THE CANTOR RIEMANNIUM

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Abstract. The Riemann surface of a holomorphic germ is the space generated by its Weierstrass analytic continuation. The Riemannium space of a holomorphic germ is the space generated by its Borel monogenic continuation. Riemannium spaces are metric, path connected, Gromov length spaces, not necessarily σ-compact. We construct an example of Riemannium space: The Cantor Riemannium.

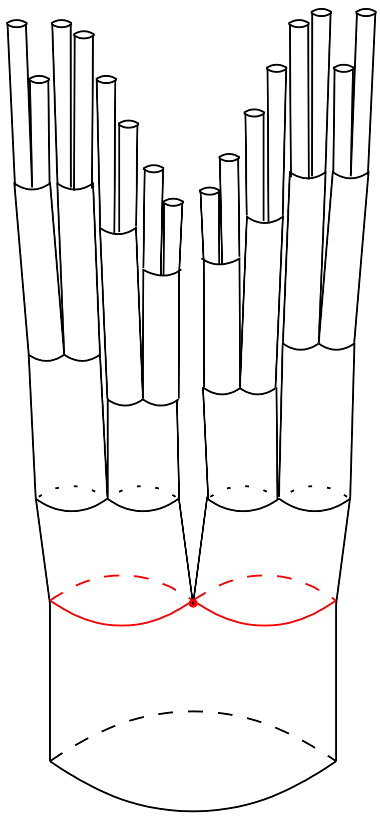

Figure 1. The Krueger hand: Principal sheet of the Cantor Riemannium.

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1. Introduction.

The main construction in this article is inspired from an old forgotten article of Giulio Vivanti (1888, [25]) about the cardinal of the values taken by a multivalued[1] analytic function at a given point, this is, all the values taken by all branches at this point.

In this distant past, the notion of analytic continuation was not properly defined and subsequently Vivanti's article was dismissed and misunderstood. If one adopts Weierstrass notion of analytic continuation, then the Cantor-Vivanti-Poincaré-Volterra Theorem[2] states that the cardinal of branches at a point is always countable. The first published proof published by Vivanti (1888, [26]) was dismissed as erroneous. It suffers from the very same problems than Poincaré's earlier articles on general Riemann surfaces (for example [20]). Vivanti's first article [25] was motivated by these difficulties contained in Poincaré's article.

To understand the origin of the difficulties, we need to be aware of the original notion of Riemann surface during the XIX-th century. Riemann surfaces were born in the celebrated work of B. Riemann (1851, [21], 1857, [22]) as the minimal spaces where the analytic continuation of holomorphic germs are univalued. Riemann's Riemann surfaces had more structure that the modern notion later proposed by H. Weyl (1913, [27]) and adopted today. They are domains over $\mathbb{C}$, and come equipped with a projection in $\mathbb{C}$. They also contain "ramification points". The structure and proper definition of the ramification locus was only well understood for finite ramifications, which was enough to deal with the theory of algebraic curves and function ([22]), and isolated logarithmic ramifications, which appear in the fuchsian class of functions by L. Fuchs (1866, [12]). There was no clear understanding, nor a good definition, of more complicated ramification locus. This difficulty represents the central point of Hilbert's 22nd problem (1900, [15]). Few people nowadays are aware of this because Weyl's modern notion of Riemann surface excluded ramification points from its structure[3].

Some years later, Émile Borel discovered non-Weierstrassian analytic continuations and he introduced the notion of monogenic functions and Borel continuation. His insightful theory did mature from his Thesis (1894, [4]) and was finally presented in his monograph (1917, [6]). We urge the reader to not miss the introduction of Borel's book in order to understand the opposition of the Weierstrassian school, in particular of G. Mittag-Leffler, to Borel's brilliant ideas. Borel's monogenic class of functions is the precursor of quasi-analytic classes developed later by A. Denjoy (1921, [10]) and T. Carleman (1926, [8]). Borel's ideas were inspired by Cauchy's original point of view on holomorphic functions, that inspired also Riemann, characterized by its monogenic property, i.e. by $\mathbb{C}$-differentiability. Sometimes, these functions came from differential equations that yield non-Weierstrassian continuation properties. Borel insisted that the natural domain of definition for monogenic functions were domains that he named C (for Cauchy) compared with the open domains W for Weierstrassian continuation. The typical domain C that he describes in his book [6] can have empty interior. He regretted what he thought was the impossibility of the exact determination of the natural domain of extension C, contrairy to the domains W that were well defined. The type of monogenic classes of functions and domains C considered by Borel are not the most general ones, and the residual measures are atomic. It is instructive to remember what he wrote in the introduction of [6],

[1] Also named polydromic in the old literature.

[2] Nowadays improperly named Poincaré-Volterra Theorem.

[3] This total lack of reference to ramification points in Weyl's book is so blatant that one is led to suspect that the whole purpose of the new definition was to remove the uncomfortable ramification points.

La théorie des fonctions monogènes non analytiques laisse subsister, bien entendu, tout ce qu'il y a de positif dans la théorie des fonctions analytiques de Weierstrass. Mais cette partie positive de la théorie de Weierstrass ne se distingue que par des détails de la théorie de Cauchy, qui utilisait aussi le prolongement analytique par les équations différentielles; c'est à son côté négatif que la théorie de Weierstrass devait ce caractère de perfection logique tant admiré par ses adeptes; en substituant les domaines C aux domaines W, on doit renoncer à ce caractère, car on ne pourra vraisemblablement jamais fixer les limites exactes au delà desquelles une extension nouvelle est impossible.(É. Borel, 1917)

We present in this article an explicit example showing that the Borel extension of a holomorphic germ can have an uncountable number of branches that take an uncountable number of values at a point, confirming Vivanti's original intuition.

The example presented is monogenic in a maximal natural space that can be explicitly determined, contrary to Borel's beliefs. As explained before, the Weierstrass analytic continuation defines the Riemann surface associated to a holomorphic germ, and even more, a concrete Riemann surface in the original sense of Riemann, or a log-Riemann surface with the modern definition given by the author and K. Biswas (2015, [1]). The minimal space where the Borel monogenic extension of a holomorphic germ is univalued defines a unique *Riemannium space*. This natural space is not a manifold in general. It is a metric and path connected space but not in general σ-compact. It is "concrete" in the sense that it comes equipped, as log-Riemann surfaces, with a natural projection on $\mathbb{C}$. It is a $\mathbb{C}$-space in the sense of Bourbaki [7]. The Riemannium space can contain a dense open set composed by Riemann surfaces.

In this article we construction an explicit example of Riemannium: The Cantor Riemannium.

2. The Cantor tube-log Riemann surface.

2.1. **Cauchy transform of a total finite mass measure.** In this first section $K \subset \mathbb{C}$ is an arbitrary compact set. In the applications in subsequent sections K will be a well defined Cantor set. Let μ be a Borel measure on $\mathbb{C}$ with total finite mass

$$|\mu|(\mathbb{C}) < +\infty$$

and compact support $\operatorname{supp} \mu = K \subset \mathbb{C}$. We consider the Cauchy transform[4] of μ

$$f_\mu(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{d\mu(t)}{z-t} \ . \tag{1}$$

Observe that if μ is an atomic measure with a finite number of atoms, them f_μ is a linear combination of simple polar parts, the poles being the location of the atoms, and the residues the mass at each point. IF $\mu = \sum_{n=1}^{N} a_n \delta_{z_n}$, then

$$f_\mu = \sum_{n=1}^{N} \frac{a_n}{z - z_n} \ .$$

Proposition 2.1. *Let $K \subset \mathbb{C}$ be the compact support of μ. The Cauchy transform f_μ is holomorphic on $\mathbb{C} - K$.*

[4]Another normalization in the harmonic analysis literature is $C_\mu(z) = \int_{\mathbb{C}} \frac{d\mu(t)}{z-t}$ ([24] for the current theory).

Let $(\mu_n)_{n\geq 0}$ be a sequence of Borel measures with uniformly bounded total mass, i.e. there exists $M > 0$ such that for all $n \geq 0$,

$$|\mu_n|(\mathbb{C}) \leq M < +\infty$$

If $\mu_n \to \mu$ in the weak topology, and $\operatorname{supp} \mu_n \to \operatorname{supp} \mu = K$ in Hausdorff topology, then uniformly on compact sets of $\mathbb{C} - K$ we have $f_{\mu_n} \to f_\mu$.

