

# SATOSHI RISK TABLES

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ABSTRACT. We present Bitcoin Security Tables computing the probability of success  $p(z, q, t)$  of a double spend attack by an attacker controlling a share  $q$  of the hashrate after  $z$  confirmations in time  $t$ .

## 1. INTRODUCTION.

The main breakthrough in [2] is the solution to the *double spend problem* of an electronic currency unit without a central authority. Bitcoin is the first form of *peer-to-peer* (P2P) electronic currency.

A double spend attack can only be attempted with a substantial fraction of the hashrate used in the *Proof-of-Work* of the Bitcoin network. The attackers will start a *double spend race* against the rest of the network to replace the last blocks of the blockchain. The last section of [2] computes the probability that the attackers catch up. Following Nakamoto, by “success of the attackers” we mean catching up the  $z$  blocks, although to replace the blocks the attackers need to validate  $z + 1$ . In [1] we correct Nakamoto’s analysis based on an abusive approximation and give a closed-form formula for the exact probability.

**Theorem 1.** (C. Grunspan, R. Pérez-Marco, [1])

Let  $0 < q < 1/2$ , resp.  $p = 1 - q$ , the relative hash power of the group of the attackers, resp. of honest miners. After  $z$  blocks have been validated by the honest miners, the probability of success of the attackers is

$$P(z) = I_{4pq}(z, 1/2) ,$$

where  $I_x(a, b)$  is the Regularized Incomplete Beta Function

$$I_x(a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^x t^{a-1}(1-t)^{b-1} dt .$$

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We carry out a more accurate risk analysis by considering not only the number of confirmations  $z$  but also the time  $t$  it took for the last  $z$  validated blocks, which is an information that is clearly available.

In practice, in order to avoid a double spend attack, the recipient of the bitcoin transaction waits for  $z \geq 1$  confirmations. He also has the information on the time  $t$  it took to confirm the transaction  $z$  times. Obviously the probability of success of the attackers increases with  $t$  since he has more time to secretly mine his alternative blockchain. In [1] we carry out a more precise risk analysis considering this available data. The relevant dimensionless parameter introduced is the relative deviation from the expected time

$$\kappa = \frac{t}{z\tau_0} = \frac{pt}{z\tau_0} = \frac{1-q}{z} \frac{t}{\tau_0},$$

where  $\tau_0$  is the expected validation time of a new block ( $\tau_0 = 10$  min for the Bitcoin network). In [1] we give a closed-form formula for the probability  $P(z, q, \kappa)$  that the attackers catch up with the current chain.

**Theorem 2.** (C. Grunspan, R. Pérez-Marco, [1])

*We have*

$$P(z, q, \kappa) = 1 - Q(z, \kappa z q/p) + \left(\frac{q}{p}\right)^z e^{\kappa z \frac{p-q}{p}} Q(z, \kappa z).$$

After a validation  $z$  has been observed in the network and the time  $t$  being measured, and  $r = \frac{t}{z\tau_0}$ , one can compute the probabilities

$$\tilde{P}(z, q, r) = P(z, q, (1-q)r),$$

that give the probability of the attackers to catch up the current blockchain. We tabulate these probabilities for  $z = 1, 2, \dots, 10$ .

## 2. USE OF THE TABLES.

The tables are useful to determine the number of confirmations needed for a given transaction. In practice, someone receiving a bitcoin transaction will check after  $z$  confirmations the corresponding table. With the measured time  $t$ , he will compute  $r = \frac{t}{z\tau_0}$ , and check the value of  $\tilde{P}(z, q, r)$  and then asses the level of risk assumed accepting the payment.

## 3. SATOSHI RISK TABLES.

Below we give the tables for  $z = 1, 2, \dots, 9$  of  $\tilde{P}(z, q, r)$  for different values of  $q$  and  $r$  in ‰ with 2 decimal places. These are what is needed for practical applications.

|       |
|-------|
| $z=1$ |
|-------|

| $r \backslash q$ | 0.02 | 0.04  | 0.06  | 0.08  | 0.1   | 0.12  | 0.14  | 0.16  | 0.18  | 0.2   | 0.22  | 0.24  | 0.26  |
|------------------|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.1              | 2.24 | 4.55  | 6.94  | 9.42  | 12    | 14.67 | 17.44 | 20.33 | 23.34 | 26.49 | 29.77 | 33.2  | 36.8  |
| 0.2              | 2.43 | 4.93  | 7.5   | 10.14 | 12.87 | 15.68 | 18.59 | 21.6  | 24.71 | 27.94 | 31.3  | 34.79 | 38.42 |
| 0.3              | 2.63 | 5.31  | 8.05  | 10.86 | 13.74 | 16.69 | 19.72 | 22.84 | 26.05 | 29.37 | 32.79 | 36.33 | 40    |
| 0.4              | 2.82 | 5.69  | 8.6   | 11.57 | 14.6  | 17.68 | 20.84 | 24.07 | 27.37 | 30.77 | 34.25 | 37.84 | 41.54 |
| 0.5              | 3.02 | 6.06  | 9.15  | 12.28 | 15.45 | 18.67 | 21.94 | 25.27 | 28.67 | 32.14 | 35.68 | 39.32 | 43.04 |
| 0.6              | 3.21 | 6.44  | 9.69  | 12.97 | 16.29 | 19.64 | 23.02 | 26.46 | 29.94 | 33.48 | 37.08 | 40.76 | 44.5  |
| 0.7              | 3.4  | 6.81  | 10.23 | 13.67 | 17.12 | 20.59 | 24.09 | 27.62 | 31.19 | 34.8  | 38.45 | 42.16 | 45.93 |
| 0.8              | 3.6  | 7.18  | 10.77 | 14.36 | 17.95 | 21.54 | 25.15 | 28.77 | 32.42 | 36.09 | 39.79 | 43.53 | 47.32 |
| 0.9              | 3.79 | 7.56  | 11.3  | 15.04 | 18.76 | 22.48 | 26.19 | 29.9  | 33.62 | 37.35 | 41.1  | 44.87 | 48.67 |
| 1                | 3.98 | 7.92  | 11.83 | 15.72 | 19.57 | 23.4  | 27.22 | 31.02 | 34.81 | 38.6  | 42.38 | 46.18 | 49.99 |
| 1.1              | 4.17 | 8.29  | 12.36 | 16.39 | 20.37 | 24.32 | 28.23 | 32.11 | 35.97 | 39.81 | 43.64 | 47.45 | 51.27 |
| 1.2              | 4.36 | 8.66  | 12.89 | 17.05 | 21.16 | 25.22 | 29.23 | 33.19 | 37.11 | 41    | 44.86 | 48.7  | 52.52 |
| 1.3              | 4.55 | 9.02  | 13.41 | 17.71 | 21.95 | 26.11 | 30.21 | 34.25 | 38.24 | 42.17 | 46.06 | 49.92 | 53.74 |
| 1.4              | 4.75 | 9.39  | 13.93 | 18.37 | 22.72 | 26.99 | 31.18 | 35.29 | 39.34 | 43.32 | 47.24 | 51.1  | 54.93 |
| 1.5              | 4.94 | 9.75  | 14.44 | 19.02 | 23.49 | 27.86 | 32.14 | 36.32 | 40.42 | 44.44 | 48.38 | 52.26 | 56.08 |
| 1.6              | 5.13 | 10.11 | 14.95 | 19.67 | 24.25 | 28.72 | 33.08 | 37.33 | 41.48 | 45.54 | 49.51 | 53.4  | 57.21 |
| 1.7              | 5.32 | 10.47 | 15.46 | 20.31 | 25.01 | 29.57 | 34.01 | 38.33 | 42.53 | 46.62 | 50.61 | 54.5  | 58.31 |
| 1.8              | 5.5  | 10.82 | 15.97 | 20.94 | 25.75 | 30.41 | 34.93 | 39.31 | 43.55 | 47.67 | 51.68 | 55.58 | 59.38 |
| 1.9              | 5.69 | 11.18 | 16.47 | 21.57 | 26.49 | 31.24 | 35.83 | 40.27 | 44.56 | 48.71 | 52.73 | 56.63 | 60.42 |
| 2                | 5.88 | 11.53 | 16.97 | 22.2  | 27.22 | 32.06 | 36.73 | 41.22 | 45.55 | 49.73 | 53.76 | 57.66 | 61.44 |
| 2.1              | 6.07 | 11.89 | 17.47 | 22.82 | 27.95 | 32.87 | 37.6  | 42.15 | 46.52 | 50.72 | 54.77 | 58.67 | 62.43 |
| 2.2              | 6.26 | 12.24 | 17.96 | 23.43 | 28.66 | 33.68 | 38.47 | 43.07 | 47.47 | 51.7  | 55.75 | 59.65 | 63.39 |
| 2.3              | 6.44 | 12.59 | 18.45 | 24.04 | 29.37 | 34.47 | 39.33 | 43.97 | 48.41 | 52.65 | 56.71 | 60.6  | 64.33 |
| 2.4              | 6.63 | 12.94 | 18.94 | 24.65 | 30.08 | 35.25 | 40.17 | 44.86 | 49.33 | 53.59 | 57.66 | 61.54 | 65.25 |
| 2.5              | 6.82 | 13.29 | 19.42 | 25.25 | 30.77 | 36.02 | 41    | 45.74 | 50.23 | 54.51 | 58.58 | 62.45 | 66.14 |
| 2.6              | 7    | 13.63 | 19.91 | 25.84 | 31.46 | 36.78 | 41.82 | 46.6  | 51.12 | 55.41 | 59.48 | 63.34 | 67.01 |
| 2.7              | 7.19 | 13.98 | 20.38 | 26.43 | 32.14 | 37.54 | 42.63 | 47.44 | 51.99 | 56.29 | 60.36 | 64.21 | 67.85 |
| 2.8              | 7.38 | 14.32 | 20.86 | 27.02 | 32.82 | 38.28 | 43.43 | 48.28 | 52.85 | 57.16 | 61.22 | 65.06 | 68.68 |
| 2.9              | 7.56 | 14.66 | 21.33 | 27.6  | 33.49 | 39.02 | 44.22 | 49.1  | 53.69 | 58.01 | 62.07 | 65.89 | 69.48 |
| 3                | 7.75 | 15    | 21.8  | 28.18 | 34.15 | 39.75 | 44.99 | 49.91 | 54.52 | 58.84 | 62.89 | 66.7  | 70.27 |
| 3.1              | 7.93 | 15.34 | 22.27 | 28.75 | 34.8  | 40.46 | 45.76 | 50.7  | 55.33 | 59.65 | 63.7  | 67.49 | 71.03 |
| 3.2              | 8.11 | 15.68 | 22.74 | 29.32 | 35.45 | 41.18 | 46.51 | 51.49 | 56.13 | 60.45 | 64.49 | 68.26 | 71.77 |
| 3.3              | 8.3  | 16.02 | 23.2  | 29.88 | 36.1  | 41.88 | 47.25 | 52.26 | 56.91 | 61.24 | 65.26 | 69.01 | 72.5  |
| 3.4              | 8.48 | 16.35 | 23.66 | 30.44 | 36.73 | 42.57 | 47.99 | 53.01 | 57.68 | 62    | 66.02 | 69.74 | 73.2  |
| 3.5              | 8.66 | 16.69 | 24.12 | 30.99 | 37.36 | 43.26 | 48.71 | 53.76 | 58.43 | 62.76 | 66.76 | 70.46 | 73.89 |

