}\left(\frac{F_0(u)G(z_0/u)}{u}\right) \\
\eta_2 &\to \operatorname{Res}_{u=z_0}\left(\frac{F(u)G_0(z_0/u)}{u}\right) + \operatorname{Res}_{u=z_0}\left(\frac{\Delta_1 F(u)G_0(z_0/u)}{u}\right) \\
\eta_3 &\to -\operatorname{Res}_{u=1}\left(\frac{F_0(u)G(z_0/u)}{u}\right) - \operatorname{Res}_{u=1}\left(\frac{F_0(u)\Delta_1 G(z_0/u)}{u}\right) \\
\eta_4 &\to -\operatorname{Res}_{u=z_0}\left(\frac{F(u)G_0(z_0/u)}{u}\right)
\end{aligned}$$

Adding these four contributions, we have two cancellations, and only two residue terms remain, which give a total residual contribution of

$$R = -\operatorname{Res}_{u=1}\left(\frac{F_0(u)\Delta_1 G(z_0/u)}{u}\right) + \operatorname{Res}_{u=z_0}\left(\frac{\Delta_1 F(u)G_0(z_0/u)}{u}\right) \quad .$$

We can give to this expression a symmetric form using the following elementary Lemma.

Lemma 5.1. *We have*

$$\operatorname{Res}_{u=z_0}\left(\frac{\Delta_1 F(u) G_0(z_0/u)}{u}\right) = -\operatorname{Res}_{u=1}\left(\frac{G_0(u)\Delta_1 F(z_0/u)}{u}\right)$$

Proof. We consider a local loop γ with winding number 1 with respect to z_0 and use the Residue Theorem and the change of variables $v = z_0/u$, $du = -z_0 dv/v^2$, γ' the image of γ that is a local loop with winding number number with respect to 1,

$$\begin{aligned}\operatorname{Res}_{u=z_0}\left(\frac{\Delta_1 F(u) G_0(z_0/u)}{u}\right) &= \frac{1}{2\pi i}\int_\gamma \frac{\Delta_1 F(u) G_0(z_0/u)}{u}\, du \\ &= \frac{1}{2\pi i}\int_{\gamma'} \frac{\Delta_1 F(z_0/v) G_0(v)}{z_0/v}(-z_0)\,\frac{dv}{v^2} \\ &= -\frac{1}{2\pi i}\int_{\gamma'} \frac{\Delta_1 F(z_0/v) G_0(v)}{v}\, dv \\ &= -\operatorname{Res}_{u=1}\left(\frac{G_0(u)\Delta_1 F(z_0/u)}{u}\right)\end{aligned}$$

□

Finally, the residue total contribution is

$$R = -\operatorname{Res}_{u=1}\left(\frac{F_0(u)\Delta_1 G(z_0/u)}{u}\right) - \operatorname{Res}_{u=1}\left(\frac{G_0(u)\Delta_1 F(z_0/u)}{u}\right)$$

In the case of a general singularity $\gamma = \alpha\beta$, this residue contribution is

$$R = -\operatorname{Res}_{u=\alpha}\left(\frac{F_0(u)\Delta_\beta G(z_0/u)}{u}\right) - \operatorname{Res}_{u=\beta}\left(\frac{G_0(u)\Delta_\alpha F(z_0/u)}{u}\right)$$

and this gives the holomorphic monodromy formula (1).

5.2. **Formula for the exponential eñe product.** The proof for the exponential eñe product follows the same lines. The only difference is in the the sign in the convolution formula and the integrating function. Since the monodromy operator commutes with differentiation there is almost no difference in the entire argument. We need to use the following Lemma:

Lemma 5.2. *At a holomorphic singulartiy α we have*

$$(F')_0 = F_0' + \frac{1}{2\pi i}\,\frac{1}{z-\alpha}\,\Delta_\alpha F$$

Proof. We use the commutation of the monodromy operator with the differentiable operator. We differentiate the decomposition

$$F(z) = F_0(z) + \frac{1}{2\pi i}\,\log(z-\alpha)\Delta_\alpha F(z)$$

and obtain

$$F'(z) = \left(F_0'(z) + \frac{1}{2\pi i}\,\frac{1}{z-\alpha}\Delta_\alpha F(z)\right) + \frac{1}{2\pi i}\,\log(z-\alpha)\Delta_\alpha F'(z)$$

and by uniqueness of the monodromy decomposition for F' from Proposition 1.3, we get the result.

