THE EÑE PRODUCT OVER A COMMUTATIVE RING

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Abstract

We define the eñe product for the multiplicative group of polynomials and formal power series with coefficients on a commutative ring and unitary constant coefficient . This defines a commutative ring structure where multiplication is the additive structure and the eñe product is the multiplicative one. For polynomials over $\mathbb{C}$, the eñe product acts as a multiplicative convolution of their divisor. We study its algebraic properties, its relation to symmetric functions on an infinite number of variables, to tensor products, and Hecke operators. The exponential linearizes also the eñe product. The eñe product extends to rational functions and formal meromorphic functions. We also study the analytic properties over $\mathbb{C}$, and for entire functions. The eñe product respects Hadamard-Weierstrass factorization and is related to the Hadamard product. The eñe product plays a central role in predicting the phenomenon of the "statistics on Riemann zeros" for Riemann zeta function and general Dirichlet L-functions discovered by the author in [6]. It also gives reasons to believe in the Riemann Hypothesis as explained in [7].

Contents

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1. Preliminaries.

We consider a commutative ring $(A, +, .)$ with unit $1 \in A$ and the associated local ring of formal power series $A[[X]]$ with coefficients in A. When series are meant to be convergent we shall use the variable z instead of X.

Proposition 1.1. *Let $\mathcal{A} = 1 + XA[[X]]$. The multiplication of formal power series is an internal operation in $\mathcal{A}$ and $(\mathcal{A}, .)$ is an abelian group with 1 as neutral element.*

Sometimes we denote $\mathcal{A}_A$ to indicate the coefficient ring A.

We recall some basic facts about the logarithmic derivative and the exponential.

Definition 1.2. *The logarithmic derivative $\mathcal{D} : (\mathcal{A}, .) \to (A[[X]], +)$*

$$f \mapsto \mathcal{D}(f) = f'/f$$

is an isomorphism of groups.

For $f, g \in \mathcal{A}$, $a \in A$, we have

$$\begin{aligned}
\mathcal{D}(f.g) &= \mathcal{D}(f) + \mathcal{D}(g) \ , \\
\mathcal{D}(1) &= 0 \ , \\
\mathcal{D}(af) &= \mathcal{D}(f) \ , \\
\mathcal{D}(f(aX)) &= a(\mathcal{D}(f))(aX) \ .
\end{aligned}$$

From now on, in the statements where the exponential appears, we assume that $\mathbb{Q} \subset A$ (we can also work in full generality with $\mathbb{Q} \otimes A$ but some statements become more complicated).

Definition 1.3. *We assume $\mathbb{Q} \subset A$. The exponential map* $\exp : (XA[[X]], +) \to (\mathcal{A}, .)$

$$f \mapsto \exp f = e^f = \sum_{n=0}^{+\infty} \frac{f^n}{n!}$$

is an isomorphism of groups.

Observe that the exponential and the logarithmic derivative factor the usual derivative. The group isomorphism $D : (XA[[X]], +) \to (A[[X]], +)$

$$f \mapsto D(f) = f'$$

factors as

$$D = \mathcal{D} \circ \exp \ .$$

A related natural operator is the exponential logarithmic derivative.

Definition 1.4. *The exponential logarithmic derivative is the group isomorphism $\mathcal{D}_{\exp} : (\mathcal{A}, .) \to (\mathcal{A}, .)$*

$$f \mapsto \mathcal{D}_{\exp}(f) = e^{X\mathcal{D}(f)} \ .$$

2. The eñe ring. Definition and first properties.

We work in this section with an arbitrary commutative ring A. We define the eñe-product on $\mathcal{A}$.

Let $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_m)$ be two sets of variables.

For $p \leq n, m$ we define

$$\Sigma_p^{n\otimes m} = \Sigma_{(i_1,j_1),\ldots,(i_p,j_p)} (X_{i_1} Y_{j_1}) \ldots (X_{i_p} Y_{j_p}) \in \mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$$

where the pairs (i_k, j_k) in the sum are taken to be pairwise distinct. We refer to [5] or [2] for qualitative and quantitative generalizations of the following proposition using the theory of symmetric functions (for the first to an infinite uncountable number of variables and the second for explicit bounds). Here we provide a direct proof (which will be more natural after section 4).

Proposition 2.1. *For $p \leq \min\{n, m\}$, there exists a universal polynomial $Q_p \in \mathbb{Z}[X_1, \ldots, X_p, Y_1, \ldots, Y_p]$ independent of n and m such that*

$$\Sigma_p^{n\otimes m} = Q_p(\Sigma_1^X, \ldots, \Sigma_p^X, \Sigma_1^Y, \ldots, \Sigma_p^Y)$$

where the Σ_k^X and Σ_k^Y are the corresponding symmetric functions in each set of variables.

We have

$$(-1)^p Q_p(X_1, \ldots, X_p, Y_1, \ldots, Y_p) = -pX_pY_p + P_p(X_1, \ldots X_p, Y_1, \ldots, Y_p)$$

where P_p does not contain any monomial X_pY_p, and the weight on the X's and Y's of each monomial of P_p is p.

Proof. Consider the polynomials

$$f(Z) = \prod_{i=1}^{n} (1 - X_i Z) = 1 + \sum_{k=1}^{n} \Sigma_k^X \ Z^k \ ,$$

$$g(Z) = \prod_{j=1}^{m} (1 - Y_j Z) = 1 + \sum_{k=1}^{m} \Sigma_k^Y \ Z^k \ .$$

Now in the same way

$$\prod_{i,j} (1 - X_i Y_j Z) = 1 + \sum_{k=1}^{p} \Sigma_k^{n\otimes m} \ Z^k + \mathcal{O}(Z^{p+1}) \ .$$

Observe that

$$f(Z) = \exp\left(\log(1 + \sum_{k=1}^{n} \Sigma_k^X \ Z^k) \right)$$

$$= \exp\left(\sum_{k=1}^{+\infty} K_k(\Sigma_1^X, \ldots, \Sigma_k^X) Z^k \right)$$

where $K_k(U_1, \ldots, U_k)$ is a polynomial of weight k on the U variables and $K_k(U_1, \ldots, U_k) = U_k + L_k(U_1, \ldots, U_{k-1})$. Also

$$f(Z) = \exp\left(\sum_i \log(1 - X_i Z) \right)$$

$$= \exp\left(-\sum_{k=1}^{n} \frac{1}{k} \left(\sum_i X_i^k \right) Z^k \right)$$

Observe now that

$$\left(\sum_i X_i^k \right) \cdot \left(\sum_j Y_j^k \right) = \sum_{i,j} (X_i Y_j)^k$$

thus

$$\exp\left(-\sum_{k=1}^{+\infty} kK_k(\Sigma_1^X,\ldots,\Sigma_k^X)K_k(\Sigma_1^Y,\ldots,\Sigma_k^Y)Z^k\right) = 1+\sum_{k=1}^{p}\Sigma_k^{n\otimes m}Z^k+\mathcal{O}(Z^{p+1}) \ .$$