$$z = 2$$

| $r \backslash q$ | 0.02 | 0.04 | 0.06 | 0.08  | 0.1   | 0.12  | 0.14  | 0.16  | 0.18  | 0.2   | 0.22  | 0.24  | 0.26  |
|------------------|------|------|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.1              | 0.05 | 0.21 | 0.49 | 0.89  | 1.45  | 2.16  | 3.06  | 4.15  | 5.47  | 7.04  | 8.89  | 11.06 | 13.58 |
| 0.2              | 0.06 | 0.25 | 0.58 | 1.05  | 1.69  | 2.51  | 3.52  | 4.74  | 6.2   | 7.92  | 9.92  | 12.25 | 14.92 |
| 0.3              | 0.07 | 0.3  | 0.68 | 1.23  | 1.96  | 2.89  | 4.03  | 5.39  | 7     | 8.87  | 11.03 | 13.51 | 16.35 |
| 0.4              | 0.09 | 0.35 | 0.79 | 1.43  | 2.26  | 3.31  | 4.58  | 6.09  | 7.85  | 9.89  | 12.21 | 14.86 | 17.85 |
| 0.5              | 0.1  | 0.4  | 0.92 | 1.64  | 2.59  | 3.77  | 5.18  | 6.84  | 8.76  | 10.96 | 13.46 | 16.26 | 19.41 |
| 0.6              | 0.12 | 0.47 | 1.05 | 1.88  | 2.94  | 4.25  | 5.81  | 7.64  | 9.73  | 12.09 | 14.76 | 17.73 | 21.02 |
| 0.7              | 0.13 | 0.54 | 1.2  | 2.13  | 3.32  | 4.77  | 6.49  | 8.47  | 10.73 | 13.27 | 16.1  | 19.24 | 22.68 |
| 0.8              | 0.15 | 0.61 | 1.36 | 2.4   | 3.72  | 5.32  | 7.2   | 9.35  | 11.78 | 14.5  | 17.49 | 20.78 | 24.37 |
| 0.9              | 0.17 | 0.69 | 1.53 | 2.68  | 4.14  | 5.89  | 7.94  | 10.26 | 12.87 | 15.76 | 18.92 | 22.36 | 26.09 |
| 1                | 0.2  | 0.77 | 1.71 | 2.98  | 4.58  | 6.5   | 8.71  | 11.21 | 13.99 | 17.05 | 20.37 | 23.97 | 27.83 |
| 1.1              | 0.22 | 0.86 | 1.89 | 3.3   | 5.05  | 7.12  | 9.51  | 12.18 | 15.14 | 18.37 | 21.86 | 25.6  | 29.59 |
| 1.2              | 0.24 | 0.95 | 2.09 | 3.63  | 5.53  | 7.77  | 10.33 | 13.18 | 16.32 | 19.71 | 23.36 | 27.24 | 31.35 |
| 1.3              | 0.27 | 1.05 | 2.3  | 3.97  | 6.03  | 8.44  | 11.18 | 14.21 | 17.52 | 21.08 | 24.87 | 28.89 | 33.11 |
| 1.4              | 0.3  | 1.15 | 2.51 | 4.33  | 6.54  | 9.13  | 12.04 | 15.26 | 18.74 | 22.46 | 26.4  | 30.54 | 34.87 |
| 1.5              | 0.33 | 1.26 | 2.74 | 4.69  | 7.08  | 9.84  | 12.93 | 16.32 | 19.97 | 23.85 | 27.94 | 32.2  | 36.63 |
| 1.6              | 0.36 | 1.37 | 2.97 | 5.08  | 7.63  | 10.56 | 13.84 | 17.41 | 21.22 | 25.26 | 29.48 | 33.85 | 38.37 |
| 1.7              | 0.39 | 1.49 | 3.21 | 5.47  | 8.19  | 11.31 | 14.76 | 18.5  | 22.48 | 26.67 | 31.02 | 35.5  | 40.1  |
| 1.8              | 0.42 | 1.61 | 3.46 | 5.88  | 8.77  | 12.06 | 15.7  | 19.61 | 23.76 | 28.08 | 32.55 | 37.14 | 41.81 |
| 1.9              | 0.46 | 1.74 | 3.72 | 6.29  | 9.36  | 12.84 | 16.65 | 20.73 | 25.03 | 29.5  | 34.09 | 38.77 | 43.5  |
| 2                | 0.49 | 1.87 | 3.98 | 6.72  | 9.96  | 13.62 | 17.61 | 21.87 | 26.32 | 30.92 | 35.62 | 40.38 | 45.17 |
| 2.1              | 0.53 | 2    | 4.26 | 7.15  | 10.58 | 14.42 | 18.59 | 23    | 27.6  | 32.33 | 37.13 | 41.97 | 46.82 |
| 2.2              | 0.57 | 2.14 | 4.54 | 7.6   | 11.2  | 15.22 | 19.57 | 24.15 | 28.89 | 33.74 | 38.64 | 43.55 | 48.44 |
| 2.3              | 0.61 | 2.28 | 4.82 | 8.06  | 11.84 | 16.04 | 20.56 | 25.29 | 30.18 | 35.14 | 40.13 | 45.11 | 50.03 |
| 2.4              | 0.65 | 2.43 | 5.11 | 8.52  | 12.48 | 16.87 | 21.55 | 26.45 | 31.46 | 36.54 | 41.61 | 46.64 | 51.6  |
| 2.5              | 0.69 | 2.58 | 5.41 | 8.99  | 13.14 | 17.7  | 22.56 | 27.6  | 32.74 | 37.92 | 43.07 | 48.15 | 53.13 |
| 2.6              | 0.73 | 2.73 | 5.72 | 9.47  | 13.8  | 18.54 | 23.56 | 28.75 | 34.02 | 39.29 | 44.52 | 49.64 | 54.63 |
| 2.7              | 0.78 | 2.89 | 6.03 | 9.96  | 14.47 | 19.39 | 24.57 | 29.9  | 35.29 | 40.66 | 45.94 | 51.1  | 56.1  |
| 2.8              | 0.82 | 3.05 | 6.35 | 10.46 | 15.15 | 20.24 | 25.58 | 31.05 | 36.55 | 42    | 47.35 | 52.54 | 57.54 |
| 2.9              | 0.87 | 3.21 | 6.67 | 10.96 | 15.84 | 21.1  | 26.6  | 32.2  | 37.81 | 43.34 | 48.73 | 53.95 | 58.95 |
| 3                | 0.92 | 3.38 | 7    | 11.47 | 16.53 | 21.96 | 27.61 | 33.34 | 39.05 | 44.66 | 50.1  | 55.33 | 60.32 |
| 3.1              | 0.97 | 3.55 | 7.34 | 11.99 | 17.22 | 22.83 | 28.63 | 34.48 | 40.29 | 45.96 | 51.44 | 56.68 | 61.65 |
| 3.2              | 1.02 | 3.73 | 7.68 | 12.51 | 17.92 | 23.7  | 29.64 | 35.62 | 41.51 | 47.24 | 52.75 | 58    | 62.96 |
| 3.3              | 1.07 | 3.91 | 8.02 | 13.03 | 18.63 | 24.57 | 30.65 | 36.74 | 42.72 | 48.51 | 54.05 | 59.3  | 64.23 |
| 3.4              | 1.12 | 4.09 | 8.37 | 13.57 | 19.34 | 25.44 | 31.66 | 37.86 | 43.92 | 49.76 | 55.32 | 60.56 | 65.47 |
| 3.5              | 1.18 | 4.27 | 8.73 | 14.1  | 20.06 | 26.31 | 32.67 | 38.97 | 45.11 | 50.99 | 56.56 | 61.8  | 66.67 |