□

The same proof as before, starting from the exponential eñe product convolution formula, gives

$$\Delta_\gamma(F\star_e G)(z)=\sum_{\substack{\alpha,\beta\\ \alpha\beta=\gamma}}\mathrm{Res}_{u=\alpha}\left((F')_0(u)\Delta_\beta G(z/u)\right)+\sum_{\substack{\alpha,\beta\\ \alpha\beta=\gamma}}\mathrm{Res}_{u=\beta}\left(G_0(u)\Delta_\alpha F'(z/u)\right)$$
$$+\frac{1}{2\pi i}\sum_{\substack{\alpha,\beta\\ \alpha\beta=\gamma}}\int_\alpha^{z/\beta}\Delta_\alpha F'(u)\Delta_\beta G(z/u)\,du$$

Now, using the Lemma 5.2, we have

$$\begin{aligned}\mathrm{Res}_{u=\alpha}\left((F')_0(u)\Delta_\beta G(z/u)\right)&=\mathrm{Res}_{u=\alpha}\left(F_0'(u)\Delta_\beta G(z/u)\right)+\frac{1}{2\pi i}\,\mathrm{Res}_{u=\alpha}\left(\frac{1}{u-\alpha}\Delta_\alpha F(u)\Delta_\beta G(z/u)\right)\\&=\mathrm{Res}_{u=\alpha}\left(F_0'(u)\Delta_\beta G(z/u)\right)+\frac{1}{2\pi i}\Delta_\alpha F(\alpha)\Delta_\beta G(z/\alpha)\end{aligned}$$

and formula (3) follows.

This commutation property for the derivation is also the reason for the symmetry on F and G of the formula. By integration by parts we have:

Proposition 5.3. *For $\gamma=\alpha\beta$ we have*

$$\Delta_\alpha F(\alpha)\Delta_\beta G(z/\alpha)+\int_\alpha^{z/\beta}\Delta_\alpha F'(u)\Delta_\beta G(z/u)\,du=\Delta_\alpha F(z/\beta)\Delta_\beta G(\beta)+\int_\beta^{z/\alpha}\Delta_\beta G'(u)\Delta_\alpha F(z/u)\,du$$

Proof. We perform an integration by parts, and the change of variables $v=z/u$,

$$\begin{aligned}\int_\alpha^{z/\beta}\Delta_\alpha F'(u)\Delta_\beta G(z/u)\,du&=\left[\Delta_\alpha F(u)\Delta_\beta G(z/u)\right]_\alpha^{z/\beta}-\int_\alpha^{z/\beta}\Delta_\alpha F(u)\Delta_\beta G'(z/u)\left(-\frac{1}{u^2}\right)du\\&=\left[\Delta_\alpha F(u)\Delta_\beta G(z/u)\right]_\alpha^{z/\beta}+\int_\beta^{z/\alpha}\Delta_\alpha F(z/v)\Delta_\beta G'(v)\,dv\end{aligned}$$

□

We could have derived the formula for the monodromy of the exponential eñe product from formula (1) for the Hadamard product by using

$$F\star_e G=-K_0\odot F\odot G\ .$$

It is instructive to derive it. We have already noted that the effect on the monodromy of the Hadamard product with $-K_0$ is just the derivation of the monodromy, hence

$$\Delta_\gamma(F\star_e G)=(\Delta_\gamma F\odot G)'$$

thus, we only need to take the derivate of formula (1). Using the commutation with the monodromy operator and previous Lemma 5.2 we get the result again.

The proof of Corollary 1.10 is straightforward from the eñe monodromy formula.