But also the expansion on power series on Z gives

$$\exp\left(-\sum_{k=1}^{+\infty} kK_k(\Sigma_1^X,\ldots,\Sigma_k^X)K_k(\Sigma_1^Y,\ldots,\Sigma_k^Y)Z^k\right) =$$
$$= 1+\sum_{k=1}^{p}(-1)^k Q_k(\Sigma_1^X,\ldots,\Sigma_k^X,\Sigma_1^Y,\ldots,\Sigma_k^Y)Z^k+\mathcal{O}(Z^{p+1})$$

where Q_k is a polynomial with the required properties. □

Definition 2.2. *The eñe-product of $f, g \in \mathcal{A}$,*

$$f(X) = 1 + a_1X + a_2X^2 + \ldots$$
$$g(X) = 1 + b_1X + b_2X^2 + \ldots$$

is

$$f \star g(X) = 1 + c_1X + c_2X^2 + \ldots$$

where for $n \geq 1$, c_n is defined by

$$c_n = (-1)^n Q_n(a_1,\ldots,a_n,b_1,\ldots,b_n) \ ,$$

where $Q_n \in \mathbb{Z}[X_1,\ldots,X_n,Y_1,\ldots,Y_n]$ was defined in Proposition 2.1.

The following is inmediate from the definition.

Proposition 2.3. *The eñe-product is an internal operation of $\mathcal{A}$.*

If $A \subset \mathbb{C}$ and (α_i) and (β_j) are the roots of two polynomials f and g then the roots of $f \star g$ are $(\alpha_i\beta_j)_{i,j}$.

Note that the coefficient c_n only depends on the coefficients of order $\leq n$. This operation, contrary to the sum and product, is not pointwise geometric. It is geometric in the roots.

We give some values for the coefficients.

Proposition 2.4. *We have*

$$c_1 = -a_1b_1 \ ,$$
$$c_2 = -2a_2b_2 + a_2b_1^2 + a_1^2b_2 \ ,$$
$$c_3 = -3a_3b_3 + 3a_3b_1b_2 - a_3b_1^3 + 3a_1a_2b_3 - a_1a_2b_1b_2 - a_1^3b_3 \ .$$

Now the main property follows.

Theorem 2.5. ***(Distributivity of the eñe-product)*** *The eñe-product $\star$ is distributive with respect to the multiplication. If $f, g, h \in \mathcal{A}$ then*

$$(f.g) \star h = (f \star h).(g \star h) \ .$$

Proof. The n-th order coefficient of $(f.g) \star h$ (resp. $(f \star h).(g \star h)$) is a polynomial with integer coefficients on the coefficients of order $\leq n$ of f, g and h. Thus, by universality, it is enough to establish the identity when $A = \mathbb{C}$ and when f, g and h are polynomials. Because in such case the polynomials with integer coefficients giving the expressions of order n on both sides will agree on an open set of $\mathbb{C}^{n^3}$ thus are equal (we must choose f, g and h of degree larger than n).

If (α_i), (β_j) and (γ_k) are respectively the zeros of f, g and h counted with multiplicity then the zeros counted with multiplicity of $(f.g) \star h$ and $(f \star h).(g \star h)$ are $(\alpha_i \gamma_k)_{i,k} \cup (\beta_j \gamma_l)_{j,l}$. Thus these two polynomial functions have the same zeros, so they must be equal, and the result follows. □

Theorem 2.6. *The set $(\mathcal{A}, ., \star)$ is a commutative ring with zero $1 \in \mathcal{A}$ and unity $1 - X \in \mathcal{A}$. More precisely, we have*

- $(\mathcal{A}, .)$ *is an abelian group.*
- *(Distributivity) For $f, g, h \in \mathcal{A}$, $(f.g) \star h = (f \star h).(g \star h)$.*
- *(Associativity) For $f, g, h \in \mathcal{A}$, $(f \star g) \star h = f \star (g \star h)$.*
- *(Commutativity) For $f, g \in \mathcal{A}$, $f \star g = g \star f$.*
- *(Unit) For $f \in \mathcal{A}$, $f \star (1 - X) = (1 - X) \star f$.*

Proof. Before we proved the distributive property. The other properties are proved in the same way. □

So we have the usual properties of ring operation:

Corollary 2.7. *For $f, g \in \mathcal{A}$ and $n \geq 1$ we have*

- $f \star 1 = 1 \star f = 1$.
- $f \star (1/g) = (1/f) \star g = \frac{1}{f \star g}$.
- $f \star g^n = f^n \star g = (f \star g)^n$.
- $\frac{1}{f} \star \frac{1}{g} = f \star g$.
- *Newton binomial formula.*

$$(f.g)^{\star n} = \prod_{k=0}^{n} \left(f^{\star(n-k)} \star g^{\star k} \right)^{\binom{n}{k}} .$$

We have also some additional properties that are proved again as in theorem 2.5.

Theorem 2.8. *We have*

- (1) *If $f, g \in \mathcal{A}$ and $a \in A$ we have*
$$f(aX) \star g(X) = f(X) \star g(aX) = (f \star g)(aX) .$$
In particular,
$$(1 - aX) \star f(X) = f(aX) .$$
- (2) *For $f, g \in \mathcal{A}$ and $k \geq 1$ positive integer,*
$$f(X^k) \star g(X^k) = \left((f \star g)(X^k)\right)^k .$$
- (3) *For $f, g \in \mathcal{A}$ and $k, l \geq 1$ positive integers with $k \wedge l = 1$,*
$$f(X^k) \star g(X^l) = (f \star g)(X^{kl}) .$$

Proof. The proof of (1) is clear. For the proof of (2), observe that if the roots of f (resp. g) are the (α_i) (resp. (β_j)), then the roots of $f(X^k)$ (resp. $g(X^k)$, $f \star g(X^k)$, $f(X^k) \star g(X^k)$) are the $(\epsilon\alpha_i^{1/k})$ (resp. $(\epsilon\beta_j^{1/k})$, $(\epsilon\alpha_i^{1/k}\beta_j^{1/k})$, $(\epsilon\epsilon'\alpha_i^{1/k}\beta_j^{1/k})$) where ϵ (and ϵ') runs over the group $\mathbb{U}_k$ of k-roots of 1. Now, the morphism $\mathbb{U}_k^2 \to \mathbb{U}_k$, $(\epsilon, \epsilon') \mapsto \epsilon\epsilon'$ is k-to-1 and the result follows.

The proof of (3) is similar observing that the morphism $\mathbb{U}_k \times \mathbb{U}_l \to \mathbb{U}_{kl}$, $(\epsilon, \epsilon') \mapsto \epsilon\epsilon'$, is an isomorphism when $k \wedge l = 1$. □

Theorem 2.9. *We assume that A is a subring of $\mathbb{C}$. If $f, g \in \mathcal{A}$ are polynomials or entire functions of order < 1 with respective zeros (α_i) and (β_j), we have*

$$f \star g(z) = \prod_{i,j} \left(1 - \frac{z}{\alpha_i\beta_j}\right) = \prod_j f\left(\frac{z}{\beta_j}\right) = \prod_i g\left(\frac{z}{\alpha_i}\right) \ .$$

This last result extends to arbitrary entrie functions for each product that is converging.