$$z = 3$$

| $r \backslash q$ | 0.02 | 0.04 | 0.06 | 0.08 | 0.1   | 0.12  | 0.14  | 0.16  | 0.18  | 0.2   | 0.22  | 0.24  | 0.26  |
|------------------|------|------|------|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.1              | 0    | 0.01 | 0.03 | 0.08 | 0.17  | 0.32  | 0.54  | 0.85  | 1.28  | 1.87  | 2.65  | 3.68  | 5.01  |
| 0.2              | 0    | 0.01 | 0.04 | 0.11 | 0.22  | 0.4   | 0.66  | 1.04  | 1.55  | 2.24  | 3.14  | 4.3   | 5.78  |
| 0.3              | 0    | 0.02 | 0.06 | 0.14 | 0.28  | 0.5   | 0.82  | 1.27  | 1.87  | 2.67  | 3.7   | 5.01  | 6.66  |
| 0.4              | 0    | 0.02 | 0.07 | 0.18 | 0.35  | 0.62  | 1     | 1.54  | 2.25  | 3.17  | 4.34  | 5.82  | 7.65  |
| 0.5              | 0    | 0.03 | 0.09 | 0.22 | 0.44  | 0.76  | 1.22  | 1.85  | 2.68  | 3.74  | 5.07  | 6.72  | 8.75  |
| 0.6              | 0    | 0.03 | 0.12 | 0.27 | 0.54  | 0.93  | 1.48  | 2.22  | 3.17  | 4.39  | 5.89  | 7.73  | 9.95  |
| 0.7              | 0.01 | 0.04 | 0.14 | 0.34 | 0.65  | 1.12  | 1.77  | 2.63  | 3.73  | 5.1   | 6.79  | 8.83  | 11.26 |
| 0.8              | 0.01 | 0.05 | 0.18 | 0.41 | 0.79  | 1.34  | 2.09  | 3.09  | 4.34  | 5.9   | 7.78  | 10.03 | 12.67 |
| 0.9              | 0.01 | 0.06 | 0.21 | 0.49 | 0.94  | 1.58  | 2.46  | 3.6   | 5.02  | 6.76  | 8.85  | 11.31 | 14.17 |
| 1                | 0.01 | 0.08 | 0.25 | 0.58 | 1.11  | 1.86  | 2.86  | 4.15  | 5.76  | 7.7   | 9.99  | 12.67 | 15.76 |
| 1.1              | 0.01 | 0.09 | 0.3  | 0.69 | 1.3   | 2.16  | 3.3   | 4.76  | 6.56  | 8.7   | 11.21 | 14.12 | 17.43 |
| 1.2              | 0.01 | 0.11 | 0.35 | 0.8  | 1.5   | 2.49  | 3.79  | 5.42  | 7.41  | 9.77  | 12.5  | 15.63 | 19.16 |
| 1.3              | 0.02 | 0.13 | 0.41 | 0.93 | 1.73  | 2.85  | 4.31  | 6.13  | 8.32  | 10.9  | 13.86 | 17.21 | 20.96 |
| 1.4              | 0.02 | 0.15 | 0.48 | 1.07 | 1.98  | 3.23  | 4.86  | 6.88  | 9.28  | 12.08 | 15.27 | 18.85 | 22.81 |
| 1.5              | 0.02 | 0.17 | 0.55 | 1.22 | 2.25  | 3.65  | 5.46  | 7.67  | 10.3  | 13.32 | 16.74 | 20.54 | 24.7  |
| 1.6              | 0.03 | 0.2  | 0.63 | 1.39 | 2.54  | 4.1   | 6.09  | 8.51  | 11.36 | 14.62 | 18.26 | 22.27 | 26.63 |
| 1.7              | 0.03 | 0.23 | 0.71 | 1.57 | 2.84  | 4.57  | 6.76  | 9.39  | 12.47 | 15.95 | 19.82 | 24.05 | 28.59 |
| 1.8              | 0.03 | 0.26 | 0.8  | 1.76 | 3.17  | 5.07  | 7.46  | 10.31 | 13.62 | 17.33 | 21.42 | 25.85 | 30.57 |
| 1.9              | 0.04 | 0.29 | 0.9  | 1.96 | 3.52  | 5.6   | 8.19  | 11.27 | 14.81 | 18.75 | 23.06 | 27.68 | 32.57 |
| 2                | 0.04 | 0.33 | 1    | 2.18 | 3.89  | 6.16  | 8.96  | 12.27 | 16.03 | 20.2  | 24.72 | 29.53 | 34.58 |
| 2.1              | 0.05 | 0.36 | 1.12 | 2.41 | 4.28  | 6.74  | 9.76  | 13.3  | 17.29 | 21.68 | 26.41 | 31.39 | 36.59 |
| 2.2              | 0.06 | 0.4  | 1.23 | 2.65 | 4.69  | 7.35  | 10.59 | 14.36 | 18.58 | 23.19 | 28.11 | 33.27 | 38.59 |
| 2.3              | 0.06 | 0.45 | 1.36 | 2.9  | 5.12  | 7.98  | 11.44 | 15.44 | 19.9  | 24.72 | 29.83 | 35.14 | 40.58 |
| 2.4              | 0.07 | 0.49 | 1.49 | 3.17 | 5.56  | 8.64  | 12.33 | 16.56 | 21.24 | 26.27 | 31.56 | 37.01 | 42.56 |
| 2.5              | 0.08 | 0.54 | 1.63 | 3.46 | 6.03  | 9.31  | 13.24 | 17.7  | 22.6  | 27.83 | 33.29 | 38.88 | 44.53 |
| 2.6              | 0.08 | 0.59 | 1.78 | 3.75 | 6.51  | 10.02 | 14.17 | 18.87 | 23.98 | 29.41 | 35.02 | 40.74 | 46.46 |
| 2.7              | 0.09 | 0.65 | 1.94 | 4.06 | 7.01  | 10.74 | 15.13 | 20.05 | 25.38 | 30.99 | 36.76 | 42.58 | 48.38 |
| 2.8              | 0.1  | 0.71 | 2.1  | 4.38 | 7.53  | 11.48 | 16.1  | 21.25 | 26.79 | 32.57 | 38.48 | 44.41 | 50.26 |
| 2.9              | 0.11 | 0.77 | 2.27 | 4.71 | 8.07  | 12.25 | 17.1  | 22.47 | 28.21 | 34.16 | 40.2  | 46.22 | 52.11 |
| 3                | 0.12 | 0.83 | 2.45 | 5.06 | 8.62  | 13.03 | 18.11 | 23.71 | 29.64 | 35.75 | 41.91 | 48    | 53.92 |
| 3.1              | 0.13 | 0.9  | 2.63 | 5.41 | 9.19  | 13.83 | 19.14 | 24.95 | 31.07 | 37.33 | 43.6  | 49.75 | 55.7  |
| 3.2              | 0.14 | 0.97 | 2.82 | 5.78 | 9.78  | 14.64 | 20.19 | 26.21 | 32.51 | 38.91 | 45.28 | 51.48 | 57.43 |
| 3.3              | 0.15 | 1.04 | 3.02 | 6.16 | 10.38 | 15.48 | 21.25 | 27.47 | 33.95 | 40.48 | 46.93 | 53.18 | 59.13 |
| 3.4              | 0.16 | 1.12 | 3.23 | 6.56 | 10.99 | 16.32 | 22.32 | 28.75 | 35.38 | 42.04 | 48.57 | 54.84 | 60.78 |
| 3.5              | 0.18 | 1.2  | 3.44 | 6.96 | 11.62 | 17.18 | 23.41 | 30.03 | 36.82 | 43.59 | 50.18 | 56.47 | 62.39 |

