On the definition of Euler Gamma function

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"On the definition of Euler Gamma function", ArXiv:2001.04445.

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Definition (Euler Gamma function)

The unique solution is Euler Gamma function.

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$$g(s) = s^{-1} \prod_{n=1}^{+\infty} \left(1 + \frac{s}{n}\right)^{-1} e^{s/n}$$

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- Let $Q \in \mathbb{C}[s]$ such that $\Delta Q(s) = Q(s+1) - Q(s) = P(s)$ and $\exp Q(0) = g(1)^{-1}$. Then $\Gamma(s) = e^{-Q(s)} g(s)$ is a solution.

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• Therefore $F(s) = \exp A(s) = e^a = 1$ and $f = \Gamma$.

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• Looking carefully the existence proof we prove that $Q(s) = \gamma s$ where γ is Euler constant.

• We do not need to assume Γ meromorphic on all $\mathbb{C}$ but just on a region Ω which is transitive for the translation $s \mapsto s+1$, and with finite order estimates.

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$$\Gamma(s) = \lim_{n\to+\infty} \frac{n!}{s(s+1)\ldots(s+n)}\, n^s$$

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- Birkhoff asymptotic (1913):

$$\Gamma(s) = \lim_{n \to +\infty} \frac{\varphi(x+n+1)}{s(s+1)\dots(s+n)}$$

where

$$\varphi(s) = \sqrt{2\pi} s^{s-1/2} e^{-s} \ .$$

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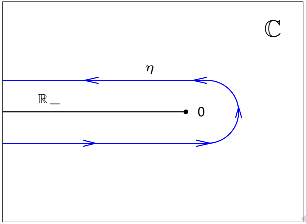

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• We could always multiply a solution of the functional equation by the exponential of any $\mathbb{Z}$-periodic function vanishing at the integers...

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- Laugwith and Rodewald (1987):

$$\lim_{n\to+\infty} (\log\Gamma(s+n) - \log\Gamma(n) - s\log(n+1)) = 0$$

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- Fuglede (2008), weakens Wielandt's boundedness allowing exponential growth (of order 1) in the vertical strip,

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- Smith (2006), enough to have $f(s)^{-1} = \mathcal{O}(exp(cs))$ with $c < \pi$.

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Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ *be a positive function,* $f(1) = 1$*, such that* $\log f$ *is convex and* $f(x+1) = xf(x)$ *then* $f = \Gamma_{/\mathbb{R}}$*.*

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- Very popular among algebrists. Remmert was very critical.
- Only natural from the real analysis point of view, but not from the complex analysis viewpoint.

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The map $\Gamma : \mathcal{E}_\omega \to \mathcal{E}_\omega$, $\Gamma(f) = \Gamma^f$ is an injective continuous morphism of group morphism that solves the cohomological equation

$$T(g).g^{-1} = f$$

where T is the shift operator, $T(g(s)) = g(s+1)$.

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1. $\Gamma_0^f(s) = f(s)$,
2. $\Gamma_N^f(1) = 1$,
3. $\Gamma_{N+1}^f(s+1) = \Gamma_N^f(s)^{-1}\,\Gamma_{N+1}^f(s)$, *for* $N \geq 0$,
4. Γ_N^f *is a meromorphic function of finite order,*
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Theorem (Barnes higher Gamma functions)

For $f(s) = s$ we obtain Barnes higher Gamma functions.

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Thank you!