3. Main formula and first applications.

The following fundamental relation relates the exponential, the logarithmic derivative and the eñe-product.

Theorem 3.1. ***(Main Formula).*** *For $f, g \in \mathcal{A}$ we have*

$$\exp\left(X\mathcal{D}(f \star g)\right) = g \star \exp(X\mathcal{D}(f)) = f \star \exp(X\mathcal{D}(g)) \ .$$

Or, in terms of the exponential logarithmic derivative,

$$\mathcal{D}_{\exp}(f \star g) = f \star \mathcal{D}_{\exp}(g) = \mathcal{D}_{\exp}(f) \star g \ .$$

Proof. We observe again that it is enough to prove the result for f and g polynomials with complex coefficients.

Let us consider f and g polynomials with respective sets of zeros (α_i) and (β_j). Observe that

$$(f \star g)(z) = \prod_{i,j} \left(1 - \frac{z}{\alpha_i\beta_j}\right) = \prod_j f\left(\frac{z}{\beta_j}\right) \ ,$$

thus

$$\mathcal{D}(f \star g)(z) = \sum_j \frac{1}{\beta_j}(\mathcal{D}f)(z/\beta_j) \ ,$$

so

$$z\mathcal{D}(f \star g)(z) = \sum_j \frac{z}{\beta_j}(\mathcal{D}f)(z/\beta_j) \ ,$$

and using theorem 2.9

$$e^{z\mathcal{D}(f\star g)} = \prod_j e^{\frac{z}{\beta_j}(\mathcal{D}f)(z/\beta_j)} = g \star e^{z\mathcal{D}(f)(z)} \ .$$

□

Corollary 3.2. *Let $f \in \mathcal{A}$, $f(X) = 1 + f_1X + \ldots$, and $a \in A$. We have*

$$f \star e^{aX} = e^{-af_1X} \ .$$

Proof. Put $g(X) = e^{aX}$ in the Main Formula. Observe that

$$\mathcal{D}(e^{aX}) = a \ .$$

We get

$$e^{X\mathcal{D}(f \star e^{aX})} = f \star e^{X\mathcal{D}(e^{aX})} = f \star e^{aX} \ .$$

Thus $F(X) = f \star e^{aX}$ satisfies the differential equation

$$F = e^{X\mathcal{D}(F)} = e^{XF'/F} \ .$$

We define $G = \log F$ then $G' = F'/F$ and G satisfies the differential equation

$$G' = \frac{1}{X} G \ .$$

The only formal solutions are $G(X) = a_0 X$ for some constant $a_0 \in A$. So finally

$$F(X) = f \star e^{aX} = e^{a_0 X} \ .$$

To determine a_0, using the formula for c_1, we observe that

$$\begin{aligned} f \star e^{aX} &= (1 + f_1 X + \ldots) \star (1 + aX + \ldots) \\ &= 1 - af_1 X + \ldots \end{aligned}$$

Thus $a_0 = -f_1 a$. □

More generally we have the following result.

Corollary 3.3. *Let $f \in \mathcal{A}$, $f(X) = 1 + f_1 X + \ldots$, $a \in A$, and $n \geq 1$ positive integer. We have*

$$f \star e^{aX^n} = e^{a\tilde{Q}_n(f_1,\ldots,f_n)X} \ ,$$

where

$$\tilde{Q}_n(f_1, \ldots, f_n) = (-1)^n Q_n(f_1, \ldots, f_n, 0, \ldots, 0, a)/a = -nf_n + P_n(f_1, \ldots f_{n-1})$$

is a polynomial vanishing when $f_1 = f_2 = \ldots = f_n = 0$.

Proof. As before, using the main formula we get

$$\begin{aligned} e^{X\mathcal{D}(f \star e^{aX^n})} &= f \star e^{X\mathcal{D}(e^{aX^n})} \\ &= f \star e^{naX^n} \\ &= \left(f \star e^{aX^n} \right)^n \end{aligned}$$

Then $F(X) = f \star e^{aX^n}$ satisfies the differential equation

$$F^n = e^{XF'/F}$$

Thus $G = \log F$ satisfies

$$nG = XG'$$

which has only formal solutions $G(X) = a_0 X^n$, $a_0 \in A$. To determine the constant a_0 we write the first term of the expansion

$$\begin{aligned} f \star e^{aX^n} &= (1 + f_1 X + \ldots) \star (1 + aX^n + \ldots) \\ &= 1 + (-1)^n Q_n(f_1, \ldots, f_n, 0, \ldots, 0, a) X^n + \ldots \end{aligned}$$

Thus

$$a_0 = (-1)^n Q_n(f_1, \ldots, f_n, 0, \ldots, 0, a) = (-nf_n + \ldots)a$$

where the quantity between brackets is independent of a and has monomials of weight n on the coefficients (f_i) (see proposition 2.1). □

Corollary 3.4. *For $n, m \geq 1$ positive integers, and $a, b \in A$, we have if $n \neq m$,*

$$e^{aX^n} \star e^{bX^m} = 1$$

and for $n = m$,

$$e^{aX^n} \star e^{bX^n} = e^{-nabX^n} \ .$$

4. Exponential form and applications.

We can summarize the previous discussion with the following key result. It shows that the eñe-product operator $\star$ has a very simple expression in exponential form, or, in other words, we have the remarkable property that the exponential linearizes the eñe product.

Theorem 4.1. ***(Exponential form).*** *Let $f, g \in \mathcal{A}$. Using the isomorphism given by the exponential map, we can write*

$$f = e^F = e^{F_1X + F_2X^2 + F_3X^3 + \ldots}$$
$$g = e^G = e^{G_1X + G_2X^2 + G_3X^3 + \ldots}$$

where $F, G \in A[[X]]$.

We have

$$\begin{aligned} f \star g &= \exp\left(F_1X + F_2X^2 + F_3X^3 + \ldots\right) \star \exp\left(G_1X + G_2X^2 + G_3X^3 + \ldots\right) \\ &= \exp\left(-F_1G_1X - 2F_2G_2X^2 - 3F_3G_3X^3 + \ldots\right) \ . \end{aligned}$$

We denote by $\star_e$ the exponential form of the eñe-product

$$F \star_e G = -F_1G_1X - 2F_2G_2X^2 - 3F_3G_3X^3 + \ldots \ .$$

Proof. We simply use the distributivity of $\star$ and the previous corollary:

$$\begin{aligned} f \star g &= \exp\left(\sum_{i=1}^{+\infty} F_i X^i\right) \star \exp\left(\sum_{j=1}^{+\infty} G_j X^j\right) \\ &= \left(\prod_{i=1}^{+\infty} \exp(F_i X^i)\right) \star \left(\prod_{j=1}^{+\infty} \exp(G_j X^j)\right) \\ &= \prod_{i,j=1}^{+\infty} \exp(F_i X^i) \star \exp(G_j X^j) \\ &= \prod_{i=1}^{+\infty} \exp(-iF_iG_iX^i) \\ &= \exp\left(-\sum_{i=1}^{+\infty} iF_iG_iX^i\right) \end{aligned}$$

□

Using this formula we can now determine exactly which elements of the ring $(\mathcal{A}, ., \star)$ are divisors of zero (the zero is the constant series 1).