Proof. Outside of K, since μ has total finite mass, the integral (1) defining the Cauchy transform is absolutely convergent, and for $z \in \mathbb{C} - K$

$$\bar{\partial}_z f_\mu(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \bar{\partial}_z \left(\frac{1}{z-t} \right) d\mu(t) = 0$$

Alternatively, we can argue that atomic measures with a finite number of atoms is a weak-dense set in the space of measures with uniformly bounded total mass. Also we can approximate μ by such measures (μ_n) with support in K. The Cauchy transforms f_{μ_n} are meromorphic functions, holomorphic on $\mathbb{C} - K$, and by Lebesgue dominated convergence Theorem they converge uniformly on compact sets of $\mathbb{C} - K$ to f_μ, hence f_μ is holomorphic on $\mathbb{C} - K$.

For the second statement we observe that we have also $f_{\mu_n} \to f_\mu$ uniformly on compact sets of $\mathbb{C} - K$ by Lebesgue dominated convergence Theorem. □

Thus the integral defines a locally holomorphic function at each point of $\mathbb{C} - K$.

Proposition 2.2 (Monogenic Residue Formula). *Let Ω be a Jordan domain such that $\gamma = \partial\Omega \subset \mathbb{C} - K$. Then we have*

$$\int_\gamma f_\mu(z)\, dz = \mu(\Omega)$$

Proof. The classical residue formula for the meromorphic functions f_{μ_n} approximating f_μ with atomic measures (μ_n) gives

$$\int_\gamma f_{\mu_n}(z)\, dz = \mu_n(\Omega)$$

and we pass to the limit $n \to +\infty$. □

With some extra assumptions we can extend this result to the case when $\gamma \cap K \neq \emptyset$.

Proposition 2.3. *Let Ω be a Jordan domain and $\gamma = \partial\Omega$. Let (μ_n) be a sequence of measure converging to μ with uniformly bounded mass. If $V_\epsilon(\gamma)$ is the ϵ-neighborhood of γ we assume that*

$$\lim_{n\to+\infty} \mu_n(V_\epsilon(\gamma)) = 0 \ .$$

Then, if (μ_n) has no atoms on γ, we have

$$\lim_{n\to+\infty} \int_\gamma f_{\mu_n}(z)\, dz = \mu(\Omega)$$

In particular, if γ is rectifiable and (f_{μ_n}) converges uniformly to a continuous function f_μ on γ, then

$$\int_\gamma f_{\mu_n}(z)\, dz = \nu_n(\Omega)$$

Proof. Under the previous assumptions, we have $\mu_n(\Omega) \to \mu(\Omega)$ and the result follows. □

2.2. **The triadic Cantor set with its equilibrium measure.** We study an explicit example of the above. Let $C \subset [0,1]$ be the usual triadic Cantor set. It can be generated by the two affine maps $A_0(x) = 3x$ and $A_1(x) = 3x - 2$ as the set of points

$$C = \bigcup_{n \geq 0} \bigcap_{\epsilon \in (\mathbb{Z}/2\mathbb{Z})^n} \left(A_{\epsilon_1} \circ \dots A_{\epsilon_n}\right)^{-1} ([0,1])$$

We can identify the Cantor set C to the ring of 2-adic integers $\mathbb{Z}_2$. The Haar measure $\mu_{\mathbb{Z}_2}$ gives the equilibrium probability measure μ on C. Observe that we can obtain μ as the weak limit of purely atomic probability measures μ_n that are the Haar measures of $\mathbb{Z}_2/2^n\mathbb{Z}_2 \approx \mathbb{Z}/2^n\mathbb{Z}$. Considering the standard representation of the triadic Cantor set, these are atomic measures composed by atoms of equal mass 2^{-n} at the 2^n end-points of the removed intervals from $[0,1]$ at depth $n \geq 1$. For instance, we have

$$\begin{aligned}
\mu_1 &= \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1 \\
\mu_2 &= \frac{1}{4}\delta_0 + \frac{1}{4}\delta_{1/3} + \frac{1}{4}\delta_{2/3} + \frac{1}{4}\delta_1 \\
\mu_3 &= \frac{1}{8}\delta_0 + \frac{1}{8}\delta_{1/9} + \frac{1}{8}\delta_{2/9} + \frac{1}{8}\delta_{1/3} + \frac{1}{8}\delta_{2/3} + \frac{1}{8}\delta_{7/9} + \frac{1}{8}\delta_{8/9} + \frac{1}{8}\delta_1 \\
&\vdots
\end{aligned}$$

The triadic Cantor set C just described has Hausdorff dimension $\dim_H C = \frac{\log 2}{\log 3}$. The Hausdorff measure of this dimension coincides with μ.

2.3. **Monodromy of the primitive of the Cauchy transform of the equilibrium measure.** We consider the translated Cantor set $K = T_{-1/2}(C)$, where $T_a(z) = z + a$ is the translation, so that the origin $z = 0$ is the center of symmetry. Let $\left(T_{-1/2}\right)_* \mu$ be the transported measure that we still denote μ.

We consider the Cauchy transform

$$f_\mu(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{d\mu(t)}{z - t} \ .$$

By symmetry f_μ is odd and we have $f_\mu(0) = 0$. We consider the primitive

$$F_\mu(z) = \int_0^z f_\mu(t)\, dt$$

which defines a holomorphic germ in a neighborhood of 0 and is multivalued on $\mathbb{C} - K$. Given a loop $\gamma \subset \mathbb{C} - K$, With base point at 0 and homotopy class $[\gamma] \in \pi_1(\mathbb{C} - K, 0)$, by analytic continuation we associate a value of the branch of F_μ along γ that only depends on its homotopy class,

$$\gamma \mapsto \mathrm{Mon}_\gamma(F_\mu) \ .$$

The monodromy map $\mathrm{Mon} : \pi_1(\mathbb{C} - K, 0) \to \mathbb{C}$ constructed in that way is a group morphism.

Proposition 2.4. *The monodromy of F_μ at $z = 0$ is the group*

$$\mathrm{Mon}_{z=0}\ F_\mu = \bigoplus_{k \geq 0} 2^{-k}\mathbb{Z} = 2^{-\infty}\mathbb{Z}$$

Proof. Consider a Jordan loop $0 \in \gamma \subset \mathbb{C} - K$ containing the base point $z = 0$ and not homotopic to 0. Then, taking $k \geq 1$ large enough, γ encloses in its bounded component exactly $1 \leq n(\gamma) \geq 2^k - 1$ pieces of level k of the Cantor set K. Then, using the Monogenic Residue Formula from Proposition 2.2 we have

$$\int_\gamma f_\mu(t)\, dt = n(\gamma) 2^{-k}$$

and this proves that the monodromy is of this form. On the other hand enclosing a single fundamental piece at level k and winding around the appropriate number of times we can achieve any monodromy of the form $n2^{-k}$. □

2.4. **The Cantor tube-log Riemann surface.** We refer to [1] and [2] for the definition of log-Riemann surfaces and to [3] for tube-log Riemann surfaces and a general description of tube-log Riemann surfaces associated to primitives of rational functions.

The Cantor tube-log Riemann surface $\mathcal{S}_\mu$ is the tube-log Riemann surface associated to F_μ. This means that the map F_μ is the uniformization of $\mathbb{C} - K$ into a tube-log Riemann surface $\mathcal{S}_\mu$ that we are now describing. According to [3], to each rational function $R \in \mathbb{C}(z)$, with set of poles and critical points $\mathcal{P}_R$ (we assume $0 \notin \mathcal{P}_R$), there corresponds a unique, up to normalization, tube-log Riemann surface $(\mathcal{S}_R, z_0)$ such that if

$$F(z) = \int_0^z R(t)\, dt$$

the germ of F at 0 extends into the uniformization $F : \overline{\mathbb{C}} - \mathcal{P}_R \to \mathcal{S}_R$.

The rational function f_{μ_n} has only simple poles of equal residues 2^{-n}. The description of the monodromy of F_{μ_n} proves that the associated tube-log Riemann surface $\mathcal{S}_{\mu_n}$ is obtained by pasting a sequence of copies of annuli A_m, $0 < m \leq +\infty$,

$$A_m = \{z \in \mathbb{C}; 0 < \operatorname{Re} z < m\}/(2\pi i)\mathbb{Z}$$

We use $1 + 2 + \ldots + 2^n = 2^{n+1}$ annulus glued together as a dyadic tree, by gluing their boundary by a translation map (the reader can find a similar construction in [19]) and we obtain the surface in Figure 2. The first and the last annuli have infinite modulus and the others finite. All critical points of F_{μ_n} are real and there is exactly one critical point of F_{μ_n} inside each interval determined by consecutive poles.