$$z = 4$$

| $r \backslash q$ | 0.02 | 0.04 | 0.06 | 0.08 | 0.1  | 0.12  | 0.14  | 0.16  | 0.18  | 0.2   | 0.22  | 0.24  | 0.26  |
|------------------|------|------|------|------|------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.1              | 0    | 0    | 0    | 0.01 | 0.02 | 0.05  | 0.09  | 0.17  | 0.3   | 0.5   | 0.79  | 1.22  | 1.85  |
| 0.2              | 0    | 0    | 0    | 0.01 | 0.03 | 0.06  | 0.12  | 0.23  | 0.39  | 0.63  | 0.99  | 1.51  | 2.24  |
| 0.3              | 0    | 0    | 0    | 0.02 | 0.04 | 0.09  | 0.17  | 0.3   | 0.5   | 0.8   | 1.24  | 1.85  | 2.71  |
| 0.4              | 0    | 0    | 0.01 | 0.02 | 0.05 | 0.12  | 0.22  | 0.39  | 0.64  | 1.01  | 1.54  | 2.27  | 3.27  |
| 0.5              | 0    | 0    | 0.01 | 0.03 | 0.07 | 0.15  | 0.29  | 0.5   | 0.82  | 1.27  | 1.91  | 2.77  | 3.93  |
| 0.6              | 0    | 0    | 0.01 | 0.04 | 0.1  | 0.2   | 0.37  | 0.64  | 1.03  | 1.59  | 2.35  | 3.36  | 4.7   |
| 0.7              | 0    | 0    | 0.02 | 0.05 | 0.13 | 0.26  | 0.48  | 0.81  | 1.3   | 1.96  | 2.87  | 4.05  | 5.59  |
| 0.8              | 0    | 0    | 0.02 | 0.07 | 0.17 | 0.34  | 0.61  | 1.02  | 1.61  | 2.41  | 3.47  | 4.84  | 6.6   |
| 0.9              | 0    | 0.01 | 0.03 | 0.09 | 0.21 | 0.43  | 0.77  | 1.27  | 1.97  | 2.92  | 4.16  | 5.74  | 7.73  |
| 1                | 0    | 0.01 | 0.04 | 0.12 | 0.27 | 0.54  | 0.95  | 1.55  | 2.39  | 3.5   | 4.94  | 6.74  | 8.97  |
| 1.1              | 0    | 0.01 | 0.05 | 0.15 | 0.34 | 0.66  | 1.16  | 1.89  | 2.87  | 4.16  | 5.81  | 7.85  | 10.34 |
| 1.2              | 0    | 0.01 | 0.06 | 0.18 | 0.42 | 0.81  | 1.41  | 2.26  | 3.41  | 4.9   | 6.77  | 9.07  | 11.83 |
| 1.3              | 0    | 0.02 | 0.08 | 0.22 | 0.51 | 0.98  | 1.69  | 2.69  | 4.02  | 5.72  | 7.83  | 10.39 | 13.42 |
| 1.4              | 0    | 0.02 | 0.09 | 0.27 | 0.61 | 1.17  | 2     | 3.16  | 4.68  | 6.61  | 8.98  | 11.81 | 15.12 |
| 1.5              | 0    | 0.02 | 0.11 | 0.33 | 0.73 | 1.39  | 2.35  | 3.68  | 5.42  | 7.58  | 10.21 | 13.32 | 16.91 |
| 1.6              | 0    | 0.03 | 0.14 | 0.39 | 0.86 | 1.63  | 2.74  | 4.26  | 6.21  | 8.63  | 11.53 | 14.92 | 18.79 |
| 1.7              | 0    | 0.04 | 0.16 | 0.46 | 1.01 | 1.9   | 3.17  | 4.88  | 7.07  | 9.75  | 12.93 | 16.6  | 20.75 |
| 1.8              | 0    | 0.04 | 0.19 | 0.54 | 1.18 | 2.19  | 3.64  | 5.56  | 7.99  | 10.94 | 14.4  | 18.35 | 22.77 |
| 1.9              | 0    | 0.05 | 0.22 | 0.63 | 1.37 | 2.52  | 4.14  | 6.29  | 8.98  | 12.2  | 15.95 | 20.18 | 24.86 |
| 2                | 0    | 0.06 | 0.26 | 0.73 | 1.57 | 2.87  | 4.69  | 7.07  | 10.02 | 13.53 | 17.55 | 22.06 | 27    |
| 2.1              | 0    | 0.07 | 0.3  | 0.84 | 1.79 | 3.25  | 5.28  | 7.9   | 11.12 | 14.91 | 19.22 | 24    | 29.17 |
| 2.2              | 0.01 | 0.08 | 0.35 | 0.96 | 2.03 | 3.66  | 5.9   | 8.78  | 12.28 | 16.35 | 20.94 | 25.98 | 31.38 |
| 2.3              | 0.01 | 0.09 | 0.4  | 1.08 | 2.29 | 4.1   | 6.57  | 9.71  | 13.48 | 17.84 | 22.71 | 28    | 33.61 |
| 2.4              | 0.01 | 0.1  | 0.45 | 1.22 | 2.56 | 4.57  | 7.27  | 10.68 | 14.74 | 19.38 | 24.52 | 30.04 | 35.85 |
| 2.5              | 0.01 | 0.12 | 0.51 | 1.38 | 2.86 | 5.07  | 8.02  | 11.69 | 16.04 | 20.97 | 26.36 | 32.11 | 38.09 |
| 2.6              | 0.01 | 0.13 | 0.58 | 1.54 | 3.18 | 5.59  | 8.79  | 12.75 | 17.39 | 22.59 | 28.23 | 34.19 | 40.34 |
| 2.7              | 0.01 | 0.15 | 0.65 | 1.71 | 3.52 | 6.15  | 9.61  | 13.85 | 18.77 | 24.24 | 30.13 | 36.28 | 42.57 |
| 2.8              | 0.01 | 0.17 | 0.72 | 1.9  | 3.88 | 6.74  | 10.46 | 14.99 | 20.19 | 25.93 | 32.04 | 38.37 | 44.78 |
| 2.9              | 0.01 | 0.19 | 0.8  | 2.1  | 4.26 | 7.35  | 11.35 | 16.16 | 21.64 | 27.63 | 33.96 | 40.46 | 46.97 |
| 3                | 0.02 | 0.21 | 0.89 | 2.31 | 4.66 | 7.99  | 12.26 | 17.36 | 23.12 | 29.36 | 35.89 | 42.53 | 49.13 |
| 3.1              | 0.02 | 0.24 | 0.98 | 2.54 | 5.08 | 8.66  | 13.21 | 18.6  | 24.63 | 31.1  | 37.82 | 44.59 | 51.26 |
| 3.2              | 0.02 | 0.26 | 1.08 | 2.78 | 5.53 | 9.36  | 14.19 | 19.86 | 26.16 | 32.86 | 39.75 | 46.63 | 53.35 |
| 3.3              | 0.02 | 0.29 | 1.18 | 3.03 | 5.99 | 10.08 | 15.2  | 21.15 | 27.71 | 34.62 | 41.67 | 48.65 | 55.39 |
| 3.4              | 0.02 | 0.32 | 1.3  | 3.29 | 6.47 | 10.83 | 16.24 | 22.47 | 29.27 | 36.38 | 43.57 | 50.63 | 57.39 |
| 3.5              | 0.03 | 0.35 | 1.41 | 3.57 | 6.98 | 11.61 | 17.3  | 23.8  | 30.85 | 38.15 | 45.46 | 52.58 | 59.34 |