Theorem 4.2. *The divisors of zero in the eñe-ring $(\mathcal{A}, ., \star)$ are exactly those $f \in \mathcal{A}$ such that if we write*

$$f = e^F = \exp\left(\sum_{i=1}^{+\infty} F_i X^i\right)$$

there is some coefficient F_i that is 0 or a divisor of 0 in A. Thus, if A is an integral domain, only those $f \in \mathcal{A}$ for which some $F_i = 0$ are divisors of 0.

The elements $f \in \mathcal{A}$ that are not divisors of zero are eñe-invertible, i.e. are units of the ring $\mathcal{A}$, if and only if each one of its exponential coefficients has an inverse, the inverse being

$$g = e^G = \exp\left(\sum_{i=1}^{+\infty} G_i X^i\right)$$

with

$$G_i = \frac{1}{i^2} F_i^{-1} \ .$$

Proof. The eñe-identity $1 - X$ has the exponential form

$$1 - X = \exp(\log(1-X)) = \exp\left(-\sum_{i=1}^{+\infty} \frac{1}{i} X^i\right) \ .$$

and the result follows. □

Remark 4.3. Notice that when $A \subset \mathbb{C}$, if f has infinite radius of convergence (i.e. it is an entire function) and is eñe-invertible, then its eñe-inverse has zero radius of convergence.

5. Some eñe products.

The next result shows that the eñe-product of rational functions is a rational function.

Theorem 5.1. *The eñe-product leaves invariant the the multiplicative group $(1 + XA[X])/(1 + XA[X]) \subset A(X)$ of rational functions quotient of polynomials in $1+XA[X]$. Thus $(1+XA[X])/(1+XA[X]) \subset \mathcal{A}$ is a sub-ring of the eñe-ring. More precisely, et $R_1(X), R_2(X) \in A(X)$ with*

$$R_1(X) = \frac{P_1(X)}{Q_1(X)}$$

$$R_2(X) = \frac{P_2(X)}{Q_2(X)}$$

with $P_1(X), P_2(X), Q_1(X), Q_2(X) \in 1 + XA[X]$. Then

$$R_1(X) \star R_2(X) = \frac{(P_1(X) \star P_2(X)) \, . \, (Q_1(X) \star Q_2(X))}{(P_1(X) \star Q_2(X)) \, . \, (Q_1(X) \star P_2(X))} \ .$$

Thus

$$\deg(R_1(X) \star R_2(X)) = \deg R_1(X) . \deg R_2(X) \ .$$

We observe that when $A \subset \mathbb{C}$, the zeros of $R_1(z) \star R_2(z)$ are the products of zeros of R_1 and R_2 or the product of poles of R_1 and R_2. Also the poles of $R_1(z) \star R_2(z)$ are the products of a pole and a zero of R_1 and R_2. In short we can write

$$\begin{aligned} \text{zero} \star \text{zero} &= \text{zero} \\ \text{pole} \star \text{pole} &= \text{zero} \\ \text{zero} \star \text{pole} &= \text{pole} \\ \text{pole} \star \text{zero} &= \text{pole} \end{aligned}$$

Notice that this information, with multipliciities and the normalization of 1 at 0, is enough to uniquely determine $R_1 \star R_2$.

Proof. It is just simple distributivity of the eñe-product. For the degrees we recall that if $R(X) \in A(X)$ with

$$R(X) = \frac{P(X)}{Q(X)}$$

with $P(X), Q(X) \in A[X]$ then by definition

$$\deg R(X) = \max(\deg P(X), \deg Q(X)) \ .$$

Thus the formula for the degree follows from the formula for the eñe-multiplication. □

The previous observations do extend to meromorphic functions on $\mathbb{C}$ quotient of two entire functions, when $A \subset \mathbb{C}$.

Theorem 5.2. *We have the same formula as before for the eñe-product.*

If $A \subset \mathbb{C}$ and $f_1, f_2 \in \mathcal{A}$ are meromorphic functions, quotient of entire functions of order < 1 with coefficients in A, then $f_1 \star f_2$ is a meromorphic function quotient of entire functions of order < 1 given by the above formula, and whose zeros are the products of zeros of f_1 and f_2, or the product of poles of f_1 and f_2, and whose poles are the products of a zero of f_1 (resp. f_2) and a pole of f_2 (resp. f_1).

As we prove in section 9 this result extends to arbitrary entire functions.

The eñe-product by $\exp(-X/(1-X))$ has an interesting property.

Theorem 5.3. ***(Convolution formula).*** *For $f \in \mathcal{A}$*

$$e^{-\frac{X}{1-X}} \star f = e^{X\mathcal{D}(f)} = \mathcal{D}_{exp}(f) \ .$$

In particular, if f is an entire function of order < 1 with zeros (α_i) (in particular when f is a polynomial), then the eñe-multiplication by the function $\exp(-z/(1-z))$ *creates a funtion with essential isolated singularities at the α_i's :*

$$e^{-\frac{z}{1-z}} \star f = \exp\left(\sum_i \frac{z}{z-\alpha_i}\right) \ .$$

Proof. This results from the main formula. We have

$$\begin{aligned} e^{X\mathcal{D}(f)} &= e^{X\mathcal{D}(f\star(1-X))} \\ &= e^{X\mathcal{D}(1-X)} \star f \\ &= e^{-\frac{X}{1-X}} \star f \end{aligned}$$

□

In the next theorem we have a list of computations of various eñe-products.

Theorem 5.4. ***(Some computations).*** *We have*

- *For* $a, b \in A$,
$$\frac{1}{1-aX} \star \frac{1}{1-bX} = (1-aX) \star (1-bX) = 1 - abX \ .$$
- *Let* $P \in A[X]$ *and* $f \in \mathcal{A}$ *then*
$$e^{XP(X)} \star f = e^{XQ(X)} \ ,$$
where $Q(X) \in A[X]$ *is a polynomial with* $\deg Q \leq \deg P$.
- *For* $N \geq 1$ *positive integer, let* $E_N(X)$ *denote the Weierstrass factor*
$$\begin{aligned} E_N(X) &= (1-X)\exp\left(X + \frac{X^2}{2} + \ldots + \frac{X^N}{N}\right) \\ &= \exp\left(-\frac{X^{N+1}}{N+1} - \frac{X^{N+2}}{N+2} - \ldots\right) \end{aligned}$$
For $f \in \mathcal{A}$ *we have*
$$E_N \star f = f.T_N^e(1/f) \ ,$$
where T_N^e *is the exponential* N*-truncation operator, i.e.*
$$T_N^e\left(\exp\left(\sum_{i=1}^{+\infty} F_i X^i\right)\right) = \exp\left(T_N\left(\sum_{i=1}^{+\infty} F_i X^i\right)\right) = \exp\left(\sum_{i=1}^{N} F_i X^i\right) \ .$$
- *For* $N, M \geq 1$ *positive integers, we have*
$$E_N \star E_M = E_{\max(N,M)} \ .$$
- *For* $N \geq 1$ *we define*
$$I_N(X) = 1 - X^N \ .$$
For $f(X) \in \mathcal{A}$, $f(X) = \exp(\sum_{i=1}^{+\infty} F_i X^i)$, *we have*
$$I_N \star f(X) = \exp\left(\sum_{k=1}^{+\infty} F_{Nk} X^{Nk}\right) \ .$$
- *For* $N, M \geq 1$,
$$I_N \star I_M = I_{\text{l.c.m.}(N,M)} \ .$$
- *For* $a \in A$ *we define*
$$(1+X)^a = \sum_{n=0}^{+\infty} \frac{a(a-1)\ldots(a-n+1)}{n!} X^n \ .$$
For $f, g \in \mathcal{A}$ *and* $a \in A$,
$$f(X)^a \star g(X) = f(X) \star g(X)^a = (f(X) \star g(X))^a \ .$$