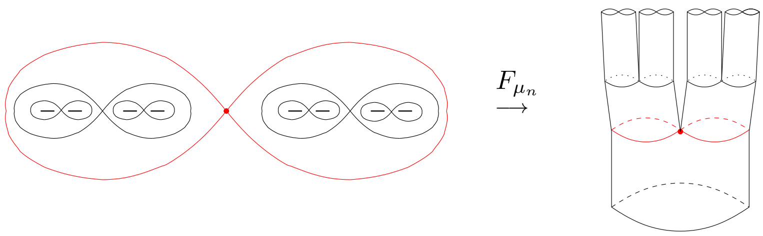

FIGURE 2. Tube-log Riemann surface of F_{μ_n}.

For the primitive F_μ, the associated tube-log-Riemann surface $\mathcal{S}_\mu$ is constructed in a similar way but with an infinite number of annuli. There is exactly one critical point inside each removed interval generating the Cantor set. The sequence of modulus along a branch that defines an end add up to infinite because of the self-similarity. Consider the two annuli, symmetric with respect to 0,

$$B_0 = B(1/2, 3/4) - \overline{B}(1/6, 1/4)$$
$$B_1 = B(1/2, 3/4) - \overline{B}(5/6, 1/4)$$

Then given the point $z_{\boldsymbol{\epsilon}} \in K$, we have a nested of essential annulus defining fundamental neighborhoods of $z_{\boldsymbol{\epsilon}}$, $B_{\epsilon_0}, A_{\epsilon_1}(B_{\epsilon_0}), A_{\epsilon_2} \circ A_{\epsilon_1}(B_{\epsilon_0}), \ldots$ All these annuli have the same modulus and

$$\sum_{n=0}^{+\infty} \quad \mathrm{mod}\ A_{\epsilon_n} \circ \ldots \circ A_{\epsilon_1}(B_{\epsilon_0}) = +\infty$$

Using a standard criterium (see [19] Lemma 2.17) we have that $\mathbb{C} - K$ is in the O_{AD} class of Riemann surfaces. This implies the following:

Proposition 2.5. *Let $z \to z_{\boldsymbol{\epsilon}} \in K$ along a path $\eta \subset \mathbb{C} - K$, then*

$$\lim_{z \to z_{\boldsymbol{\epsilon}}} \mathrm{Im}\, F_\mu(z) = +\infty$$

The tubular "fingers" have infinite height. This can also be established by a direct computation.

Green lines.

The lines corresponding to level lines of $\mathrm{Re}\, F_\mu$ are the Green lines orthogonal to the equipotentials[5] that are the level lines of $\mathrm{Im}\, F_\mu$. We observe that we have a parametrization of Green lines by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and we can normalize it so that the two Green lines hitting 0 are $1/4$ and $-1/4$. We denote $\mathbb{T}_0$ the abstract circle without the dyadic angles,

$$\mathbb{T}_0 = \mathbb{T} - \bigcup_{k \geq 0} 2^{-k}\mathbb{Z} \ .$$

The inner points $K_i \subset K$ are those points in K that are not end-points of the removed intervals. To each inner point $x \in K_i$ of K set there correspond an angle $\theta \in \mathbb{T}_0$ which is not a dyadic rational such that the Green line of θ and $-\theta$ land at x. We define the map $\sigma : \mathbb{T}_0 \to K_i$, $\theta \mapsto \sigma(\theta) = \sigma(-\theta) = x$.

We observe that the prime-ends at infinite of $\mathcal{S}_\mu$ correspond to the point ∞ and the points in the Cantor set, and the inner points in the Cantor set K correspond to two prime-ends for the lines $\mathrm{Re}\, F_\mu = \pm\theta$. So there is a natural quotient of the prime-end compactification that gives the Riemann sphere.

2.5. **Julia Cantor sets.** For those readers familiar with holomorphic dynamics, we point out that we could construct examples of Cantor sets of the above type by taking the Julia set of a quadratic polynomial $P_c(z) = z^2 + c$ with $c \notin M$ outside the Mandelbrot set, this means that the iterates of the critical point $z = 0$ escapes to infinite, $P_c^n(0) \to \infty$. In that case the Julia set is the set of points with bounded (positive) orbit,

$$J_c = \{z \in \mathbb{C}; (P_c^n(z))_{n \geq 0} \text{ bounded}\}$$

The Julia set J_c obviously contains all periodic orbits and it is the closure of the set of periodic orbits. It is a Cantor set and we take for μ the measure of maximal entropy that is equivalent to

[5] These are not the regular equipotentials of K.

the limit of equidistributed atomic measures on periodic orbits. Then the Böttcher coordinate of the basin of attraction at infinite is defined in the tube-log Riemann surface described before. The reader can consult [19] and [11] for how to determine combinatorially the tube-log Riemann surface.

3. The Cauchy transform of a singular measures and its Borel monodromy.

We define now a new measure ν on K that is singular with respect to μ. By construction ν is an atomic measure, but we could easily modify the construction to get a non-atomic measure. Let $(\lambda_n)_{n\geq 0}$ be a sequence of positive reals $\lambda_n > 0$ decreasing fast to 0, $\lambda_n \to 0$. We assume that

$$\sum_{k=0}^{+\infty} 2^k \lambda_k = 1 \ . \tag{2}$$

We consider the probability measure

$$\nu = \sum_{k=0}^{+\infty} \lambda_k \sum_{l=1}^{2^k} \delta_{x_{k,l}}$$

where $(x_{k,l})_l$ are the end-points of order $k \geq 0$ defining the Cantor set K.

Proposition 3.1. *We assume that the sequence $(\lambda_k)_{k\geq 0}$ decreases fast enough so that for some constant $\kappa > 3$*

$$\sum_{k=0}^{+\infty} \lambda_k (2\kappa)^k < +\infty \ .$$

For a dense and F_σ set of good points $G_0 \subset K$ of full ν-measure, we have that for $x_0 \in G_0$ the primitive of the Cauchy transform F_ν is a continuous function on the vertical line $x_0 + i\mathbb{R}$. In particular, there is a finite limit $\lim_{y\to 0} F_\nu(x_0 + iy) = F_\nu(x_0)$.

Proof. Consider the set $G_0 \subset K$ of points badly approximated by end-points $(x_{k,l})$,

$$G_0 = K - \bigcap_{k_0 \geq 0} \bigcup_{k \geq k_0} \bigcup_{l=1}^{2^k} B(x_{k,l}, \kappa^{-k})$$

Then G_0 is a dense and F_σ set and of total ν-measure.

For any $x_0 \in K_0$, and $z = x_0 + iy$, $y \in \mathbb{R}$, we have for some $k_0 \geq 1$, for $k \geq k_0$, $1 \leq l \leq 2^k$,

$$\left| \frac{\lambda_k}{z - x_{k,l}} \right| \leq \left| \frac{\lambda_k}{x_0 - x_{k,l}} \right| \leq \lambda_k \kappa^k$$

and

$$\left| \sum_{l=1}^{2^k} \frac{\lambda_k}{z - x_{k,l}} \right| \leq \lambda_k (2\kappa)^k$$

thus condition (2) proves that the series

$$F_\nu(z) = \frac{1}{2\pi i} \sum_{k=0}^{+\infty} \sum_{l=1}^{2^n} \frac{\lambda_k}{z - x_{k,l}}$$

is normally convergent and bounded, and defines a continuous function on $x_0 + i\mathbb{R}$. $\square$

Proposition 3.2. *The primitive of the Cauchy transform F_ν is well defined, continuous and bounded on the full Lebesgue set $(\mathbb{C} - K) \cup G_0$ and holomorphic in $\mathbb{C} - K$.*

Proof. We only need to check continuity at points $x_0 \in G_0$ and this follows observing that the series in the precedent proof converges normally in G_0. □

So, we can extend continuously the function F_ν on vertical lines $x_0 + i\mathbb{R}$ with $x_0 \in G_0$. This holds for any path crossing the real axes at $x_0 \in G_0$ with a non-zero angle. This defines the Borel extension of F_ν on this path (see section 6). We can define the larger set $G_1 \subset K$ where this extension holds.