6. Applications.

6.1. **Application 1: Monodromy of polylogarithms.**

We use formula (1) to compute the classical monodromy of polylogarithms. This is usually done in the literature by using functional equations (as for example in [17]). The polylogarithm $\mathrm{Li}_k(z)$ for $k = 1, 2, \ldots$ can be defined for $|z| < 1$ by the converging series

$$\mathrm{Li}_k(z) = \sum_{n=1}^{+\infty} n^{-k} z^n$$

Therefore Li_1 is given by the logarithm, $\mathrm{Li}_1(z) = -\log(1-z)$ that has the principal branch with a multivalued holomorphic extension to $\mathbb{C} - \{1\}$ with a unique singularity at 1 which is of logarithmic type with a constant monodromy

$$\Delta_1 \mathrm{Li}_1 = -2\pi i \ .$$

Higher polylogarithms can also be defined inductively by integration

$$\mathrm{Li}_{k+1}(z) = \int_0^z \mathrm{Li}_k(u) \, \frac{du}{u}$$

as we readily see by integrating term by term the defining power series in its disk of convergence. Using the integral expression we prove by induction that their principal branch extends holomorphically to a multivalued function on $\mathbb{C} - \{1\}$ with a unique singularity at 1 (non-principal branches can have also singularities at 0 as we see below).

From the power series definition it is clear that higher order polylogarithms can also be defined inductivelly using Hadamard multiplication

$$\mathrm{Li}_{k+1} = \mathrm{Li}_k \odot \mathrm{Li}_1$$

The minimal Hadamard ring structure (for the sum and the Hadamard product) generated by the logarithm is obtained by adjoining higher order polylogarithms, for $k, l \geq 1$,

$$\mathrm{Li}_{k+l} = \mathrm{Li}_k \odot \mathrm{Li}_l$$

As a Corollary of monodromy formula (1) we can compute directly the monodromy at $s = 1$.

Corollary 6.1. *For $k \geq 2$ the only singularities of the analytic continuation of Li_k are located at 0 and 1. For $k \geq 1$ the monodromy at 1 of the principal branch of Li_k is holomorphic at $z = 1$ and, more precisely,*

$$\Delta_1 \mathrm{Li}_k = -\frac{2\pi i}{(k-1)!} (\log z)^{k-1}$$

Proof. For $k = 1$ this is the monodromy of the classical logarithm that is holomorphic (constant) at 1. Assuming by induction the result for $k \geq 1$, the monodromy for Li_k is holomorphic at $z = 1$, and we can use Theorem 1.8 with the formula $\mathrm{Li}_{k+1} = \mathrm{Li}_k \odot \mathrm{Li}_1$, obsrving that the singularity at $\alpha = 1$ is totally holomorphic, so we can use the simpler formula (2), and we get (using the change of variables $v = \log u$)

$$\begin{aligned}
\Delta_1 \mathrm{Li}_{k+1}(z) &= -\frac{1}{2\pi i} \int_1^z -\frac{2\pi i}{(k-1)!} (\log u)^{k-1} (-2\pi i) \, \frac{du}{u} \\
&= -\frac{2\pi i}{(k-1)!} \int_0^{\log z} v^{k-1} \, dv \\
&= -\frac{2\pi i}{k!} (\log z)^k
\end{aligned}$$

□

We can use the formula for the exponential eñe product to cross check this result since it follows from the power series expansion that, for $k, l \geq 1$,

$$\mathrm{Li}_k \star_e \mathrm{Li}_l = -\mathrm{Li}_{k+l-1}$$

and in particular,

$$\begin{aligned}\mathrm{Li}_{k+1} &= -\mathrm{Li}_k \star_e \mathrm{Li}_2 \\ \mathrm{Li}_1 &= -\mathrm{Li}_1 \star_e \mathrm{Li}_1\end{aligned}$$

hence the minimal exponential eñe ring containing Li_1 is just generated by Li_1, but the one containing Li_1 and Li_2 is generated by all the higher polylogarithms as before. Related to this we observe that we need to know the monodromy of Li_2 to start the induction with the exponential eñe product formula. The exponential eñe convolution formula and the Hadamard convolution formula are directly related using $\mathrm{Li}'_{k+1}(z) = \mathrm{Li}_k(z)/z$ since

$$\mathrm{Li}_k \star_e \mathrm{Li}_l = -\frac{1}{2\pi i}\int_\eta \mathrm{Li}'_k(u)\,\mathrm{Li}_l(z/u)\,du = -\frac{1}{2\pi i}\int_\eta \mathrm{Li}_k(u)\,\mathrm{Li}_l(z/u)\,\frac{du}{u} = -\mathrm{Li}_k \odot \mathrm{Li}_l = -\mathrm{Li}_{k+l}$$