$$z = 5$$

| $r \backslash q$ | 0.02 | 0.04 | 0.06 | 0.08 | 0.1  | 0.12 | 0.14  | 0.16  | 0.18  | 0.2   | 0.22  | 0.24  | 0.26  |
|------------------|------|------|------|------|------|------|-------|-------|-------|-------|-------|-------|-------|
| 0.1              | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.02  | 0.04  | 0.07  | 0.13  | 0.24  | 0.41  | 0.68  |
| 0.2              | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.02  | 0.05  | 0.1   | 0.18  | 0.31  | 0.53  | 0.87  |
| 0.3              | 0    | 0    | 0    | 0    | 0.01 | 0.01 | 0.03  | 0.07  | 0.13  | 0.24  | 0.41  | 0.68  | 1.1   |
| 0.4              | 0    | 0    | 0    | 0    | 0.01 | 0.02 | 0.05  | 0.1   | 0.18  | 0.32  | 0.54  | 0.89  | 1.39  |
| 0.5              | 0    | 0    | 0    | 0    | 0.01 | 0.03 | 0.07  | 0.13  | 0.25  | 0.43  | 0.72  | 1.14  | 1.76  |
| 0.6              | 0    | 0    | 0    | 0.01 | 0.02 | 0.04 | 0.09  | 0.19  | 0.34  | 0.57  | 0.93  | 1.46  | 2.22  |
| 0.7              | 0    | 0    | 0    | 0.01 | 0.03 | 0.06 | 0.13  | 0.25  | 0.45  | 0.75  | 1.21  | 1.86  | 2.77  |
| 0.8              | 0    | 0    | 0    | 0.01 | 0.04 | 0.09 | 0.18  | 0.34  | 0.59  | 0.98  | 1.55  | 2.34  | 3.43  |
| 0.9              | 0    | 0    | 0    | 0.02 | 0.05 | 0.12 | 0.24  | 0.45  | 0.77  | 1.26  | 1.96  | 2.92  | 4.21  |
| 1                | 0    | 0    | 0.01 | 0.02 | 0.07 | 0.16 | 0.32  | 0.58  | 1     | 1.6   | 2.45  | 3.6   | 5.12  |
| 1.1              | 0    | 0    | 0.01 | 0.03 | 0.09 | 0.21 | 0.41  | 0.75  | 1.26  | 2     | 3.02  | 4.39  | 6.16  |
| 1.2              | 0    | 0    | 0.01 | 0.04 | 0.12 | 0.27 | 0.53  | 0.95  | 1.58  | 2.48  | 3.69  | 5.29  | 7.34  |
| 1.3              | 0    | 0    | 0.01 | 0.05 | 0.15 | 0.34 | 0.67  | 1.19  | 1.96  | 3.03  | 4.46  | 6.31  | 8.65  |
| 1.4              | 0    | 0    | 0.02 | 0.07 | 0.19 | 0.43 | 0.83  | 1.47  | 2.39  | 3.65  | 5.32  | 7.46  | 10.1  |
| 1.5              | 0    | 0    | 0.02 | 0.09 | 0.24 | 0.53 | 1.03  | 1.79  | 2.88  | 4.36  | 6.29  | 8.72  | 11.68 |
| 1.6              | 0    | 0    | 0.03 | 0.11 | 0.3  | 0.66 | 1.25  | 2.16  | 3.44  | 5.16  | 7.36  | 10.1  | 13.39 |
| 1.7              | 0    | 0.01 | 0.04 | 0.14 | 0.37 | 0.8  | 1.51  | 2.58  | 4.07  | 6.04  | 8.53  | 11.59 | 15.22 |
| 1.8              | 0    | 0.01 | 0.05 | 0.17 | 0.45 | 0.96 | 1.8   | 3.05  | 4.76  | 7     | 9.81  | 13.19 | 17.16 |
| 1.9              | 0    | 0.01 | 0.06 | 0.21 | 0.54 | 1.15 | 2.13  | 3.57  | 5.53  | 8.05  | 11.18 | 14.9  | 19.2  |
| 2                | 0    | 0.01 | 0.07 | 0.25 | 0.64 | 1.36 | 2.5   | 4.14  | 6.36  | 9.19  | 12.64 | 16.7  | 21.34 |
| 2.1              | 0    | 0.01 | 0.08 | 0.3  | 0.76 | 1.59 | 2.9   | 4.78  | 7.27  | 10.41 | 14.19 | 18.59 | 23.55 |
| 2.2              | 0    | 0.02 | 0.1  | 0.35 | 0.89 | 1.86 | 3.35  | 5.46  | 8.25  | 11.71 | 15.83 | 20.57 | 25.84 |
| 2.3              | 0    | 0.02 | 0.12 | 0.41 | 1.04 | 2.15 | 3.84  | 6.21  | 9.29  | 13.08 | 17.55 | 22.61 | 28.19 |
| 2.4              | 0    | 0.02 | 0.14 | 0.48 | 1.21 | 2.46 | 4.37  | 7.01  | 10.4  | 14.53 | 19.33 | 24.72 | 30.58 |
| 2.5              | 0    | 0.03 | 0.16 | 0.56 | 1.39 | 2.81 | 4.95  | 7.86  | 11.58 | 16.05 | 21.19 | 26.88 | 33.01 |
| 2.6              | 0    | 0.03 | 0.19 | 0.65 | 1.59 | 3.19 | 5.56  | 8.78  | 12.82 | 17.63 | 23.1  | 29.09 | 35.46 |
| 2.7              | 0    | 0.04 | 0.22 | 0.74 | 1.81 | 3.59 | 6.22  | 9.74  | 14.12 | 19.27 | 25.06 | 31.33 | 37.92 |
| 2.8              | 0    | 0.04 | 0.25 | 0.84 | 2.04 | 4.03 | 6.93  | 10.76 | 15.47 | 20.96 | 27.06 | 33.6  | 40.39 |
| 2.9              | 0    | 0.05 | 0.29 | 0.96 | 2.3  | 4.5  | 7.68  | 11.83 | 16.88 | 22.7  | 29.1  | 35.89 | 42.85 |
| 3                | 0    | 0.06 | 0.33 | 1.08 | 2.58 | 5    | 8.46  | 12.95 | 18.34 | 24.49 | 31.17 | 38.18 | 45.29 |
| 3.1              | 0    | 0.06 | 0.37 | 1.22 | 2.87 | 5.54 | 9.3   | 14.11 | 19.85 | 26.31 | 33.26 | 40.47 | 47.7  |
| 3.2              | 0    | 0.07 | 0.42 | 1.36 | 3.19 | 6.1  | 10.17 | 15.32 | 21.39 | 28.16 | 35.37 | 42.75 | 50.09 |
| 3.3              | 0    | 0.08 | 0.48 | 1.52 | 3.53 | 6.7  | 11.08 | 16.57 | 22.98 | 30.04 | 37.48 | 45.02 | 52.43 |
| 3.4              | 0    | 0.09 | 0.53 | 1.69 | 3.89 | 7.33 | 12.03 | 17.87 | 24.6  | 31.94 | 39.59 | 47.27 | 54.72 |
| 3.5              | 0    | 0.11 | 0.59 | 1.87 | 4.28 | 8    | 13.02 | 19.19 | 26.25 | 33.86 | 41.7  | 49.49 | 56.97 |