- *Action of the Artin-Hasse exponential. We recall that*

$$\exp(X) = \prod_{n=1}^{+\infty} (1-X^n)^{\mu(n)/n}$$

where μ is the Möebius function. For a prime p, the Artin-Hasse exponential is

$$\begin{aligned}\exp_p(X) &= \prod_{n=1;n\neq kp}^{+\infty} (1-X^n)^{\mu(n)/n} \\ &= \exp\left(X + \frac{X^p}{p} + \frac{X^{p^2}}{p^2} + \ldots\right)\end{aligned}$$

If $f \in \mathcal{A}$, $f(X) = \exp\left(\sum_{i=1}^{+\infty} F_i X^i\right)$, then

$$\exp_p(X) \star f(X) = \exp\left(-\sum_{k=1}^{+\infty} F_{p^k} X^{p^k}\right) \ .$$

Proof. We prove one of the formulas and left the others as exercices. Observe that

$$\begin{aligned}&\exp\left(X + \frac{X^2}{2} + \ldots + \frac{X^N}{N}\right) \star \exp\left(X + \frac{X^2}{2} + \ldots + \frac{X^M}{M}\right) \\ &= \exp\left(-X - \frac{X^2}{2} - \ldots - \frac{X^{\min(N,M)}}{\min(N,M)}\right)\end{aligned}$$

Then we use the distributivity

$$\begin{aligned}E_N \star E_M &= (1-X).\exp\left(X + \frac{X^2}{2} + \ldots + \frac{X^M}{M}\right).\exp\left(X + \frac{X^2}{2} + \ldots + \frac{X^M}{M}\right) \\ &\quad /\exp\left(X + \frac{X^2}{2} + \ldots + \frac{X^{\min(N,M)}}{\min(N,M)}\right) \\ &= (1-X).\exp\left(X + \frac{X^2}{2} + \ldots + \frac{X^{\max(N,M)}}{\max(N,M)}\right) \\ &= E_{\max(N,M)}\end{aligned}$$

□

In view of the action of the action by eñe-product of $I_n(X) = 1 - X^n$, it is natural to define, in parallel with the theory of modular forms, the following Hecke operators.

Definition 5.5. ***(Hecke operators).*** *For $n \geq 1$ we define,*

$$T(n) : \mathcal{A} \to \mathcal{A}$$

by

$$T(n)(f)(X) = (I_n \star f)(X^{1/n}) \ ,$$

that is, if $f(X) = \exp\left(\sum_{i=1}^{+\infty} F_i X^i\right)$,

$$T(n)(f) = \exp\left(\sum_{k=1}^{+\infty} F_{nk} X^k\right) \ .$$

Note that $T(n)$ can be defined in the same way on $1 + A[[X^{1/\lambda}]]$ for $\lambda \in \mathbb{C}^*$.

We can also define the "dilatation operators" by

Definition 5.6. ***(Dilatation operators).*** *For $\lambda \in \mathbb{C}^*$ we define,*

$$R_\lambda : \mathcal{A} \to 1 + A[[X^{1/\lambda}]]$$

by

$$R_\lambda(f)(X) = f(X^{1/\lambda}) \ .$$

Note that R_λ is defined in the same way on $1 + A[[X^{1/\mu}]]$ for any $\mu \in \mathbb{C}^*$.

We observe that $T(n)$ factors.

Theorem 5.7. *We have*

$$T(n)(f) = R_n(I_n \star f) \ .$$

Note that extending properly the eñe-product to $1 + A[[X^{1/\lambda}]]$ we have commutation of R_λ and eñe-multiplication by I_n, thus we can also write

$$T(n)(f) = I_n \star (R_{1/n}(f)) \ .$$

Then we have, similar (and simpler) formulas than in the theory of modular forms (see [8] p.159 for example),

Theorem 5.8. *We have*

- *For $\lambda, \mu \in \mathbb{C}^*$,*
$$R_\lambda R_\mu = R_{\lambda\mu}$$
- *For $n \geq 1$ and $\lambda \in \mathbb{C}^*$,*
$$R_\lambda T(n) = T(n) R_\lambda \ .$$
- *For $n, m \geq 1$, $n \wedge m = 1$,*
$$T(n) \ T(m) = T(nm) \ .$$

6. Eñe ring structure for a field A.

The eñe-product of polynomials P and Q of respective degrees d_1 and d_2 is the polynomial $P \star Q$ of degree $d_1 d_2$ (because the roots of $P \star Q$ counted with multipicity are the products of a root of P with a root of Q). Thus the eñe-product does not respect the graduation by degrees. But in exponential form it does. More precisely, for $N \geq 1$, let $\mathcal{A}_N \subset \mathcal{A}$ be the subset of $\mathcal{A}$

$$\mathcal{A}_N = \{f \in \mathcal{A}; \exists P \in XA[X], \ \ \deg(P) \leq N, \ \ f = \exp(P)\} \ .$$

Observe that the exponential truncation T_N^e defines a ring homomorphism $T_N^e : \mathcal{A} \to \mathcal{A}_N$

$$f \mapsto T_N^e(f)$$

We denote $\mathcal{I}_N$ its kernel, thus

$$\mathcal{A}_N \approx \mathcal{A}/\mathcal{I}_N \ .$$

Obviously the inclusions

$$\mathcal{A}_N \hookrightarrow \mathcal{A}_{N+1}$$

are ring homomorphisms.

Theorem 6.1. *The subset $\mathcal{A}_N \subset \mathcal{A}$ is a subring of the eñe-ring $(\mathcal{A}, ., \star)$ with unit $(1-X)/E_N(X)$ and zero $E_N(X)$. Morevoer, $\mathcal{A}$ is the direct limit of the $\mathcal{A}_N$'s*

$$\mathcal{A} = \varinjlim \mathcal{A}_N \ .$$

Proof. The eñe-product preserves each $\mathcal{A}_N$ as is immediate from its simple exponential form. Also $(1-X)/E_N(X)$ is the unit since

$$(1-X)/E_N(X) = \exp\left(-X - \frac{X^2}{2} - \ldots - \frac{X^N}{N}\right) \ .$$

Also $E_N(X)$ is the zero since

$$E_N(X) = \exp(\mathcal{O}(X^{N+1})) \ .$$

The direct limit is clear. □

When A is a field the following theorem gives a description of the ideals of the eñe-ring.