Now, using the formula (4) for the eñe monodromy for the totally holomorphic singularity at $\alpha = 1$, and the change of variables $v = \log u/\log z$, we have

$$\begin{aligned}\Delta_1(\mathrm{Li}_k \star_e \mathrm{Li}_l) &= \frac{1}{2\pi i}\Delta_1\,\mathrm{Li}_k(1)\,\mathrm{Li}_l(z) + \frac{1}{2\pi i}\int_1^z \left(\frac{-2\pi i}{(k-1)!}(\log u)^{k-1}\right)' \left(\frac{-2\pi i}{(l-1)!}(\log(z/u))^{l-1}\right) du \\ &= 0 + \frac{2\pi i}{(k-2)!(l-1)!}\int_1^z (\log u)^{k-2}.(\log(z/u))^{l-1} du \\ &= \frac{2\pi i}{(k-2)!(l-1)!}(\log z)^{k+l-2}\int_0^1 v^{k-2}(1-v)^{l-1}\,dv \\ &= \frac{2\pi i}{(k-2)!(l-1)!}(\log z)^{k+l-2} B(k-1,l) \\ &= \frac{2\pi i}{(k-2)!(l-1)!}(\log z)^{k+l-2}\frac{\Gamma(k-1)\Gamma(l)}{\Gamma(k+l-1)} \\ &= \frac{2\pi i}{(k+l-2)!}(\log z)^{k+l-2} \\ &= -\Delta_1(\mathrm{Li}_{k+l-1})\end{aligned}$$

The polylogarithm ring is a good example showing that the eñe ring structure corresponds to the twisted Hadamard structure (see [18] Section 10). We note also that the monodromy formula suggests that the proper normalization of the polylogarithm functions is

$$\mathrm{li}_k(z) = -\frac{1}{2\pi i}\,\mathrm{Li}_k(z)$$

so that

$$\Delta_1\,\mathrm{li}_k = \frac{1}{(k-1)!}(\log z)^{k-1} \ .$$

6.2. **Application 2: Polynomial logarithmic monodromy class.** The simplest case of holomorphic monodromy occurs for a polynomial monodromy. We determine now the minimal Hadamard ring containing polynomial monodromies. As a Corollary of our main formula we have the following Proposition:

Proposition 6.2. *Let F and G be two holomorphic functions with polynomial monodromies, $\Delta_\alpha F, \Delta_\beta G \in \mathbb{C}[z]$. Then we have that*
$$\Delta_\gamma(F \odot G) \in \mathbb{C}[z] \oplus \mathbb{C}[z] \log z \ .$$

Proof. We write
$$\Delta_\alpha F(z) = \sum_n a_n z^n$$
$$\Delta_\beta G(z) = \sum_m b_m z^m$$
then, using the holomorphic monodromy formula, we get
$$\begin{aligned}
\Delta_\gamma(F \odot G)(z) &= -\frac{1}{2\pi i} \sum_{\substack{\alpha,\beta \\ \alpha\beta=\gamma}} \int_\alpha^{z/\beta} \Delta_\alpha F(u) \Delta_\beta G(z/u) \, \frac{du}{u} \\
&= -\frac{1}{2\pi i} \sum_{\substack{\alpha,\beta \\ \alpha\beta=\gamma}} \sum_{n,m} a_n b_m z^m \int_\alpha^{z/\beta} u^{n-m-1} \, du \\
&= -\frac{1}{2\pi i} \sum_{\substack{\alpha,\beta \\ \alpha\beta=\gamma}} \sum_{n,m,n\neq m} a_n b_m z^m \, \frac{z^{n-m-1} - \gamma^{n-m-1}}{\beta^{n-m-1}(n-m)} \\
&\quad - \frac{1}{2\pi i} \sum_{\substack{\alpha,\beta \\ \alpha\beta=\gamma}} \sum_n a_n b_n z^n (\log z - \log \gamma)
\end{aligned}$$
□

We have the following elementary Lemma:

Lemma 6.3. *Let $\alpha \in \mathbb{C}^*$, and $k, l \geq 0$. We have*
$$\int_\alpha^z u^k (\log u)^l \, du \in \mathbb{Q}[\alpha, \log \alpha][z, \log z] \ .$$

Proof. The result is obtained by recurrence on the exponent $l \geq 0$ and integration by parts. □