$$z = 6$$

| $r \backslash q$ | 0.02 | 0.04 | 0.06 | 0.08 | 0.1  | 0.12 | 0.14 | 0.16  | 0.18  | 0.2   | 0.22  | 0.24  | 0.26  |
|------------------|------|------|------|------|------|------|------|-------|-------|-------|-------|-------|-------|
| 0.1              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0.01  | 0.02  | 0.03  | 0.07  | 0.14  | 0.25  |
| 0.2              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0.01  | 0.02  | 0.05  | 0.1   | 0.19  | 0.33  |
| 0.3              | 0    | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.02  | 0.04  | 0.07  | 0.14  | 0.25  | 0.45  |
| 0.4              | 0    | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.02  | 0.05  | 0.1   | 0.19  | 0.34  | 0.59  |
| 0.5              | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.02 | 0.04  | 0.08  | 0.15  | 0.27  | 0.47  | 0.79  |
| 0.6              | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.02 | 0.05  | 0.11  | 0.21  | 0.37  | 0.63  | 1.04  |
| 0.7              | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.04 | 0.08  | 0.16  | 0.29  | 0.51  | 0.85  | 1.37  |
| 0.8              | 0    | 0    | 0    | 0    | 0.01 | 0.02 | 0.05 | 0.11  | 0.22  | 0.4   | 0.69  | 1.13  | 1.78  |
| 0.9              | 0    | 0    | 0    | 0    | 0.01 | 0.03 | 0.08 | 0.16  | 0.3   | 0.55  | 0.92  | 1.48  | 2.3   |
| 1                | 0    | 0    | 0    | 0    | 0.02 | 0.05 | 0.11 | 0.22  | 0.42  | 0.73  | 1.21  | 1.92  | 2.93  |
| 1.1              | 0    | 0    | 0    | 0.01 | 0.02 | 0.06 | 0.15 | 0.3   | 0.56  | 0.97  | 1.58  | 2.46  | 3.68  |
| 1.2              | 0    | 0    | 0    | 0.01 | 0.03 | 0.09 | 0.2  | 0.4   | 0.74  | 1.26  | 2.02  | 3.1   | 4.57  |
| 1.3              | 0    | 0    | 0    | 0.01 | 0.04 | 0.12 | 0.27 | 0.53  | 0.96  | 1.61  | 2.55  | 3.86  | 5.6   |
| 1.4              | 0    | 0    | 0    | 0.02 | 0.06 | 0.16 | 0.35 | 0.69  | 1.23  | 2.03  | 3.18  | 4.74  | 6.78  |
| 1.5              | 0    | 0    | 0.01 | 0.02 | 0.08 | 0.21 | 0.45 | 0.88  | 1.54  | 2.53  | 3.9   | 5.74  | 8.11  |
| 1.6              | 0    | 0    | 0.01 | 0.03 | 0.1  | 0.27 | 0.58 | 1.1   | 1.92  | 3.11  | 4.74  | 6.88  | 9.6   |
| 1.7              | 0    | 0    | 0.01 | 0.04 | 0.14 | 0.34 | 0.73 | 1.37  | 2.36  | 3.77  | 5.68  | 8.15  | 11.24 |
| 1.8              | 0    | 0    | 0.01 | 0.05 | 0.17 | 0.43 | 0.9  | 1.69  | 2.86  | 4.52  | 6.74  | 9.56  | 13.02 |
| 1.9              | 0    | 0    | 0.01 | 0.07 | 0.22 | 0.53 | 1.11 | 2.05  | 3.44  | 5.37  | 7.91  | 11.09 | 14.95 |
| 2                | 0    | 0    | 0.02 | 0.09 | 0.27 | 0.65 | 1.35 | 2.46  | 4.08  | 6.31  | 9.19  | 12.75 | 17    |
| 2.1              | 0    | 0    | 0.02 | 0.11 | 0.33 | 0.79 | 1.62 | 2.92  | 4.8   | 7.34  | 10.58 | 14.54 | 19.18 |
| 2.2              | 0    | 0    | 0.03 | 0.13 | 0.4  | 0.95 | 1.93 | 3.44  | 5.6   | 8.47  | 12.09 | 16.43 | 21.47 |
| 2.3              | 0    | 0    | 0.04 | 0.16 | 0.48 | 1.14 | 2.27 | 4.02  | 6.47  | 9.7   | 13.69 | 18.44 | 23.85 |
| 2.4              | 0    | 0    | 0.04 | 0.19 | 0.58 | 1.35 | 2.66 | 4.66  | 7.42  | 11.01 | 15.4  | 20.54 | 26.31 |
| 2.5              | 0    | 0.01 | 0.05 | 0.23 | 0.68 | 1.58 | 3.09 | 5.35  | 8.45  | 12.42 | 17.2  | 22.72 | 28.85 |
| 2.6              | 0    | 0.01 | 0.06 | 0.27 | 0.8  | 1.84 | 3.56 | 6.11  | 9.56  | 13.91 | 19.09 | 24.98 | 31.44 |
| 2.7              | 0    | 0.01 | 0.08 | 0.32 | 0.94 | 2.13 | 4.08 | 6.93  | 10.74 | 15.48 | 21.05 | 27.31 | 34.07 |
| 2.8              | 0    | 0.01 | 0.09 | 0.38 | 1.09 | 2.45 | 4.65 | 7.82  | 12    | 17.13 | 23.09 | 29.69 | 36.73 |
| 2.9              | 0    | 0.01 | 0.11 | 0.44 | 1.26 | 2.8  | 5.26 | 8.76  | 13.32 | 18.85 | 25.18 | 32.11 | 39.4  |
| 3                | 0    | 0.01 | 0.12 | 0.51 | 1.44 | 3.18 | 5.92 | 9.77  | 14.72 | 20.64 | 27.33 | 34.56 | 42.07 |
| 3.1              | 0    | 0.02 | 0.15 | 0.59 | 1.65 | 3.59 | 6.62 | 10.83 | 16.17 | 22.48 | 29.53 | 37.03 | 44.72 |
| 3.2              | 0    | 0.02 | 0.17 | 0.68 | 1.87 | 4.04 | 7.38 | 11.96 | 17.69 | 24.38 | 31.76 | 39.51 | 47.36 |
| 3.3              | 0    | 0.02 | 0.19 | 0.78 | 2.11 | 4.52 | 8.18 | 13.14 | 19.27 | 26.33 | 34.01 | 41.99 | 49.96 |
| 3.4              | 0    | 0.03 | 0.22 | 0.88 | 2.38 | 5.03 | 9.03 | 14.37 | 20.89 | 28.31 | 36.29 | 44.46 | 52.51 |
| 3.5              | 0    | 0.03 | 0.25 | 1    | 2.66 | 5.58 | 9.92 | 15.66 | 22.57 | 30.33 | 38.57 | 46.91 | 55.02 |