Definition 3.3. *Let $G_1 \subset K$ such that for $x_1 \in G_1$ we have that the restriction of F_ν to $(x_1 + i\mathbb{R}) - \{x_1\}$ has a continuous extension to the full vertical line $x_1 + i\mathbb{R}$. We have $G_0 \subset G_1 \subset K$ and the set $P_\nu = K - G_1$ is the monogenic polar set.*

Theorem 3.4. *The values of different branches of F_ν at $z = 0$ by Borel continuation is the Borel monodromy*

$$B_\nu = \text{B-Mon}_{z=0} \, F_\nu \subset \mathbb{R}$$

that is an uncountable additive sub-group. In particular, it contains the uncountable subset

$$\{\nu([0, x_1]); x_1 \in G_1\} \subset B_\nu$$

The subgroup $M_\nu \subset \mathbb{R}$ also contains the countable sub-group of the regular Weierstrass monodromy,

$$W_\nu = \bigoplus_{n \geq 1} 2^{-n} \lambda_n \mathbb{Z} \subset B_\nu \subset \mathbb{R} \ .$$

Proof. We can cross the Cantor set K in the uncountable set of points in G_1, hence we can split the total measure in two parts of mass $\nu([-1/2, x_1])$ and $\nu([x_1, 1/2])$ by a simple Jordan loop from 0 crossing K at $x_1 \in G_1$. If the Jordan curve crosses the real line only at one point $x_1 \in G_1$, and the Jordan curve is positively oriented, then the Monogenic Residue Formula in Proposition 2.2 proves that we have the monodromy value $\nu([0, x_1])$. The computation of the regular Weierstrass monodromy follows from the same argument as in Proposition 2.4 observing that the mass of each piece at level $n \geq 1$ building-up the Cantor set is $2^{-n}\lambda_n$. □

4. The restricted Cantor Riemannium.

4.1. **The Krueger hand, ramification locus and monogenic topology.** As for the Riemann surface associated to the Weierstrass continuation, we want to understand the minimal space where the monogenic function F_ν is univalued (or monodromic in ancient terminology). This is the Riemannium space associated to F_ν.

The Riemann surface $\mathcal{S}_\nu$ associated to Weierstrass continuation is similar to the tube-log Riemann surface $\mathcal{S}_\nu$ (see Figure 2), but this time we have an uncountable number of "finger tubes" with finite length, i.e. where the imaginary part of F_ν is bounded from above. The tips at height $\leq H$ form a closed set that corresponds to a Cantor subset in G_1. The external rays of bounded height (finite length "fingers") form a smooth fan in the sense defined in [9]. When the tips are dense (this requires some homogeneity property of the measure ν) then we have a Lelek fan which enjoys some unique topological characterization [16]. We call this a "Krueger hand" (see Figure 1). The Krueger hand plays the role of the 0-sheet of the Riemannium space associated to F_ν. The tips of the finger tubes of finite length do correspond to the perfect subset $G_1 \subset K$, so they correspond to angles θ

such that $\sigma(\theta) = x_1 \in G_1$. For these points $x_1 \in G_1$, when we continue upward along the vertical $\sigma(\theta) + i\mathbb{R}_-$ and we cross the tip finger corresponding to x_1, we appear in the corresponding upper vertical line $\sigma(\bar{\theta}) + i\mathbb{R}_+$ (note that the line orientation is reversed when we cross). But then the monodromy changes by addition of the non-dyadic rational $\theta \in \mathbb{T}_0$. Thus, we do appear in a new copy. It would be suitable to formalize the definition of $\mathcal{S}_\nu$ by means of a sheaf theory associated to a mixed sheaf structures of holomorphic germs at regular points $z \in \mathbb{C} - K$ and path germs of Borel monogenic functions at points $x_1 \in G_1$ (eventually modulo addition of a dyadic constant). In the example considered the construction is very concrete and we don't need such a general sheaf theory extension that will be developed elsewhere. The tube-log Riemann surface $\mathcal{S}_\nu$ comes equipped with a log-euclidean metric and we can define its ramification locus as its Cauchy completion as in [1], [3] and [17].

Definition 4.1 (Ramification locus)**.** *The ramification locus $\mathcal{R}_\nu$ of $\mathcal{S}_\nu$ is the completion of $\mathcal{S}_\nu$ for its log-euclidean metric. The completion $\mathcal{S}_\nu^* = \mathcal{S}_\nu \cup \mathcal{R}_\nu$ is a path connected length space extending the monogenic metric.*

Now, we can consider the monogenic metric on $\mathbb{C} - K$.

Definition 4.2 (Monogenic metric)**.** *The monogenic metric on $\mathbb{C} - K$ is the pull-back of the log-euclidean metric on $\mathcal{S}_\nu$ by the map $F_\nu : \mathbb{C} - K \to \mathcal{S}_\nu$.*

Proposition 4.3 (Monogenic completion of $\mathbb{C} - K$)**.** *Recall that $P_\nu = K - G_1$ is the polar set. We can embed $\mathcal{S}_\nu^*$ into $\mathbb{C}$ by extending the inverse $F_\nu^{-1} : \mathbb{C} - K \to \mathcal{S}_\nu$ to $F_\nu^{-1} : \mathbb{C} - P_\nu \to \mathcal{S}_\nu^*$. The monogenic topology of $\mathcal{S}_\nu^*$ defines a topology on $\mathbb{C} - P_\nu$ that is weaker that the standard topology in the plane. The space $\mathbb{C} - P_\nu$ is exhausted by the the monogenic closed metric balls $(\bar{B}(z_0, n))_{n\geq 1}$, so it is an F_σ set for the monogenic topology. The closed monogenic balls are not closed for the standard topology.*

For an approximate description of the monogenic topology see [5] (1912).

4.2. **The Restricted Riemannium space associated to F_ν.** By pasting copies of the Krueger hand, can define now a Restricted Cantor Riemannium C_ν^r (super-index r for "restricted")that is the space where the Borel extension of the monogenic function F_ν along paths that intersect G_1 in a finite set. The natural extension space is larger and we define it in the next section, but the construction in this section is very concrete and worth doing it first.

Definition 4.4 (Restricted Cantor Riemannium)**.** *We construct the Restricted Cantor Riemannium C_ν^r as follows:*

- *We consider an uncountable number of copies $(\mathcal{S}_\nu^*(\boldsymbol{\theta}))_{\boldsymbol{\theta}}$ of the completed Krueger hand $\mathcal{S}_\nu^*$ indexed by finite sequences $\boldsymbol{\theta} = (\theta_n)_{1\leq n\leq n(\boldsymbol{\theta})}$ of angles $\theta_n \in \sigma^{-1}(G_1)$. We include a copy $\mathcal{S}_\nu^*(\emptyset)$ corresponding to the empty sequence, which is named the principal sheet. Let $j_{\boldsymbol{\theta}} : \mathcal{S}_\nu^* \to \mathcal{S}_\nu^*(\boldsymbol{\theta})$ be the identification mappings.*
- *We glue to the copy $\mathcal{S}_\nu^*(\boldsymbol{\theta})$ all copies $\mathcal{S}_\nu^*(\boldsymbol{\theta'})$, such that $n(\boldsymbol{\theta'}) = n(\boldsymbol{\theta}) + 1$ at a single point by identifying $F_\nu(\sigma(\theta'_{n(\boldsymbol{\theta'})})) \in \mathcal{S}_\nu(\boldsymbol{\theta})$ to $F_\nu(\sigma(\bar{\theta}'_{n(\boldsymbol{\theta'})})) \in \mathcal{S}_\nu^*(\boldsymbol{\theta'})$ when $\sigma(\theta'_{n(\boldsymbol{\theta'})}) = x_0 \in G_1$. That is, for $x_0 \in G_1$, $x_0 = \sigma(\theta'_{n(\boldsymbol{\theta'})}) = \sigma(\bar{\theta}'_{n(\boldsymbol{\theta'})})$*

$$j_{\boldsymbol{\theta}}(F_\nu(x_0)) = j_{\boldsymbol{\theta'}}(F_\nu(x_0))$$

The space obtained $\hat{C}_\nu^r$ is path connected and has a well-defined continuous projection map

$$\pi : \hat{C}_\nu \to \mathcal{S}_\nu^*$$

such that for $z \in \mathcal{S}_\nu^*(\boldsymbol{\theta})$, $\pi(z) = j_{\boldsymbol{\theta}}(z)$.

