# Log-euclidean geometry and ”Grundlagen der Geometrie”

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**Abstract.** We define the simplest log-euclidean geometry. This geometry exposes a difficulty hidden in Hilbert's list of axioms presented in his ”Grundlagen der Geometrie”. The list of axioms appears to be incomplete if the foundations of geometry are to be independent of set theory, as Hilbert intended. In that case we need to add a missing axiom. Log-euclidean geometry satisfies all axioms but the missing one, the fifth axiom of congruence and Euclid's axiom of parallels. This gives an elementary proof (with no need of Riemannian geometry) of the independence of these axioms from the others.

## 1. Log-euclidean geometry.

### 1.1. A metric space.

Consider two pointed euclidean planes, that we can identify with the complex plane $\mathbf{C}-\{0\}$, slitted along an euclidean half-line, say along the negative real axes $]-\infty, 0[$. We glue by the identity on charts the upper, resp. lower, boundary of the slit of one plane with the lower, resp. lower, boundary of the slit in the other plane. We add the missing point at 0, that we continue to denote by 0. The distance between two points in this space is the infimum of lengths of paths joining these two points. The length is being computed in each euclidean chart. We denote by $\mathcal{X}$ the metric space constructed. Conformally the concrete Riemann surface† obtained is the Riemann surface associated to $z \mapsto \sqrt{z}$. This is why we refer to 0 as the ramification point. Note that it is necessary to add a the ramification point 0 in order to have a complete space. It is elementary to prove:

**Proposition.** *Given two distinct points $A, B \in \mathcal{X}$ there is a unique geodesic path joining them. This geodesic path is an euclidean segment, or a broken line formed by two euclidean segments meeting at the ramification point* 0.

### 1.2. Log-euclidean points and lines.

In log-euclidean geometry a point is an element of the space $\mathcal{X}$. Lines in log-euclidean geometry are defined to be euclidean lines or two half-lines meeting at the ramification point 0 (this point being included).

There exists a log-euclidean geometry in each log-Riemann surface. These are Riemann surfaces with a set of canonical charts of a certain type. Log-euclidean metric

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† A concrete Riemann surface is a Riemann surface endowed with a canonical system of charts.

characterizes the log-Riemann surface structure (see [Bi-PM1] and [Bi-PM2] for more information on log-Riemann surfaces).

## 2. Hilbert axioms.

### 2.1) Preliminary comments.

The purpose of the "Grundlagen der Geometry" is to give a complete list of independent axioms upon which euclidean geometry can be build. This philosophy is more than 2300 years old and is borrowed from "The Elements" of Euclid ([Eu]) which marks the birth of formal Mathematics. The purpose of the "Grundlagen" is to add those axioms that were assumed to be self-evident in "The Elements" and to prove the independence of the axioms. The problem of the independence of Euclid's fifth postulate on parallels has kept busy generations of mathematicians (Saccheri, Lambert, Legendre,...) until the creation of non-euclidean geometries by Gauss, Bolyai, Lobatchevski and their analytic realization. We refer to [Bo] and [Ef] for a delightful historical survey.

Hilbert's Grundlagen sets the axioms for three dimensional euclidean geometry. A subset of these axioms (named planar axioms) compose the axiomatic system of two dimensional euclidean geometry. We restrict to discuss planar geometry in this article.

As Hilbert states in his first sentence, he considers in planar geometry two types of objects: *points*, denoted with capital letters $A, B, C, \ldots$, and *lines*, denoted by lower case letters $a, b, c, \ldots$ The goal of the axioms is to describe relations between these objects. Thus, with the system of axioms at hand, a blind person can prove theorems in euclidean geometry. An important subtle point of Hilbert's conception is that the nature of "points" or "lines" is irrelevant. A famous quote of Hilbert that report several sources ([We], [Re], [BB], see also [Hi-Ce]) is the following

*"...One should be able to talk about chairs, tables and mugs of beer instead of points, lines or planes..."*

All of this discusion shows that in Hilbert's system **lines are not considered to be sets of points** to start with (as Hilbert would say "tables are not sets of chairs".) There is a very good reason for this point of view. Hilbert carefully avoids to base axiomatic euclidean geometry on axiomatic set theory. He is well aware (since the work of Cantor) that axiomatic set theory is far more complex to build.

### 2.2) Hilbert's axioms.

Hilbert's axioms are divided into five distinct groups:

I. Axioms of connection.
II. Axioms of order.
III. Axioms of congruence.
IV. Axiom of parallels.
V. Axioms of continuity.

There have been ten editions of the Grundlagen (see [Hi-Ro]), seven during Hilbert's lifetime. The set of axioms evolved to reach the final form after the seventh edition. In particular in the second edition one axiom became a theorem proved by E. H. Moore [Mo].

We first focuss on the first group of axioms of connection in planar geometry.

### 2.3) Log-euclidean geometry and the axioms of connexion.

#### 2.3.1) Axioms of connection.

In the words of Hilbert, the axioms of connection build connections or links between point objects and line objects.

The first two planar axioms are the following:

**I-1. Given two points $A$ and $B$, there always exists a line $a$ that corresponds to the two points $A$ and $B$.**

**I-2. Given two points $A$ and $B$, there is only one line that corresponds to the two points $A$ and $B$.**

Hilbert takes good care in indicating that "two points" means "two distinct points"; and that "corresponds" will be or can be replaced by "goes through", or "$A$ is located in $a$", or "$A$ is a point of $a$", or "$A$ belongs to $a$", etc

As explained above "$A$ belongs to $a$" cannot be taken in a set theoretical sense. This seems to have caused some psychological difficulty: According to [Hi-Ro] after the sixth edition of the Grundlagen "belongs to" was replacing "corresponds to" from the previous edition. And this term is translated in French as "appartient". But in [Hi-Ce], which is a translation of the seventh edition, it is translated in Spanish as "se corresponde". The German word used in the text is "zusammengehören".