Theorem 6.2. *We assume that A is a field. The maximal ideals of $\mathcal{A}$ are*

$$\mathcal{J}_n = \{f \in \mathcal{A}; f(X) = \exp(F_1 X + F_2 X^2 + \ldots) \ \text{ such that } F_n = 0\} \ .$$

In particular, $\mathcal{A}_N$ is a quasi-local ring, i.e. it has a finite number of maximal ideals.

Proof. Given an ideal $\mathcal{J} \subset \mathcal{A}$, we consider the set of integers

$$\mathbb{N}_{\mathcal{J}} = \{n \geq 1; \exists f(X) = \exp\left(\sum_{i\geq 1} F_i X^i\right) \in \mathcal{J} \ , \quad \text{with} \quad F_n \neq 0\}$$

If $\mathbb{N}_{\mathcal{J}} = \emptyset$ then $\mathcal{J} = \{1\}$. Otherwise we have

$$\mathcal{J} \subset \bigcap_{n \notin \mathbb{N}_{\mathcal{J}}} \mathcal{J}_n$$

□

7. Eñe-product and tensor product.

We have the well known formal relation for $M \in M_n(A)$,

$$\det(I - MX) = \exp\left(-\sum_{k=1}^{+\infty} \text{Tr}(M^k) \frac{X^k}{k}\right) \ .$$

We recall that given $M, N \in M_n(A)$ the tensor product $M \otimes N \in M_{n^2}(A)$ is defined by

$$(M \otimes N)(x \otimes y) = (Mx) \otimes (Ny) \ .$$

In terms of the coefficients of the matrices

$$(M \otimes N)_{(j,k)(i,l)} = M_{ij} N_{kl} \ .$$

Thus in particular

$$\text{Tr}(M \otimes N) = \text{Tr}(M).\text{Tr}(N) \ .$$

From these observations and the exponential form of the eñe-product we get

Theorem 7.1. *For $M, N \in M_n(A)$ we have*

$$\det(I - MX) \star \det(I - NX) = \det(I - (M \otimes N)X) \ .$$

This last result provides a linear algebra procedure to compute the eñe-product. Notice that if $P(X) = 1 + a_1 X + a_2 X^2 + \ldots + a_d X^d$ we have

$$P(X) = \det \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & (-1)^{d-1} a_d X \\ -X & 1 & 0 & \cdots & 0 & (-1)^{d-2} a_{d-1} X \\ 0 & -X & 1 & \cdots & 0 & (-1)^{d-3} a_{d-2} X \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & \ddots & 1 & -a_2 X \\ 0 & \cdots & \cdots & 0 & -X & 1 + a_1 X \end{bmatrix} = \det(I - M_P X)$$

where

$$M_P = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & (-1)^d a_d \\ 1 & 0 & 0 & \cdots & 0 & (-1)^{d-1} a_{d-1} \\ 0 & 1 & 0 & \cdots & 0 & (-1)^{d-2} a_{d-2} \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & \ddots & 0 & a_2 \\ 0 & \cdots & \cdots & 0 & 1 & -a_1 \end{bmatrix}$$

Thus we get

Theorem 7.2. *We have*

$$P(X) \star Q(X) = \det(I - (M_P \otimes M_Q) X) \ .$$

Notice that the extension of the eñe-product to formal power series indicates that theorem 7.2 remains valid for infinite matrices. Also theorem 7.1 makes sense for infinite matrices once the tensor product and the determinant are properly defined (one can also define the infinite determinant the other way around).

8. Analytic properties of the eñe product.

The eñe-product satisfies remarkable analytic properties. We assume in this section that $A \subset \mathbb{C}$ and we study the convergence properties of series. Recall Hadamard formula for the radius of convergence of $f \in \mathcal{A}$, $f(z) = 1 + \sum_{i=1}^{+\infty} f_i z^i$,

$$\frac{1}{R(f)} = \limsup_{i \to +\infty} |f_i|^{1/i} \ .$$

It is convenient to introduce the eñe-radius of convergence of f as

$$\tilde{R}(f) = \min_i (|\alpha_i|, R(f))$$

where (α_i) are the zeros of f. Since $f(0) = 1$ we have

$$R(f) \geq \tilde{R}(f) > 0 \ .$$

We observe that if we write $f = \exp(F)$ with $F = \log f$, then

$$R(F) = \tilde{R}(f) \ .$$

The first basic result is that the eñe-product of series with positive radius of convergence is a series with positive radius of convergence :

Theorem 8.1. *Let $f, g \in \mathcal{A}$. We have*

$$R(f \star g) \geq \tilde{R}(f).\tilde{R}(g) \ .$$

In particular, the eñe-product of two series with positive radius of convergence has positive radius of convergence.

Remark 8.2. We will improve this result and show that indeed

$$\tilde{R}(f \star g) \geq \tilde{R}(f).\tilde{R}(g) \ .$$

Proof. We write f and g in exponential form

$$f(z) = \exp(F(z)) = \exp\left(\sum_{i=1}^{+\infty} F_i z^i\right)$$

$$g(z) = \exp(G(z)) = \exp\left(\sum_{i=1}^{+\infty} G_i z^i\right)$$

Using Hadamard formula we get

$$\begin{aligned}\frac{1}{R(F \star_e G)} &= \limsup_{i\to+\infty} (i|F_i||G_i|)^{1/i} \\ &= \limsup_{i\to+\infty} (|F_i||G_i|)^{1/i} \\ &\leq \left(\limsup_{i\to+\infty} |F_i|^{1/i}\right).\left(\limsup_{i\to+\infty} |G_i|^{1/i}\right) \\ &= \frac{1}{R(F)}.\frac{1}{R(G)}\end{aligned}$$

Thus

$$R(f \star g) \geq R(F \star_e G) \geq R(F).R(G) = \tilde{R}(f).\tilde{R}(g) \ .$$

□

We state next a continuity property:

Theorem 8.3. *We consider the space $\tilde{\mathcal{A}}_{R_0}$ of power series $f \in \mathcal{A}$ with $\tilde{R}(f) \geq R_0$, i.e convergent and with no zeros on the disk $\mathbb{D}_{R_0}$ of center 0 and radius $R_0 > 0$. We consider also the space $\mathcal{A}_{R_0}$ of power series $f \in \mathcal{A}$ with $R(f) \geq R_0$. We endow this spaces with the topology of uniform convergence on compact subsets of $\mathbb{D}_R$. The eñe-product $\star : \tilde{\mathcal{A}}_{R_1} \times \tilde{\mathcal{A}}_{R_2} \to \mathcal{A}_{R_1 R_2}$*

$$(f, g) \mapsto f \star g$$

is continuous.

Proof. Any function $f \in \tilde{\mathcal{A}}_{R_0}$ can be written $f = e^F$ with $F = \log f$ having radius of convergence at least $R_0 > 0$. The linear expression of the eñe-product on the coefficients of F shows the continuity. □

We can now improve theorem 8.1.