$$z = 7$$

| $r \backslash q$ | 0.02 | 0.04 | 0.06 | 0.08 | 0.1  | 0.12 | 0.14 | 0.16  | 0.18  | 0.2   | 0.22  | 0.24  | 0.26  |
|------------------|------|------|------|------|------|------|------|-------|-------|-------|-------|-------|-------|
| 0.1              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0     | 0     | 0.01  | 0.02  | 0.05  | 0.09  |
| 0.2              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0     | 0.01  | 0.01  | 0.03  | 0.06  | 0.13  |
| 0.3              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0     | 0.01  | 0.02  | 0.05  | 0.09  | 0.18  |
| 0.4              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0.01  | 0.01  | 0.03  | 0.07  | 0.13  | 0.25  |
| 0.5              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0.01  | 0.02  | 0.05  | 0.1   | 0.19  | 0.35  |
| 0.6              | 0    | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.02  | 0.04  | 0.07  | 0.15  | 0.27  | 0.49  |
| 0.7              | 0    | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.02  | 0.05  | 0.11  | 0.21  | 0.39  | 0.68  |
| 0.8              | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.02 | 0.04  | 0.08  | 0.16  | 0.31  | 0.55  | 0.93  |
| 0.9              | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.02 | 0.06  | 0.12  | 0.24  | 0.43  | 0.75  | 1.25  |
| 1                | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.04 | 0.08  | 0.17  | 0.34  | 0.6   | 1.03  | 1.67  |
| 1.1              | 0    | 0    | 0    | 0    | 0.01 | 0.02 | 0.05 | 0.12  | 0.25  | 0.47  | 0.83  | 1.38  | 2.2   |
| 1.2              | 0    | 0    | 0    | 0    | 0.01 | 0.03 | 0.08 | 0.17  | 0.34  | 0.64  | 1.11  | 1.82  | 2.85  |
| 1.3              | 0    | 0    | 0    | 0    | 0.01 | 0.04 | 0.11 | 0.24  | 0.47  | 0.86  | 1.47  | 2.36  | 3.63  |
| 1.4              | 0    | 0    | 0    | 0    | 0.02 | 0.06 | 0.15 | 0.32  | 0.63  | 1.14  | 1.9   | 3.02  | 4.57  |
| 1.5              | 0    | 0    | 0    | 0.01 | 0.03 | 0.08 | 0.2  | 0.43  | 0.83  | 1.47  | 2.43  | 3.8   | 5.66  |
| 1.6              | 0    | 0    | 0    | 0.01 | 0.04 | 0.11 | 0.27 | 0.57  | 1.08  | 1.88  | 3.06  | 4.71  | 6.92  |
| 1.7              | 0    | 0    | 0    | 0.01 | 0.05 | 0.15 | 0.35 | 0.74  | 1.38  | 2.37  | 3.8   | 5.77  | 8.35  |
| 1.8              | 0    | 0    | 0    | 0.02 | 0.07 | 0.19 | 0.46 | 0.94  | 1.73  | 2.94  | 4.66  | 6.97  | 9.94  |
| 1.9              | 0    | 0    | 0    | 0.02 | 0.09 | 0.25 | 0.58 | 1.18  | 2.15  | 3.6   | 5.63  | 8.31  | 11.71 |
| 2                | 0    | 0    | 0.01 | 0.03 | 0.11 | 0.32 | 0.73 | 1.47  | 2.64  | 4.36  | 6.73  | 9.81  | 13.64 |
| 2.1              | 0    | 0    | 0.01 | 0.04 | 0.14 | 0.4  | 0.91 | 1.8   | 3.2   | 5.22  | 7.95  | 11.44 | 15.72 |
| 2.2              | 0    | 0    | 0.01 | 0.05 | 0.18 | 0.49 | 1.12 | 2.18  | 3.83  | 6.18  | 9.3   | 13.22 | 17.95 |
| 2.3              | 0    | 0    | 0.01 | 0.06 | 0.22 | 0.61 | 1.36 | 2.62  | 4.55  | 7.24  | 10.77 | 15.14 | 20.31 |
| 2.4              | 0    | 0    | 0.01 | 0.08 | 0.28 | 0.74 | 1.63 | 3.12  | 5.34  | 8.41  | 12.36 | 17.18 | 22.79 |
| 2.5              | 0    | 0    | 0.02 | 0.1  | 0.34 | 0.9  | 1.95 | 3.68  | 6.22  | 9.68  | 14.07 | 19.34 | 25.38 |
| 2.6              | 0    | 0    | 0.02 | 0.12 | 0.41 | 1.07 | 2.3  | 4.3   | 7.19  | 11.06 | 15.89 | 21.6  | 28.05 |
| 2.7              | 0    | 0    | 0.03 | 0.14 | 0.49 | 1.27 | 2.7  | 4.98  | 8.24  | 12.53 | 17.82 | 23.96 | 30.8  |
| 2.8              | 0    | 0    | 0.03 | 0.17 | 0.59 | 1.5  | 3.15 | 5.73  | 9.38  | 14.11 | 19.84 | 26.41 | 33.59 |
| 2.9              | 0    | 0    | 0.04 | 0.21 | 0.7  | 1.75 | 3.64 | 6.55  | 10.6  | 15.77 | 21.95 | 28.92 | 36.43 |
| 3                | 0    | 0    | 0.05 | 0.25 | 0.82 | 2.04 | 4.18 | 7.44  | 11.9  | 17.52 | 24.13 | 31.49 | 39.29 |
| 3.1              | 0    | 0    | 0.06 | 0.29 | 0.95 | 2.35 | 4.76 | 8.39  | 13.29 | 19.36 | 26.39 | 34.1  | 42.15 |
| 3.2              | 0    | 0.01 | 0.07 | 0.34 | 1.11 | 2.69 | 5.4  | 9.41  | 14.75 | 21.26 | 28.7  | 36.73 | 45    |
| 3.3              | 0    | 0.01 | 0.08 | 0.4  | 1.28 | 3.07 | 6.09 | 10.5  | 16.28 | 23.24 | 31.06 | 39.39 | 47.83 |
| 3.4              | 0    | 0.01 | 0.09 | 0.46 | 1.46 | 3.49 | 6.84 | 11.66 | 17.88 | 25.27 | 33.46 | 42.04 | 50.62 |
| 3.5              | 0    | 0.01 | 0.11 | 0.54 | 1.67 | 3.93 | 7.63 | 12.88 | 19.55 | 27.36 | 35.88 | 44.69 | 53.36 |

$$z = 8$$

| $r \backslash q$ | 0.02 | 0.04 | 0.06 | 0.08 | 0.1  | 0.12 | 0.14 | 0.16  | 0.18  | 0.2   | 0.22  | 0.24  | 0.26  |
|------------------|------|------|------|------|------|------|------|-------|-------|-------|-------|-------|-------|
| 0.1              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0     | 0     | 0     | 0.01  | 0.01  | 0.03  |
| 0.2              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0     | 0     | 0     | 0.01  | 0.02  | 0.05  |
| 0.3              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0     | 0     | 0.01  | 0.02  | 0.03  | 0.07  |
| 0.4              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0     | 0     | 0.01  | 0.02  | 0.05  | 0.11  |
| 0.5              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0     | 0.01  | 0.02  | 0.04  | 0.08  | 0.16  |
| 0.6              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0     | 0.01  | 0.03  | 0.06  | 0.12  | 0.23  |
| 0.7              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0.01  | 0.02  | 0.04  | 0.09  | 0.18  | 0.33  |
| 0.8              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0.01  | 0.03  | 0.07  | 0.14  | 0.26  | 0.48  |
| 0.9              | 0    | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.02  | 0.05  | 0.1   | 0.2   | 0.38  | 0.68  |
| 1                | 0    | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.03  | 0.07  | 0.15  | 0.3   | 0.55  | 0.96  |
| 1.1              | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.02 | 0.05  | 0.11  | 0.23  | 0.43  | 0.77  | 1.32  |
| 1.2              | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.03 | 0.07  | 0.16  | 0.33  | 0.61  | 1.07  | 1.78  |
| 1.3              | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.04 | 0.11  | 0.23  | 0.46  | 0.84  | 1.45  | 2.36  |
| 1.4              | 0    | 0    | 0    | 0    | 0.01 | 0.02 | 0.06 | 0.15  | 0.33  | 0.64  | 1.14  | 1.93  | 3.09  |
| 1.5              | 0    | 0    | 0    | 0    | 0.01 | 0.03 | 0.09 | 0.21  | 0.45  | 0.86  | 1.52  | 2.52  | 3.96  |
| 1.6              | 0    | 0    | 0    | 0    | 0.01 | 0.05 | 0.13 | 0.29  | 0.61  | 1.15  | 1.99  | 3.24  | 5     |
| 1.7              | 0    | 0    | 0    | 0    | 0.02 | 0.06 | 0.17 | 0.4   | 0.81  | 1.5   | 2.56  | 4.1   | 6.22  |
| 1.8              | 0    | 0    | 0    | 0.01 | 0.03 | 0.09 | 0.23 | 0.53  | 1.06  | 1.92  | 3.24  | 5.1   | 7.62  |
| 1.9              | 0    | 0    | 0    | 0.01 | 0.04 | 0.12 | 0.31 | 0.69  | 1.36  | 2.43  | 4.03  | 6.26  | 9.21  |
| 2                | 0    | 0    | 0    | 0.01 | 0.05 | 0.15 | 0.4  | 0.88  | 1.72  | 3.03  | 4.95  | 7.58  | 10.99 |
| 2.1              | 0    | 0    | 0    | 0.01 | 0.06 | 0.2  | 0.51 | 1.12  | 2.14  | 3.73  | 6     | 9.06  | 12.94 |
| 2.2              | 0    | 0    | 0    | 0.02 | 0.08 | 0.26 | 0.65 | 1.4   | 2.64  | 4.53  | 7.19  | 10.7  | 15.08 |
| 2.3              | 0    | 0    | 0    | 0.02 | 0.11 | 0.33 | 0.82 | 1.72  | 3.22  | 5.44  | 8.51  | 12.49 | 17.38 |
| 2.4              | 0    | 0    | 0    | 0.03 | 0.13 | 0.41 | 1.01 | 2.1   | 3.87  | 6.46  | 9.98  | 14.45 | 19.84 |
| 2.5              | 0    | 0    | 0.01 | 0.04 | 0.17 | 0.51 | 1.24 | 2.54  | 4.61  | 7.6   | 11.57 | 16.54 | 22.43 |
| 2.6              | 0    | 0    | 0.01 | 0.05 | 0.21 | 0.63 | 1.5  | 3.04  | 5.44  | 8.85  | 13.3  | 18.78 | 25.15 |
| 2.7              | 0    | 0    | 0.01 | 0.06 | 0.26 | 0.77 | 1.8  | 3.6   | 6.36  | 10.21 | 15.16 | 21.14 | 27.97 |
| 2.8              | 0    | 0    | 0.01 | 0.08 | 0.32 | 0.92 | 2.15 | 4.23  | 7.38  | 11.69 | 17.14 | 23.61 | 30.87 |
| 2.9              | 0    | 0    | 0.01 | 0.1  | 0.39 | 1.11 | 2.53 | 4.93  | 8.49  | 13.27 | 19.23 | 26.17 | 33.84 |
| 3                | 0    | 0    | 0.02 | 0.12 | 0.47 | 1.31 | 2.97 | 5.7   | 9.69  | 14.97 | 21.42 | 28.82 | 36.85 |
| 3.1              | 0    | 0    | 0.02 | 0.14 | 0.56 | 1.55 | 3.45 | 6.54  | 10.98 | 16.76 | 23.71 | 31.54 | 39.89 |
| 3.2              | 0    | 0    | 0.03 | 0.17 | 0.66 | 1.81 | 3.98 | 7.46  | 12.37 | 18.64 | 26.07 | 34.3  | 42.93 |
| 3.3              | 0    | 0    | 0.03 | 0.21 | 0.78 | 2.11 | 4.57 | 8.45  | 13.84 | 20.62 | 28.51 | 37.1  | 45.96 |
| 3.4              | 0    | 0    | 0.04 | 0.25 | 0.91 | 2.43 | 5.21 | 9.52  | 15.39 | 22.67 | 31    | 39.92 | 48.96 |
| 3.5              | 0    | 0    | 0.05 | 0.29 | 1.06 | 2.79 | 5.91 | 10.66 | 17.03 | 24.8  | 33.53 | 42.74 | 51.92 |