- *We consider a copy $\mathcal{S}^*(\boldsymbol{\theta})$ in $\hat{C}_\nu^r$. It can be connected with the principal sheet $\mathcal{S}_\nu^*(\emptyset)$ by a path $\gamma_{\boldsymbol{\theta}}$ starting at $j_\emptyset(0) \in \mathcal{S}_\nu^*(\emptyset)$, ending at $j_{\boldsymbol{\theta}}(0) \in \mathcal{S}_\nu^*(\boldsymbol{\theta})$, going through $\mathcal{S}_\nu^*(\boldsymbol{\theta}_1) = \mathcal{S}_\nu^*(\boldsymbol{\theta}_2), \ldots, \mathcal{S}_\nu^*(\boldsymbol{\theta}_{n(\boldsymbol{\theta})})$, where $\boldsymbol{\theta}_j = (\theta_1, \ldots, \theta_j)$, and crossing the sewing points transversally to what corresponds to the real axes in each copy.*

  We denote by $M(\boldsymbol{\theta}) \in \mathbb{C}$ the Borel monodromy of F_ν along $\pi(\gamma_{\boldsymbol{\theta}})$ that is well defined by virtue of Theorem 3.4.
- *We identify the copies $\mathcal{S}^*(\boldsymbol{\theta})$ and $\mathcal{S}^*(\boldsymbol{\theta}')$ if and only if $M(\boldsymbol{\theta})$ and $M(\boldsymbol{\theta}')$ are equal modulo the dyadic rationals,*

$$M(\boldsymbol{\theta}) \equiv M(\boldsymbol{\theta}') \quad \left[mod \ \bigoplus_{k \geq 0} 2^{-k} \lambda_k \mathbb{Z} \right] \tag{3}$$

  This defines an equivalence relation in $\hat{C}_\nu^r$ and the topological quotient $C_\nu^r = \hat{C}_\nu^r / \sim$ is the Restricted Cantor Riemannium.

Observe that the projection mapping is compatible with the equivalence relation (equivalent classes are contained in the fibers of π) and defines a projection mapping (still denoted by π),

$$\pi : C_\nu^r \to \mathcal{S}_\nu^* \approx \mathbb{C} - P_\nu$$

Proposition 4.5. *The Restricted Cantor Riemannium C_ν^r is a Hausdorff topological space that is path connected and not σ-compact. It is a metric length space. The Restricted Cantor Riemannium has an open dense part composed by an uncountable number of disjoint log-Riemann surfaces for the projection π,*

$$C_\nu^0 = \pi^{-1}(\mathcal{S}_\nu) \subset C_\nu^r$$

We call C_ν^0 the regular part of the Riemannium. The projection mapping $\pi : C_\nu^r \to \mathcal{S}_\nu^*$ is a local holomorphic diffeomorphism and, for the log-euclidean metric, a local isometry on the regular part, and a contraction elsewhere.

Proof. The space is not σ-compact since we attach to $\mathcal{S}_\nu(0)$ an uncountable number of copies. The log-euclidian metric of $\mathcal{S}_\nu$ defines the length space metric in the copies that build C_ν. The connections between copies are through a single point so the length metric extends along the crossing paths and in the resulting length space the copies are embedded isometrically. The other properties are clear. □

Observe that $\pi : C_\nu^r \to \mathcal{S}_\nu^*$ is not a classical covering of topological spaces since for any $\tilde{x}_1 \in \pi^{-1}(x_1)$ for $x_1 \in G_1$ no neighborhood of $\tilde{x}_1$ is homeomorphic to a neighborhood of x_1. Even worse, we have infinite local degree.

We recall that the Uniformization Theorem holds in the category of tube-log-Riemann surfaces. Let $\tilde{\mathcal{S}}_\nu$ be the universal covering of the tube-log-Riemann surface $\mathcal{S}_\nu$ which means that $\tilde{\mathcal{S}}_\nu$ is a simply connected log-Riemann surface and we have a covering $p_0 : \tilde{\mathcal{S}}_\nu \to \mathcal{S}_\nu$ compatible with the projection mappings $\pi_0 : \tilde{\mathcal{S}}_\nu \to \mathbb{C}$, which means that locally $p_0 \circ \pi_0^{-1}$ is a translation, and p_0 is a local isometry for their log-euclidean metrics.

Another way to understand $\tilde{\mathcal{S}}_\nu$ is to "unfold" the cylinders of $\mathcal{S}_\nu$ and it is a log-Riemann surface that can be constructed by means of straight cuts and pastings. In Figure 3 we show a fundamental domain with the obvious identifications (indicated by dotted lines) to get $\tilde{\mathcal{S}}_\nu$ (for $\tilde{\mathcal{S}}_\nu$ is similar, with widths corresponding to the mass ν). The full 0-sheet with cuts is obtaining by extending this fundamental domain by $\mathbb{Z}$-periodicity. This 0-sheet corresponds to the "unfolding" of the root-cylinder. Note that the set of cuts in this 0-sheet for $\tilde{\mathcal{S}}_\nu$ do not have a discrete set of end-points (for $\tilde{\mathcal{S}}_\nu$ this set is discrete in the 0-sheet) .

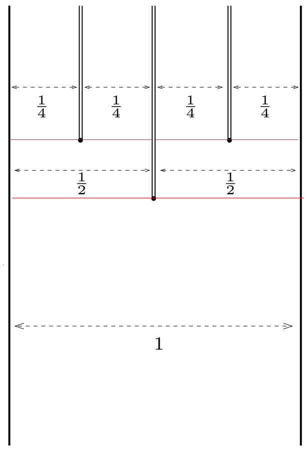

FIGURE 3. Fundamental domain for $\tilde{\mathcal{S}}_\nu$.

As usual for log-Riemann surfaces we can consider its Cauchy completion for the log-euclidean metric. Then we can start with $\tilde{\mathcal{S}}_\nu^*$ and carry out the same Cantor Riemannium construction as before. After the first two steps we get the space $\check{C}_\nu^r$ that is a simply connected topological space (all loops are contractible). The pasting construction is compatible with the projection $\pi_0 : \tilde{\mathcal{S}}_\nu^* \to \mathbb{C}$ in the copies and we get a projection $\check{\pi} : \check{C}_\nu^r \to \mathbb{C}$. The regular part of $\check{C}_\nu^0$ is defined as before. Despite having the projection $\check{C}_\nu^0 \to \mathbb{C} - P_\nu$, it cannot really be identified with the universal cover of $\mathbb{C} - P_\nu$ because the lift property for paths does not hold. We can only lift paths in $\mathbb{C} - P_\nu$ such that when we remove its end-point have a discrete intersection with G_1 transversal to $\mathbb{R}$. If we consider loops in $\mathbb{C} - P_\nu$ with base point at a regular point $z_0 \in \mathbb{C} - P_\nu$, for example $z_0 = 0$, with a finite intersection with G_1 and transversal with $\mathbb{R}$ we have a subgroup $\pi_1^r(\mathbb{C} - P_\nu, z_0)$ of $\pi_1(\mathbb{C} - P_\nu, z_0)$. The group of "deck transformations" of the map $\check{C}_\nu \to \tilde{\mathcal{S}}_\nu^* \to \mathcal{S}_\nu^* \approx \mathbb{C} - P_\nu$ (it is not a classical cover as observed before) is the quotient group $\pi_1(\mathbb{C} - P_\nu, z_0)/\pi_1^r(\mathbb{C} - P_\nu, z_0)$.

Then we can consider the quotient of $\check{C}_\nu$ as in the last point of the construction in Definition 4.4 and get

$$\tilde{C}_\nu^r = \check{C}_\nu^r / \sim$$

where this time the equivalence relation $\mathcal{S}^*(\boldsymbol{\theta}) \sim \mathcal{S}^*(\boldsymbol{\theta'})$ holds if and only if $M(\boldsymbol{\theta}) = M(\boldsymbol{\theta'})$.