Corollary 8.4. *Let $f, g \in \mathcal{A}$. We have*

$$\tilde{R}(f \star g) \geq \tilde{R}(f).\tilde{R}(g) \ .$$

Proof. We only need to show that any zero ξ of $f \star g$ with $|\xi| < R(f \star g)$ satisfies $|\xi| \geq \tilde{R}(f)\tilde{R}(g)$. If $|\xi| \geq R(f)R(g) \geq \tilde{R}(f)\tilde{R}(g)$ we are done. We assume then $|\xi| < R(f)R(g)$. By the previous theorem we have that

$$T_N(f) \star T_N(g) \to f \star g$$

when $N \to +\infty$ uniformly on compact sets in $\mathbb{D}_{R(f)R(g)}$. Thus, in particular, we have uniform convergence in a compact neighborhood of ξ. Then by Hurwitz theorem ξ must be the limit of zeros of $T_N(f) \star T_N(g)$. Any such zero is of the form $\alpha_N \beta_N$ where α_N (resp. β_N) is a zero of the polynomial $T_N(f)$ (resp. $T_N(g)$). Since $\alpha_N \beta_N \to \xi$ we must have that the sequences (α_N) and (β_N) are bounded (if $\alpha_{N_k} \to \infty$ then $\beta_{N_k} \to 0$ which is impossible). We can extract converging subsequences $\alpha_{N_k} \to \alpha$ and $\beta_{N_k} \to \beta$. Finally we observe that $|\alpha| \geq \tilde{R}(f)$ and $|\beta| \geq \tilde{R}(g)$. Because if $|\alpha| < R(f)$ then since $T_N(f) \to f$ in $\mathbb{D}_{R(f)}$ then α would be a zero of f thus $|\alpha| \geq \tilde{R}(f)$. The same argument applies to β. We conclude

$$|\xi| = |\alpha||\beta| \geq \tilde{R}(f)\tilde{R}(g) \ ,$$

as we wanted to show. □

Remark 8.5. We show in section 10 as an application of Hadamard multiplication theorem that we do have the equality

$$\tilde{R}(f \star g) = \tilde{R}(f).\tilde{R}(g) \ .$$

9. Eñe product and entire functions.

It is not difficult to see, from the interpretation involving the zeros, that the eñe-product does extend from polynomials to entire functions of order < 1 and leaves this space invariant. The next result shows that we have even better, Weierstrass factors cause no trouble and the eñe-product extends to functions of finite order.

Theorem 9.1. *We consider here $A = \mathbb{C}$, and $0 \leq \lambda < +\infty$. We define the space $\mathcal{E}_\lambda \subset \mathcal{A}$ of entire functions of order $< \lambda$ with constant coefficient* 1*.*

The eñe-product is an internal operation in $\mathcal{E}_\lambda$ and $(\mathcal{E}_\lambda, ., \star)$ is a subring of $\mathcal{A}$.

Moreover, the eñe-product $\star : \mathcal{E}_\lambda \times \mathcal{E}_\lambda \to \mathcal{E}_\lambda$, $(f, g) \to f \star g$ is continuous for the topology of uniform convergence on compact subsets.

This theorem results from the next result that is more general and that shows that the eñe-product respects Hadamard-Weierstrass factorization of entire functions of finite genus. We recall that the genus ρ of en entire function f is the minimal integer so that f can be written

$$f(z) = e^{F(z)} \prod_i E_\rho \left(\frac{z}{\alpha_i} \right)$$

where $F \in \mathbb{C}[z]$ is a polynomial of degree $\leq \rho$, and the infinite product is uniformly convergent on compact subsets of $\mathbb{C}$. The factorization for general entire functions is due to Weierstrass. The above factorization for functions of finite genus is due to Hadamard. If no such ρ exists then the genus is infinite. We have $\rho \leq \lambda \leq \rho + 1$ (see for example [1] p. 209).

Theorem 9.2. *Let f and g be entire functions of finite genus $0 \leq \rho < +\infty$ with respective sets of zeros (α_i) and (β_j). We assume that* 0 *is not a zero of f or g, and more precisely f and g have* *constant coefficient $f(0) = g(0) = 1$. We consider the Hadamard-Weierstrass factorizations*

$$f(z) = e^{F(z)} \prod_i E_\rho \left(\frac{z}{\alpha_i} \right)$$
$$g(z) = e^{G(z)} \prod_j E_\rho \left(\frac{z}{\beta_j} \right)$$

where, by definition of the genus, F and G are polynomials vanishing at 0 of degree $\leq \max(\deg F, \deg G) \leq \rho$. Then $F \star_e G$ is a polynomial of degree $\leq \rho$ and we have

$$f \star g(z) = e^{F \star_e G(z)} \prod_{i,j} E_\rho \left(\frac{z}{\alpha_i \beta_j} \right) \ .$$

Proof. The exponential form of the eñe-product shows that $F \star_e G$ is a polynomial of degree $\leq \rho$.

Now, working on the ring $\mathcal{A}$ (thus we do not need to pay attention to questions of convergence for the moment and the computations are done at the formal level), we have, using distributivity,

$$f \star g(z) = (e^F \star e^G). \left(e^F \star \prod_j E_\rho \left(\frac{z}{\beta_j} \right) \right) . \left(\prod_i E_\rho \left(\frac{z}{\alpha_i} \right) \star e^G \right) .$$
$$\left(\prod_i E_\rho \left(\frac{z}{\alpha_i} \right) \star \prod_j E_\rho \left(\frac{z}{\beta_j} \right) \right)$$

Now, we have

$$e^F \star e^G = e^{F \star_e G} \ .$$

Also

$$\begin{aligned} e^F \star \prod_j E_\rho \left(\frac{z}{\beta_j} \right) &= \prod_j e^F \star E_\rho \left(\frac{z}{\beta_j} \right) \\ &= \prod_j e^{F(z/\beta_j)} \star E_\rho(z) \\ &= \prod_j e^{F(z/\beta_j)} . T_\rho^e \left(e^{-F} \right) (z/\beta_j) \\ &= \prod_j e^{(F - T_\rho(F))(z/\beta_j)} \\ &= 1 \end{aligned}$$

because $\deg F \leq \rho$ thus $F - T_\rho(F) = 0$. By the same reasons

$$\prod_i E_\rho \left(\frac{z}{\alpha_i} \right) \star e^G = 1 \ .$$

And finally,

$$\begin{aligned}
&\prod_i E_\rho\left(\frac{z}{\alpha_i}\right) \star \prod_j E_\rho\left(\frac{z}{\beta_j}\right) \\
&= \prod_{i,j} E_\rho\left(\frac{z}{\alpha_i}\right) \star E_\rho\left(\frac{z}{\beta_j}\right) \\
&= \prod_{i,j} E_\rho(z) \star E_\rho\left(\frac{z}{\alpha_i\beta_j}\right) \\
&= \prod_{i,j} E_\rho\left(\frac{z}{\alpha_i\beta_j}\right) .T_N^e\left(1/E_\rho\left(\frac{z}{\alpha_i\beta_j}\right)\right) \\
&= \prod_{i,j} E_\rho\left(\frac{z}{\alpha_i\beta_j}\right)
\end{aligned}$$

where the last equality is obtained observing that

$$T_N^e\left(1/E_\rho\left(\frac{z}{\alpha_i\beta_j}\right)\right) = 1 \ .$$

Thus we have established the formal Weierstrass factorization for $f \star g$. We only need to check that the product of Weierstrass factors is uniformly convergent on compact subsets of $\mathbb{C}$. This follows from the continuity of the eñe-product for the topology of uniform convergence on compact sets on a domain where the functions have no zeros. Given a compact set in the plane, we consider a ball $\mathbb{D}_R$ of center 0 and radius $R > 0$ large enough to contain the compact set. Consider only those Weierstrass factors having zeros out of this ball, we observe that their product converges uniformly as well as they eñe-product by theorem 8.3. The remaining Weierstrass factors are finite. □

Remark 9.3. This previous result has a generalization to arbitrary entire functions of infinite genus. We must then choose the orders in the Weierstrass factors large enough (depending on f and g) in order not to introduce other terms in the exponential besides $F \star_e G$.