As a Corollary we obtain the stability of the ring $\mathbb{C}[z, \log z]$ by the holomorphic monodromy formula, and the same computation as for Proposition 6.2 proves the general Theorem:

Theorem 6.4. *Let F and G be two holomorphic functions with monodromies, $\Delta_\alpha F, \Delta_\beta G \in \mathbb{C}[z, \log z]$. Then we have that*
$$\Delta_\gamma(F \odot G) \in \mathbb{C}[z, \log z] \ .$$

This motivates the definition of the *polynomial logarithmic monodromy class.*

Definition 6.5. *The polynomial logarithmic monodromy (or PLM)class is the class of holomorphic functions in a neighborhood of 0 having only singularities with monodromies in the ring $\mathbb{C}[z, \log z]$.*

We have proved:

Proposition 6.6. *The PLM class is closed under the Hadamard product, and it is the minimal Hadamard ring containing the subclass of functions with polynomial monodromies.*

We can be more precise keeping track of the previous computations.

Definition 6.7. *Let $K \subset \mathbb{C}$ be a field such that $2\pi i \in K$. The $PLM(K)$ class is composed by holomorphic functions in a neighborhood of* 0*, having only singularities in the ring $K[z, \log z]$, with singularities $\alpha \in K$.*

Theorem 6.8. *The $PLM(K)$ class is closed under the Hadamard product, and it is the minimal Hadamard ring containing the subclass of functions with polynomial monodromies in $K[z]$ with singularities in K.*

We have the same result for the eñe product.

6.3. **Application 3: Divisor interpretation of the eñe product.** As explained in the introduction, the exponential eñe product linearizes is the exponential form of the eñe product. It is remarkable that the formulas for the eñe product are linearized through the exponential function.

Using the monodromy formulas we can prove directly the divisor interpretation of the eñe product. Thus this gives an alternative definition of the eñe product and its properties. Starting from two meromorphic functions in the plane, holomorphic at 0, normalized such that $f(0) = g(0) = 1$, we can consider their exponential form:

$$f(z) = \exp(F(z)) = 1 + a_1 z + a_2 z^2 + \ldots = 1 + \sum_{n\geq 1} a_n\, z^n$$
$$g(z) = \exp(G(z)) = 1 + b_1 z + b_2 z^2 + \ldots = 1 + \sum_{n\geq 1} b_n\, z^n$$

The functions $F = \log f$ and $G = \log g$ are holomorphic germs at $z = 0$. Their singularities are located at the zeros of f and g respectively. We have constant monodromies,

$$\Delta_\alpha F = 2\pi i\, n_\alpha$$
$$\Delta_\beta G = 2\pi i\, n_\beta$$

where n_α, resp. n_β, is the multiplicity of the zero or pole of f, resp. g (negative multiplicity for poles). We can define the eñe product by the exponential eñe product,

$$f \star g = \exp(F \star_e G)$$

this defines a holomorphic germ near 0, with $f \star g(0) = 1$. We know that the singularities of $f \star g$ are located at the singularities of $F \star_e G$, i.e. at the points $\gamma = \alpha\beta$.

The monodromies are totally holomorphic, hence we can use formula (4) and compute the monodromies of the singularities γ, which gives

$$\Delta_\gamma(F \star_e G) = \frac{1}{2\pi i} \sum_{\substack{\alpha,\beta \\ \alpha\beta=\gamma}} (2\pi i\, n_\alpha)(2\pi i\, n_\beta) = (2\pi i) \sum_{\substack{\alpha,\beta \\ \alpha\beta=\gamma}} n_\alpha n_\beta$$

This proves that γ is a zero or pole for $f \star g$ of multiplicity

$$n_\gamma = \sum_{\substack{\alpha,\beta \\ \alpha\beta=\gamma}} n_\alpha n_\beta$$

hence recovering the definition given in [18].

It is instructive to note, in view of the infinite divisor interpretation of the eñe product given in [19], that isolated essential singularities of f and g do correspond to regular poles of F and G, that are a particular case of regular isolated singularities. If we use the analogue of Borel Theorem for regular isolated singularities, we can see that the divisor eñe product interpretation extends further to singularities of f and g of infinite order, for example of the form

$$f(z) = e^{1-e^{\frac{z}{z-1}}} .$$

These "higher order divisor" extension are left for future work.

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