We get in this way a restricted Riemannium space $\tilde{C}_\nu$ with a projection mapping

$$\tilde{\pi} : \tilde{C}_\nu^r \to \mathbb{C}$$

and we have a fiber-compatible mapping

$$\tilde{p} : \tilde{C}_\nu^r \to C_\nu$$

that is a sort of "universal cover Riemannium". We have the correspondence of the regular parts

$$\tilde{p}(\tilde{C}_\nu^0) = C_\nu^0$$

and $\tilde{p}$ is a local isometry on the regular part for the log-euclidean metrics. We have the commutative diagram

$$\begin{array}{ccccc} \check{C}^r_\nu & \longrightarrow & \tilde{C}^r_\nu & \longrightarrow & \tilde{\mathcal{S}}^*_\nu \\ \downarrow & & \downarrow & & \downarrow \\ \hat{C}^r_\nu & \longrightarrow & C^r_\nu & \longrightarrow & \mathcal{S}^*_\nu \end{array}$$

We would like to carry out a similar construction as the classical algebraic topology construction of the universal cover but we have several difficulties. The first one is related to the non-local simple connectivity of $\mathbb{C} - P_\nu$ at points of G_1. We can construct Hawaian earrings at each point $x_1 \in G_1$ by taking a bouquet of circles with base point at x_1 and crossing transversally $\mathbb{R}$ at another point of G_1. But more importantly, we don't have the path lift property: There are paths that cannot be lifted to the Cantor Riemannium. For example, take a path γ that wiggles between the upper and lower half planes determined by $\mathbb{R}$ and converging to a point $x_\infty \in G_1$ and after reaching x_∞ goes into the upper half plane (see Figure 4).

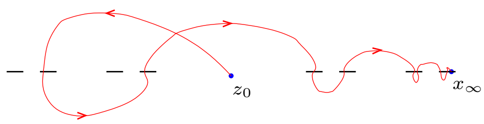

FIGURE 4. Wiggling path.

There is no lift of γ in $\check{C}_\nu$ because there is no sheet where the end-point of the lift belongs to.

The reason for this phenomenon is related to the fact that the Riemanniums $\hat{C}^r_\nu$ and $\tilde{C}^r_\nu$ are not complete for the log-euclidean metric, despite that the sheets $\tilde{\mathcal{S}}^*_\nu$ and $\mathcal{S}^*_\nu$ are complete.

Proposition 4.6. *The restricted Riemanniums $\hat{C}^r_\nu$ and $\tilde{C}^r_\nu$ are not complete.*

Proof. We can take a rectifiable path γ of finite length for the log-Euclidean metric as before. And a monotone Cauchy sequence $(z_n) \subset$ converging in $\mathbb{C} - P_\nu$ to x_1 for the log-euclidean metric in $\mathbb{C} - P_\nu$. There is a unique lift of $\gamma - \{x_\infty\}$ in $\hat{C}_\nu$ (or $\tilde{C}_\nu$) but it has no limit point because of the discrete topology on the fiber of x_∞. $\square$

The proof gives a hint on how to solve this problem: We must enrich the topology of $\hat{C}^r_\nu$ and introduce a natural non-discrete topology on the fibers of $\hat{C}^r_\nu \to \mathbb{C} - G_1$.

To each point in a fiber $z \in \pi^{-1}(x_1)$ with $x_1 \in G_1$ there corresponds a sequence $\boldsymbol{\theta}(z)$ in the construction. We define a distance on the fiber by

$$d_0(z_1, z_2) = |M(\boldsymbol{\theta}(z_1)) - M(\boldsymbol{\theta}(z_2)|$$

This defines a metric on each fiber that makes it complete. And the fiber is no longer discrete as for a classical covering.

Now, let $x_\infty \in G_1$ and a path γ as before, starting at the regular base point z_0 and ending at x_∞ avoiding P_ν, thus $\gamma \subset \mathbb{C} - P_\nu$, and crossing transversally the real axes except at the end-point. We can homotop γ in $\mathbb{C} - K$ into γ' so that $\gamma \cap G_1 = \gamma' \cap \mathbb{R}$, by just moving the connected components of $\gamma - \Gamma_1$ that intersect the real line into the upper or lower half plane. Now, for the sequence of intersection points $(x_n) \in \gamma' \cap G_1$, $x_n \to x_\infty$ we know in which sheet we are by looking at the value of the corresponding $M(\boldsymbol{\theta}_n)$. The limit of the sequence $(M(\boldsymbol{\theta}_n))_{n\geq 1}$ indicated in which sheet is the limit in the fiber $\pi^{-1}(x_\infty)$, and we have

$$\lim_{n\to+\infty} M(\boldsymbol{\theta}_n) = M(\theta_\infty)$$

Then the path passes through the sheet determined by $M(\theta_\infty)$ and connects to the one corresponding to $M(\bar{\theta}_\infty)$. This suggests that we have to do a further quotient of $\check{C}^r_\nu$ on the fibers $\pi^{-1}(x_\infty)$. This will give the unrestricted Cantor Riemannium $\check{C}^r_\nu$.

But we would like to have this extension constructed not only for paths with a countable transversal intersection with $\mathbb{R}$ but those with a non-discrete intersection with G_1. The same procedure works, but for this purpose we can do the general construction of the next section.

5. The Cantor Riemannium.

5.1. **Monogenic algebraic topology.** We carry out a similar construction[6] of the universal covering in algebraic topology but for the non-locally simply connected space $X = \mathbb{C} - P$. We refer to [14], [13], and in particular to [7] for minimal hypothesis for classical results for relation of fundamental groups and coverings. The natural topology on the fundamental group, is the *admissible topology* defined by N. Bourbaki[7] in [7], Chap. III, 5.4, p.315. In our non locally simply connected situation, the admissible topology is not discrete. Also we don't have coverings with discrete fibers, but a general projection with non-discrete fibers. The key property to carry out a similar construction of the universal covering is to use projections with the path-lifting property. After figuring this out we learned about the excellent exposition by E.H. Spanier in his classical book [23], in particular chapter 2, where all the strength of the path-lifting property is used for Hurewicz fibrations. But the projections we have are not a priori Hurewicz fibrations. The situation we have is closer to Serre fibrations in one topological dimension where we have the lifting of path homotopies.

We remind that the space $X = \mathbb{C} - P$ (we simplify the notation in this section by removing the reference to ν that is fixed once for all) can be endowed with the regular planar topology or with the monogenic topology, that is more natural for our purposes. For both topologies, the space X is path-connected, hence connected, separable, and locally path-connected, but not semilocally simply connected, not even locally simply connected. More precisely, each point $x_1 \in G_1$ is the base point for a uncountable Hawaian earring as is easily seen by taking a family of circles $(C_{x_1,x_2})_{x_2\in G_1}$ cutting perpendicularly the real axes at $x_1, x_2 \in G_1$ (See Figure 5).

A path γ in $\mathbb{C} - P$ is essential or we say that γ has an essential intersection at $x_1 \in \gamma \cap G_1$ if the point x_1 does not disconnect γ into two connected components, such that the closure of one is not a loop η such that $\eta - \{x_1\} \subset \mathbb{C} - K$ (see Figure 6).

We observe that if γ_1 and γ_2 are two homotopic paths in X with fixed end-points, $\gamma_1 \sim \gamma_2$, and with essential intersections with G_1, then

$$\gamma_1 \cap G_1 = \gamma_2 \cap G_1$$

because all points of G_1 are inner points of the Cantor set K that are "squeezed" between a left and right sequences of points in P. For non-essential intersections at x_1, we could locally slide

[6] I am indebted to K. Biswas to point out this approach.
[7] Special thanks to André Gramain.

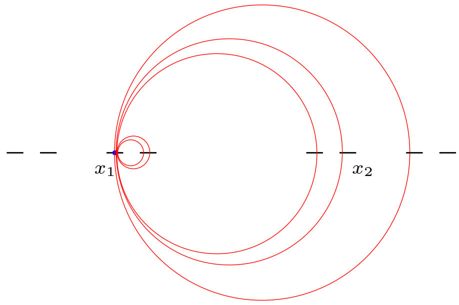

FIGURE 5. Hawaian earring based on x_1.

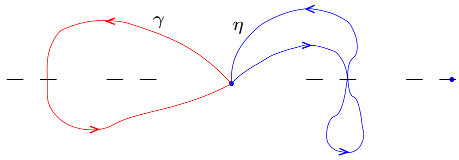

FIGURE 6. An essential loop γ and a non-essential one η.

homotopically the path to leave the intersection at x_1. Hence, under these conditions, γ_1 and γ_2 are homotopic in X if and only if they are homotopic relative to G_1. Any continuous path for the monogenic topology is also continuous or the regular topology, and Borel homotopic equivalence is equivalent to homotopic equivalence for the regular topology.

We can define in general:

Definition 5.1. *Let X be a path connected topological space and $z_0 \in X$ a base point. We define*

$$\tilde{X} = \{[\gamma]; \gamma : [0,1] \to X \text{ continuous}, \gamma(0) = z_0\}$$

where $[\gamma]$ denotes the homotopy class of γ fixing its end-points.