10. Eñe-product and Hadamard multiplication.

We recall the definition of Hadamard multiplication (see [4]).

Definition 10.1. ***(Hadamard multiplication)*** *Let A be an arbitrary ring. The Hadamard multiplication of $f(X), g(X) \in A[[X]]$,*

$$f(X) = \sum_{n=0}^{+\infty} f_n X^n$$

$$g(X) = \sum_{n=0}^{+\infty} g_n X^n$$

is

$$f \odot g(X) = \sum_{n=0}^{+\infty} f_n g_n X^n \ .$$

Note that the unit for the Hadamard multiplication is

$$\frac{X}{1-X} = \sum_{n=1}^{+\infty} X^n \ .$$

More precisely, we have

Theorem 10.2. *The sum and the Hadamard multiplication are internal operations of* $A[X]$, $\mathcal{A}$ *and* $A[[X]]$, *and* $(A[[X]], +, \odot)$ *is a commutative ring if* A *is commutative.*

The Hadamard multiplication has similar properties than the exponential eñe-product $\star_e$. For example, we have

Theorem 10.3. *If* $P(X) \in A[X]$ *and* $f(X) \in A[[X]]$ *then* $P \odot f \in A[X]$, *and*

$$\deg(P) = \deg(P \odot f) \ .$$

The relation to the exponential eñe-product is clear from the definition.

Theorem 10.4. *We have for* $F(X), G(X) \in A[[X]]$,

$$F \star_e G = -K_0 \odot F \odot G$$

where

$$K_0(X) = X + 2X^2 + 3X^3 + \ldots = \frac{X}{(1-X)^2}$$

is the Koébé function (that plays a central role in univalent function theory, being extremal for many problems).

This means that the exponential eñe-product structure is the Hadamard ring structure twisted by $-K_0$. Note that the inverse of $-K_0$ for the Hadamard multiplication (that is the unit for the twisted structure) is

$$-X - \frac{1}{2}X^2 - \frac{1}{3}X^3 - \ldots = \log(1-X) \ ,$$

i.e. it is also the unit for the exponential eñe-product.

Also directly from the definition we get

Theorem 10.5. *Let* $F, G \in A[[X]]$. *We have*

$$D(F \star_e G) = -D(F) \odot D(G) \ .$$

More precisely, $-D : (XA[[X]], +, \star_e) \to (A[[X]], +, \odot)$, $F \mapsto -D(F)$ *is a ring isomorphism between the exponential eñe ring structure and the Hadamard ring structure.*

As corollary we get the direct relation to the eñe-product.

Theorem 10.6. *Let* $f, g \in \mathcal{A}$. *We have*

$$\mathcal{D}(f \star g) = -\mathcal{D}(f) \odot \mathcal{D}(g) \ .$$

Proof. Write

$$f = \exp(F)$$
$$g = \exp(G)$$

and observe that

$$\mathcal{D}(f \star g) = D(F \star_e G) = -D(F) \odot D(G) = -\mathcal{D}(f) \odot \mathcal{D}(g) \ .$$

□

11. Extension of the eñe-product and inversion.

We extend the definition of eñe-product to the ring of non zero polynomials $A[X]$ by using the interpretation with roots.

Definition 11.1. *For $P(X), Q(X) \in A[X]$, P and Q non zero, with*

$$P(X) = a_0 X^n P_0(X)$$
$$Q(X) = b_0 X^m Q_0(X)$$

where $a_0, b_0 \in A - \{0\}$, and $P_0(X), Q_0(X) \in 1 + X\ A[X]$ we define

$$(P \star Q)(X) = X^{n \deg(Q_0) + m \deg(P_0) + nm} (P_0 \star Q_0)(X) \ .$$

We denote by $\mathbb{P}(A[X])$ the set of polynomials with lower coefficient 1 and $\neq 0$ (when A is a field, this is the projective space of the ring $A[X]$). We have

Proposition 11.2. *We have that $(\mathbb{P}(A[X]), ., \star)$ is a commutative ring.*

Now we can extend the eñe product to non-zero rational functions.

Definition 11.3. *We extend the ene product to non-zero rational functions quotients of elements in $\mathbb{P}(A[X])$. If*

$$R_1(X) = \frac{P_1(X)}{Q_1(X)}$$
$$R_2(X) = \frac{P_2(X)}{Q_2(X)}$$

then we put

$$(R_1 \star R_2)(X) = \frac{(P_1 \star P_2).(Q_1 \star Q_2)}{(P_1 \star Q_2).(Q_1 \star P_2)} \ .$$

Next, we have a main property of the extension of the eñe product. It shows that the points at 0 and ∞ in the Riemann sphere play a symmetric role.

Theorem 11.4. *The eñe-product is invariant by inversion. More precisely, let $P(X), Q(X) \in A[X]$ be non-zero polynomials with lower degree and leading coefficient 1, then*

$$P(\frac{1}{X}) \star Q(\frac{1}{X}) = (P \star Q)(\frac{1}{X}) \ .$$

Proof. Write

$$P(\frac{1}{X}) = \frac{\hat{P}(X)}{X^n} \ ,$$
$$Q(\frac{1}{X}) = \frac{\hat{Q}(X)}{X^m} \ ,$$

where $\hat{P}(X), \hat{Q}(X) \in \mathcal{A}$. Observe that if (α_i) are the zeros of P then the zeros of $\hat{P}$ are (α_i^{-1}). From this observation it follows that

$$(P \star Q)(\frac{1}{X}) = \frac{(\hat{P} \star \hat{Q})(X)}{X^{nm}}$$

thus

$$(P \hat{\star} Q) = \hat{P} \star \hat{Q} \ .$$

We have now

$$
\begin{aligned}
P(\frac{1}{X}) \star Q(\frac{1}{X}) &= \frac{\hat{P}(X)}{X^n} \star \frac{\hat{Q}(X)}{X^m} \\
&= \frac{(\hat{P} \star \hat{Q}).(X^n \star X^m)}{(\hat{P} \star X^m).(\hat{Q} \star X^n)} \\
&= \frac{\hat{P} \star \hat{Q}}{X^{nm}} \\
&= (P \star Q)(\frac{1}{X})
\end{aligned}
$$

□

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