$$z = 9$$

| $r \backslash q$ | 0.02 | 0.04 | 0.06 | 0.08 | 0.1  | 0.12 | 0.14 | 0.16 | 0.18  | 0.2   | 0.22  | 0.24  | 0.26  |
|------------------|------|------|------|------|------|------|------|------|-------|-------|-------|-------|-------|
| 0.1              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0     | 0     | 0     | 0     | 0.01  |
| 0.2              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0     | 0     | 0     | 0.01  | 0.02  |
| 0.3              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0     | 0     | 0.01  | 0.01  | 0.03  |
| 0.4              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0     | 0     | 0.01  | 0.02  | 0.05  |
| 0.5              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0     | 0.01  | 0.01  | 0.03  | 0.07  |
| 0.6              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0     | 0.01  | 0.02  | 0.05  | 0.11  |
| 0.7              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0.01  | 0.02  | 0.04  | 0.08  | 0.17  |
| 0.8              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0.01  | 0.03  | 0.06  | 0.13  | 0.25  |
| 0.9              | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.02  | 0.04  | 0.1   | 0.19  | 0.37  |
| 1                | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.03  | 0.07  | 0.15  | 0.29  | 0.55  |
| 1.1              | 0    | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.02 | 0.05  | 0.11  | 0.23  | 0.43  | 0.79  |
| 1.2              | 0    | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.03 | 0.08  | 0.17  | 0.34  | 0.63  | 1.11  |
| 1.3              | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.02 | 0.05 | 0.11  | 0.25  | 0.49  | 0.89  | 1.54  |
| 1.4              | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.03 | 0.07 | 0.17  | 0.36  | 0.69  | 1.24  | 2.09  |
| 1.5              | 0    | 0    | 0    | 0    | 0    | 0.01 | 0.04 | 0.11 | 0.24  | 0.51  | 0.96  | 1.68  | 2.78  |
| 1.6              | 0    | 0    | 0    | 0    | 0    | 0.02 | 0.06 | 0.15 | 0.35  | 0.7   | 1.3   | 2.24  | 3.63  |
| 1.7              | 0    | 0    | 0    | 0    | 0.01 | 0.03 | 0.08 | 0.21 | 0.48  | 0.95  | 1.73  | 2.92  | 4.65  |
| 1.8              | 0    | 0    | 0    | 0    | 0.01 | 0.04 | 0.12 | 0.3  | 0.65  | 1.26  | 2.26  | 3.75  | 5.87  |
| 1.9              | 0    | 0    | 0    | 0    | 0.01 | 0.05 | 0.16 | 0.4  | 0.86  | 1.65  | 2.9   | 4.73  | 7.27  |
| 2                | 0    | 0    | 0    | 0    | 0.02 | 0.08 | 0.22 | 0.53 | 1.12  | 2.12  | 3.66  | 5.88  | 8.89  |
| 2.1              | 0    | 0    | 0    | 0.01 | 0.03 | 0.1  | 0.29 | 0.7  | 1.44  | 2.68  | 4.55  | 7.2   | 10.7  |
| 2.2              | 0    | 0    | 0    | 0.01 | 0.04 | 0.14 | 0.38 | 0.9  | 1.83  | 3.34  | 5.59  | 8.69  | 12.72 |
| 2.3              | 0    | 0    | 0    | 0.01 | 0.05 | 0.18 | 0.49 | 1.14 | 2.28  | 4.11  | 6.76  | 10.36 | 14.93 |
| 2.4              | 0    | 0    | 0    | 0.01 | 0.07 | 0.23 | 0.63 | 1.43 | 2.82  | 4.99  | 8.09  | 12.2  | 17.33 |
| 2.5              | 0    | 0    | 0    | 0.02 | 0.09 | 0.29 | 0.79 | 1.77 | 3.44  | 5.99  | 9.56  | 14.22 | 19.91 |
| 2.6              | 0    | 0    | 0    | 0.02 | 0.11 | 0.37 | 0.98 | 2.16 | 4.14  | 7.11  | 11.19 | 16.39 | 22.63 |
| 2.7              | 0    | 0    | 0    | 0.03 | 0.14 | 0.46 | 1.21 | 2.62 | 4.94  | 8.36  | 12.96 | 18.72 | 25.5  |
| 2.8              | 0    | 0    | 0    | 0.04 | 0.17 | 0.57 | 1.47 | 3.14 | 5.83  | 9.73  | 14.87 | 21.19 | 28.47 |
| 2.9              | 0    | 0    | 0.01 | 0.05 | 0.22 | 0.7  | 1.77 | 3.73 | 6.83  | 11.22 | 16.92 | 23.78 | 31.54 |
| 3                | 0    | 0    | 0.01 | 0.06 | 0.27 | 0.85 | 2.12 | 4.39 | 7.92  | 12.84 | 19.09 | 26.48 | 34.68 |
| 3.1              | 0    | 0    | 0.01 | 0.07 | 0.33 | 1.03 | 2.51 | 5.13 | 9.12  | 14.57 | 21.38 | 29.27 | 37.87 |
| 3.2              | 0    | 0    | 0.01 | 0.09 | 0.4  | 1.23 | 2.95 | 5.94 | 10.42 | 16.42 | 23.77 | 32.14 | 41.08 |
| 3.3              | 0    | 0    | 0.01 | 0.11 | 0.48 | 1.45 | 3.45 | 6.83 | 11.82 | 18.37 | 26.26 | 35.06 | 44.29 |
| 3.4              | 0    | 0    | 0.02 | 0.13 | 0.57 | 1.71 | 4    | 7.81 | 13.31 | 20.43 | 28.82 | 38.02 | 47.48 |
| 3.5              | 0    | 0    | 0.02 | 0.16 | 0.67 | 1.99 | 4.6  | 8.86 | 14.9  | 22.57 | 31.45 | 40.99 | 50.64 |

## REFERENCES

- [1] GRUNSPAN, C.; PÉREZ-MARCO, R.; *Double spend races*, arXiv:1702.02867, hal-01456773, mp-arc 17-17, February 2017.
- [2] NAKAMOTO, S.; *Bitcoin: A Peer-to-Peer Electronic Cash System*, [www.bitcoin.org/ bitcoin.pdf](http://www.bitcoin.org/bitcoin.pdf), 2009.

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