We endow $\tilde{X}$ with the topology generated by the open sets

$$U_{[\gamma]} = \{[\gamma.\eta]; \eta([0,1]) \subset U \text{ loop with } \eta(0) = \eta(1) = \gamma(1)\}$$

where U is an open set of X and γ is a path in X.

The topological space constructed does not depend on the base point for a path connected space X. If $z_0' \in X$ is another base point, then a path γ from z_0 to z_0' defines a homeomorphism depending only on the homotopy class of γ, $h_{[\gamma]} : \tilde{X}(z_0) \to \tilde{X}(z_0')$, by

$$h_{[\gamma]}([\eta]) = [\eta.\gamma]$$

The topological space $\tilde{X}$ is path connected, and even locally path-connected by construction. We will see shortly that it is simply connected.

We use this definition for $X = \mathbb{C} - P$ endowed with the richer monogenic topology. By the precedent remarks, we don't really need Borel's topology to define *the set* $\tilde{X}$. But the topology defined by taking a bases with the $(U_{[\gamma]})$ when U runs over monogenic open sets endows $\tilde{X}$ with a richer topology.

Observe that since the monogenic topology is generated by the log-euclidean metric on $\mathbb{C} - P$, we could have taken as basis of neighborhoods

$$V_{[\gamma]}(\epsilon) = \{[\gamma.\eta]; \eta \text{ loop with } \eta(0) = \eta(1) = \gamma(1), \mathrm{diam}(\eta) < \epsilon\}$$

where $\mathrm{diam}(\eta)$ is the log-euclidean diameter. This comes down to request that $d_H(\gamma, \gamma.\eta) < \epsilon$ for the Hausdorff distance.

Proposition 5.2. *The projection mapping $\pi : \tilde{X} \to X$ defined by*

$$\pi([\gamma]) = \gamma(1)$$

is a continuous map and has the path-lifting property, and unique path-lifting property.

The path lifting and unique path-lifting property follows almost tautologically from the construction of $\tilde{X}$.

Corollary 5.3. *The space $\tilde{X}$ is simply connected, i.e. any loop is contractible.*

This result is independent of the base point, hence we can take a loop starting and ending at z_0 and carry out the same proof as in [14] p.65. The homotopy to the null loop is obtained by using the parametrization.

The projection π makes of $\tilde{X}$ a X-space in the sense of Bourbaki ([7], p.1). In the non-semilocally simply connected situation the projection π is not in general a local homeomorphism. In general, we have:

Proposition 5.4. *For an open set $U \subset X$, and $[\gamma] \in \tilde{X}$, with $\gamma(1) \in U$, we have that $\pi : U_{[\gamma]} \to U$ is continuous and surjective.*

In our particular situation, if $U \cap G_1 \neq \emptyset$ then we have that $\pi : V \to U$ has infinite degree (even uncountable, as we have an uncountable number of pre-images generated by the uncountable Hawaiian earrings). We say that π has local infinite degree at $\tilde{x}_1 \in p^{-1}(\{x_1\})$.

We are particularly interested in our particular situation in the case when X is a Gromov length space.

Proposition 5.5. *If X is a Gromov length space, then $\tilde{X}$ is also a Gromov length space and π is a contraction.*

Proof. The path-lifting property allows to pull-back the Gromov length distance into a Gromov length distance on $\tilde{X}$. □

Also, as in the classical situation, the group $\pi_1(X, z_0)$ acts on $\tilde{X}$. For a loop γ, starting and ending at z_0, the map $h_{[\gamma]}$ is a homeomorphism of $\tilde{X}$ that respects the fibers of π, so $\pi_1(X, z_0)$ acts on fibers. When we endow $\pi_1(X, z_0)$ with Bourbaki's admissible topology ([7], III, p.315) it becomes a topological group and this action is continuous and faithful on the fiber $\pi^{-1}(z_0)$.

Observe also that the fibers of π are all homeomorphic. For another point $z_1 \in X$, consider a path γ from the base point z_0 to z_1. Then $h_{[\gamma]}$ gives an homeomorphism from $\pi^{-1}(z_0)$ to $\pi^{-1}(z_1)$.

We observe that in our particular situation, $\pi_1(X, z_0)$ is not generated by a countable bases, as show the circles in a local uncountable Hawaiian earrings mentioned before.

In the classical theory, for X semi-locally simply connected, to each subgroup H of $\pi_1(X, z_0)$ it corresponds a quotient $X_H = \tilde{X}/\sim$ and a base point $\tilde{z}_0 \in X_H$, such that for the resulting covering $\pi_H : X_H \to X$, $(\pi_H)_*(\pi_1(X_H, \tilde{z}_0)) = H$. We have the same type of result in our general situation:

Proposition 5.6. *Let X be path connected and locally path-connected. For every subgroup $H \subset \pi_1(X, z_0)$ there is an equivalence relation $\sim$ on $\tilde{X}$ respecting fibers of π, such that for the quotient $X_H = \tilde{X}/\sim$ and the resulting projection $\pi_H : X_H \to X$, there is a base point $\tilde{z}_0 \in X_H$, $\pi_H(\tilde{z}_0) = z_0$, such that*

$$(\pi_H)_* (\pi_1(X_H, \tilde{z}_0)) = H$$

Proof. The proof is the same as Proposition 1.36 in [14]. For $[\gamma], [\gamma'] \in \tilde{X}$, we define $[\gamma] \sim [\gamma']$ if and only if $\gamma(1) = \gamma'(1)$ and $[\gamma.(\gamma')^{-1}] \in H$. This is an equivalence relation because H is a subgroup. Note also that if $\gamma(1) = \gamma'(1)$ we have $[\gamma] \sim [\gamma']$ if and only if $[\gamma.\eta] \sim [\gamma'.\eta]$ for any loop η with $\eta(0) = \eta(1) = \gamma(1) = \gamma'(1)$. So, if two points in $U_{[\gamma]}$ and $U_{[\gamma']}$ are identified, then the whole neighborhoods $U_{[\gamma]}$ and $U_{[\gamma']}$ are identified by the equivalence relation and the open sets $U_{[\gamma]}/\sim$ are a bases for the topology of the quotient X_H. The equivalence relation respects fibers of π since if $[\gamma] \sim [\gamma']$ then $\gamma(1) = \gamma'(1)$, thus $\pi_H : X_H \to X$ defined by $\pi_H(\overline{[\gamma]}) = \pi([\gamma])$ is a well defined continuous projection. Let $\tilde{z}_0 \in \tilde{X}$ be the point corresponding to the class $[z_0]$ of the constant path in X equal to the base point z_0. Choose for the base point in X_H, the image of $\tilde{z}_0$ by the quotient (that we still denote in the same way) . Now, the image $(\pi_H)_* (\pi_1(X_H, \tilde{z}_0))$ is exactly H, because a loop γ in X based at z_0 lifts to $\tilde{X}$ into a path starting at $[z_0]$ and ending at $[\gamma]$, thus the image of this lifted path in X_H is a loop if and only if $[\gamma] \sim [z_0]$, or $[\gamma] \in H$. □

5.2. **Construction of the Cantor Riemannium.** We consider in what follows $X = \mathbb{C} - P$ and we set $z_0 = 0$ for example. We have different relevant subgroups of $\pi_1(X, 0) = \pi_1(X)$.

We have a representation of $\pi_1(X)$ in the group of translation of $\mathbb{C}$, $\rho : \pi_1(X) \to \mathbb{C}$ given by

$$[\gamma] \mapsto \int_\gamma F_\nu(z)\, dz$$

The image $B_\nu = \rho(\pi_1(X))$ is an additive subgroup of $\mathbb{R}$ which represents the possible values of the different branches at 0, i.e.

$$B_\nu = \text{B-Mon}_{z=0}\, F_\nu$$

Let $H_0 = \operatorname{Ker} \rho$ be the kernel of this representation. We have that $H_0 \triangleleft \pi_1(X)$ is a normal closed subgroup of $\pi_1(X)$. Hence, the equivalence relation associated to H_0 is closed and the quotient X_{H_0} is separated.

Definition 5.7. *The Cantor Riemannium $\check{C}_\nu$ is the quotient $X_{H_0} = \tilde{X}/\sim$ corresponding to the subgroup H_0. It is the minimal space where the Borel extension of F_ν is univalued (or monodrome in ancient terminology).*

Let $H_1 = \pi_1^r(X) \subset \pi_1(X)$ be the subgroup of loops in $\mathbb{C} - P$ which have an essential intersection with G_1 (we add the null loop also to have a subgroup). This subgroup is closed.