But what word to use, or in its place to use a potato to mark the connection, is after all just a matter of semantics and is irrelevant.

#### 2.3.2) Log-euclidean geometry.

We have to define the correspondence stated in axioms **I-1** and **I-2** in log-euclidean geometry. The line corresponding to two points $A$ and $B$ will be the euclidean line passing through these points if an euclidean segment is the geodesic joining these two points (including the case when this euclidean segment contains the ramification point 0). In the case when a broken segment is the geodesic joining these two points then the corresponding line is obtained by extending the two segments in the oposite direction of the ramification point 0.

It is clear that such correspondence satisfies the axioms **I-1** and **I-2**. Moreover the third planar connection axiom is also trivially verified:

**I-3. Any line goes through at least two points. There are at least three points not in a line.**

Again here the "goes through" and "belonging" statement are to be taken in the sense specified by Hilbert's remarks. Thus, for example, to say that a line goes through at least two points $A$ and $A'$ means that there are points $B$, distinct from $A$, and $B'$, distinct from $A'$, such that the line is associated to $A$ and $B$, and also to $A'$ and $B'$.

**2.3.3) Log-euclidean geometry and the planar part of the first Theorem of the Grundlagen.**

After stating the axioms of this first group, Hilbert states without proof his first Theorem. The first part of the theorem is a planar statement that supposedly follows from the planar axioms:

**Theorem 1 (Planar part).** *Two lines have one or no points in common.*

In an equivalent form, if two lines have two distinct points in common then they must be equal.

Also in [Ef] p.43 this part of Theorem 1 is stated to follow trivially from axiom **I-2** and does not deserve a proof.

Now consider two points $A$ and $B$ in log-euclidean geometry joined by a geodesic which is a broken segment. The line $a$ associated to $A$ and $B$, and the line $b$ associated to $A$ and the ramification point 0, do have a half line in common!

**Conclusion: Theorem 1 does not hold in log-euclidean geometry, thus theorem 1 cannot follow from axioms I-1, I-2 and I-3.**

**3) Solution of the paradox.**

The confusion in Hilbert's text is created by the terminology "belonging to a line" that is not to be taken in a set theoretic sense, but apparently is taken in this sense in order to prove Theorem 1. If we assume set theory, and we formulate the second axiom as

**I-2'. Given two points $A$ and $B$ there exists at most one line containing $A$ and $B$.**

where we understand that "containing" is taken in a set theoretic sense (thus this implies that the objects "lines" are sets containing some "point" objects), then the proof of theorem 1 is straithforward.

This appears to be a subtle point, but log-euclidean geometry shows that one must be careful on this point.

One needs to add a missing axiom in order to exclude log-euclidean geometry. We propose:

**I-2bis. If a line $a$ corresponds to $A$ and $C$, and also to $B$ and $D$, then if $A$ and $B$ are distinct points, the line $a$ also corresponds to $A$ and $B$.**

Note that axiom **I-2bis** is not fulfilled by log-euclidean geometry. Just consider a broken line with $A$ and $B$ in one of the half lines determined by the ramification point 0 and and $C$ and $D$ in the other half line.

We can now prove theorem 1.

**Proof of Theorem 1.**

Assume that the lines $a$ and $a'$ have two points $A$ and $B$ in common. This means that there exist points $C$ and $D$ such that the line $a$ corresponds to $A$ and $C$, and it also corresponds to $B$ and $D$. Thus using the axiom **I-2bis** it also corresponds to $A$ and $B$. With this same argument the line $a'$ also corresponds to $A$ and $B$. Thus we conclude that the lines $a$ and $a'$ coincide using axiom **I-2**.$\diamond$

It is curious to note that the weaker statement:

**"If the line $a$ corresponds to $A$ and $B$, and also to $A$ and $C$, and if $B$ is distinct from $C$ then $a$ corresponds to $B$ and $C$"**

apparently was included in axiom **I-2** of the first edition of the Grundlagen according to [Hi-Ro]. This still seems insufficient to prove Theorem 1 (but is enough to exclude log-euclidean geometry).

In conclusion we want to insist that the subtle point raised here is not irrelevant. If one aims to build an axiomatic system of euclidean geometry not relying on axiomatic set theory (as Hilbert intended), one should add an axiom as **I-2bis**.

### 4) Log-euclidean geometry and the other axioms.

It is straightforward to check that the axioms of order (group II) and continuity (group V) are satisfied by log-euclidean geometry. Also, taking as lenght of segments in log-euclidean geometry the euclidean lengths, then the axioms of congruence relative to segments (these are **III-1**, **III-2**, **III-3**) are fulfilled.

In order to check the axioms of congruence of angles **III-4** and **III-5** we need to define the magnitude of angles in log-euclidean geometry. Lines meeting at a point distinct from 0 form an angle whose magnitude is the one given by euclidean geometry. For angles having the ramification point 0 as vertex, we define its magnitude as the euclidean angle formed by the angle formed by the bissectors of the broken lines at 0 pointing towards the region where a full sheet is attached. With this convention axiom **III-4** is fulfilled.

But axiom **III-5** on congruence of triangles is not fulfilled as well as Euclid's axiom of parallels **IV-1**. Any line has infinitely many parallel lines going through an external point.

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