Proposition 5.8. *The quotient space $X_{H_1} = \tilde{X}/\sim$ corresponding to the subgroup H_1 is homeomorphic to $\mathbb{C} - P \approx \mathcal{S}_\nu^*$.*

Let $H_2 \subset \pi_1(X)$ be the subgroup of loops in $\mathbb{C} - P$ which have an essential finite non-empty intersection with G_1 (we add the null loop also to have a subgroup). This subgroup is not closed.

Proposition 5.9. *The quotient space* $X_{H_2} = \tilde{X}/\sim$ *corresponding to the subgroup* H_2 *is the restricted Cantor Riemannium* $\tilde{C}_\nu^r$.

Let $H_3 \subset \pi_1(X)$ be the subgroup of loops in $\mathbb{C} - P$ which have an essential finite intersection with G_1 (that can be empty and we add the null loop also to have a subgroup). This subgroup is not closed.

Proposition 5.10. *The quotient space* $X_{H_3} = \tilde{X}/\sim$ *corresponding to the subgroup* H_3 *is the restricted Cantor Riemannium* C_ν^r.

Another interesting closed subgroup is $H_4 \subset \pi_1(X)$, the subgroup of loops contained in $\mathbb{C} - K$. The quotient space X_{H_4} is in some sense the "universal cover" of the non-regular part.

As in the classical situations, we can study the classification of "universal projections" above X, $Y \to X$, but we leave this for future work.

6. Borel extension of a holomorphic germ.

Our aim in this section is not to define the most general notion of Borel type extension, but just one that is has the uniqueness properties for the Cantor Riemannium construction.

We consider a holomorphic germ $F : U \to \mathbb{C}$ defined in a neigborhood U of $z_0 \in \mathbb{C}$ and a continuous path $\gamma : [0,1] \to \mathbb{C}$ without self-intersections such that

$$\gamma(0) = z_0$$
$$\gamma(1) = z_1$$

Definition 6.1. *Let* $\epsilon > 0$. *A continuous path* $\eta : [0,1] \to \mathbb{C}$ *without self-intersections such that*

$$\eta(0) = z_0$$
$$\eta(1) = z_1$$

is ϵ*-Jordan homotopic to* γ *if* $\eta([0,1])$ *and* $\gamma([0,1])$ *are* ϵ*-close in Hausdorff topology, and* $\eta([0,1]) \cup \gamma([0,1])$ *is a Jordan domain.*

Definition 6.2 (Polar measure)**.** *Let* $R \in \mathbb{C}(z)$ *be a rational function. The polar measure of* R *in the Riemann sphere* $\overline{\mathbb{C}}$ *is the atomic complex measure*

$$\mu_R = \sum_\rho (Res_\rho R)\ \delta_\rho$$

i.e. the purely atomic measure with support at the poles with complex atomic mass the residue at this point.

Definition 6.3. *For* $M > 0$, *we consider the space* $\mathcal{M}(M)$ *of rational functions with a polar measure of total mass bounded by* $M > 0$,

$$|\mu_R|(\mathbb{C}) \leq M < +\infty\ .$$

Definition 6.4. *Let γ be a path as before. An asymptotically holomorphic sequence (f_n) on γ is a sequence of meromorphic functions in $\mathcal{M}(M)$ for some $M > 0$, with simple poles out of $\gamma([0,1])$, such that for any $\epsilon > 0$ and neighborhood $V_\epsilon(\gamma([0,1]))$ we have*

$$\lim_{n\to+\infty} \mu_{f_n}(V_\epsilon(\gamma([0,1]))) = 0$$

where $V_\epsilon(\gamma([0,1]))$ is the ϵ-neighborhood of the support of γ.

Definition 6.5. *Let $F : U \to \mathbb{C}$ be an holomorphic germ and a path γ as before. If we have an asymptotically holomorphic sequence (f_n) on γ such that (f_n) converges uniformly on $\gamma([0,1])$ to a continuous function $f : \gamma([0,1]) \to \mathbb{C}$ such that $f_{/U} = F'$, then we define*

$$F(z_1) = \int_\gamma f(t)\, dt = \lim_{n\to+\infty} \int_\gamma f_n(t)\, dt$$

to be the Borel extension of F at z_1 along γ for the approximating sequence (f_n).

Proposition 6.6. *If (γ_k) is a sequence of ϵ_k-Jordan homotopic paths to γ with $\epsilon_k \to 0$, then*

$$\lim_{n,k\to+\infty} \int_{\gamma_k} f_n(t)\, dt = \int_\gamma f(t)\, dt$$

Proof. Let Ω_k be the Jordan domain with boundary $\gamma([0,1]) \cup \gamma_k([0,1])$. We have by the Monogenic Residue Formula, Proposition 2.2, when $k \to +\infty$,

$$\int_{\gamma_k} f_n(t)(t)\, dt - \int_{\gamma_k} f_n(t)(t)\, dt = \mu_{f_n}(\Omega_k) \to 0$$

and the result follows. $\square$

The definition in general does depend on the choice of the asymptotically holomorphic sequence. In the case where F' is the Cauchy transform of a measure μ supported on a Cantor set K, more precisely, and we approximate by a sequence (f_n) of Cauchy transforms then we have the following Theorem:

Proposition 6.7. *We assume that*

$$F'(z) = f_\mu(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{d\mu(t)}{z - t}$$

i.e. $\bar\partial\partial F = \mu$ in the sense of distributions is a finite total mass measure supported on $K = \operatorname{supp}\mu$ that is a Cantor set. We also assume that $\mu(\gamma([0,1])) = 0$.

If we take an asymptotically holomorphic sequence (f_n) in Definition 6.5 such that $F(z_1) \neq \infty$, then the extension $F(z_1)$ is independent of the chosen sequence.

More generally, we can take a sequence (f_n) of Cauchy transforms of measures $(\mu_{f_n}) \subset \mathcal{M}(M)$, for some $M > 0$, such that $\mu_{f_n} \to \mu$ and $\operatorname{supp}(\mu_{f_n}) \to \operatorname{supp}(\mu)$.

Lemma 6.8. *Given a Cantor set $K \subset \mathbb{C}$ and γ a continuous non-self-intersecting path, for any $\epsilon > 0$ we can find an ϵ-Jordan homotopic path such that it does not intersect K except eventually at the end-points.*

Proof. We use the useful result that homeomorphisms of the plane act transitively on Cantor subsets (see [18] Theorem 7 p.93). Note that this is false in dimension 3 as the Antoine necklace $A \subset \mathbb{R}^3$

cannot be transported a Cantor set C on a segment since $\pi_1(\mathbb{R}^3 - A)$ is not trivial and $\pi_1(\mathbb{R}^3 - C)$ is trivial.

Therefore, we can find a global homeomorphism $h : \mathbb{C} \to \mathbb{C}$ that transforms K into the standard triadic Cantor set on $C \subset [0,1] \subset \mathbb{C}$. Then the result is easy to prove, we first perturb $h(\gamma)$ into a Jordan equivalent path that only intersects the real line at isolated points, and then we do a second perturbation locally at each such point that intersects C so that the new path does not intersect $C = h(K)$. The final observation is that Jordan small perturbation can be transported by the homeomorphism h. □

Proof. Using the previous Lemma we are reduced to the case when γ intersects K only at its end-point. Consider another such sequence (g_n). We have

$$\mu_{f_n - g_n} = \mu_{f_n} - \mu_{g_n}$$

thus $\mu_{f_n - g_n} \to 0$. By Lebesgue dominated convergence, the Cauchy transforms $f_{\mu_{f_n}}$ and $g_{\mu_{g_n}}$ converges pointwise to F' on $\mathbb{C} - K$. Since they are uniformly bounded on compact subsets of $\mathbb{C} - K$, they also converge uniformly on compact sets of $\mathbb{C} - K$ to F'. Again by Lebesgue dominated convergence on the integral along γ (we can replace γ by a Jordan homotopic rectifiable path) we have

$$F(z_1) = \lim_{n\to+\infty} \int_\gamma f_n(t)\, dt = \lim_{n\to+\infty} \int_\gamma g_n(t)\, dt = \int_\gamma F'(t)\, dt$$

□

When we consider the Borel extension along a path γ with self-intersections, but intersecting C at isolated points transversally to $\mathbb{R}$ where F_{ν_1} is finite, the relative homotopy class $[\gamma]_{\gamma\cap\mathbb{R}}$ (the intersections points being fixed) can be decomposed into the sum of homotopy classes of simple loops without self-intersections with the same intersections on C.

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