A simple dynamical model leading to Pareto distribution and stability

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Pareto distribution and stability

Pareto distribution

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$$\lim_{x \to +\infty} -\log f(x) / \log x = \alpha$$

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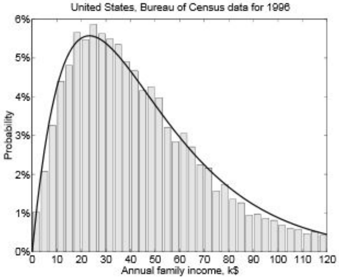
Vilfredo Pareto (1848-1923)



USA annual famiiy income

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- Etc, etc,...

Universality

An idea hinting at the universality:

Pareto distribution is typical from competitive systems

where the reward is proportional to the actual wealth.

Dynamical Stability

Pareto did conjeture the **Dynamical Stability** of the distribution:

"Si, par exemple, on enlevait tout leur revenu aux citoyens les plus riches, en supprimant la queue de la figure des revenus, celle-ci ne conserverait pas cette forme, mais tôt ou tard elle se rétablirait suivant une forme semblable à la première."

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"If, for instance, we confiscate all income to the richests citizens, thus erasing the tail of income distribution, this shape will not persist and sooner or later it will evolve to a similar shape of the original."

The game

Wealth evolution happens by rounds. The model contains three parameters: $0 < p < 1$, $\gamma > 0$, $\kappa > 0$.

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- Wins with probability $0 < p < 1$, then wealth is multiplied by $1 + \gamma$,
- Looses with probability $0 < q = 1 - p < 1$, then wealth is divided by $1 + \gamma$.
- Global taxes drains individual wealth by a factor $\kappa > 1$.

Wealth operator

The distribution of wealth $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, positive, assymptotically decreasing to 0 at $+\infty$, such that $f(x)dx$ is the number of individuals with wealth in the interval $[x, x + dx[$.

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In one round, the wealth distribution is transformed by the wealth operator:

$$\mathcal{W}_\kappa(f) = \frac{p}{\kappa(1+\gamma)} \, f(x/(1+\gamma)) + \frac{(1-p)(1+\gamma)}{\kappa} \, f(x(1+\gamma))$$

Wealth preserving model

We have

$$||\mathcal{W}(f)||_{L^1} = \frac{1}{\kappa}||f||_{L^1} \ ,$$

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The condition $\kappa > 1$ is equivalent to $p > 1/2$.

Equilibrium equation

The invariant distributions satisfy the **Equilibrium Equation**:

$$f(x) = \frac{p}{\kappa(1+\gamma)} f(x/(1+\gamma)) + \frac{(1-p)(1+\gamma)}{\kappa} f(x(1+\gamma)) \quad (1)$$

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Changing variables $x \mapsto e^x$, we consider $F(x) = f(e^x)$. Then the equilibrium equation becomes the following functional equation:

$$aF(x+\lambda) - F(x) + bF(x-\lambda) = 0 \ , \quad (2)$$

where $a, b, \lambda > 0$ and $a = (1-p)(1+\gamma)/\kappa$, $b = p/(\kappa(1+\gamma))$, $\lambda = \log(1+\gamma)$.

Solution

Theorem

The general solution of the Equilibrium Equation (2),

$$a\,F(x+\lambda) - F(x) + b\,F(x-\lambda) = 0\ ,$$

(with a, b, λ as before) is

$$F(x) = e^{\rho_0 x} L_0(x/\lambda) + e^{\rho_1 x} L_1(x/\lambda)$$

where L_0 and L_1 are $\mathbb{Z}$-periodic functions, and $e^{\lambda\rho_0}$ and $e^{\lambda\rho_1}$ are the two real solutions of the second degree equation $aX^2 - X + b = 0$, where $\rho_0 < 0 < \rho_1$.

Admissible solutions

The discriminant is positive since $\kappa > 1$ and

$$\Delta = 1 - 4ab = 1 - 4\frac{p(1-p)}{\kappa^2} > 1 - 4p(1-p) > 0 \ .$$

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The only admissible solutions with $\lim_{x \to +\infty} f(x) = 0$ are $F(x) = e^{\rho_0 x} L_0(x/\lambda)$ or

$$f(x) = x^{\rho_0} L_0(\log x/\lambda) \, .$$

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Therefore they all have Pareto assymptotic:

$$\lim_{x\to+\infty} -\frac{\log f(x)}{\log x} = -\rho_0 = \alpha \ .$$

Exponent formula

We can compute in closed form the Pareto exponent:

Corollary

The Pareto exponent is given by

$$\alpha = -\rho_0 = -\lambda^{-1} \log \left(\frac{1 - \sqrt{1 - 4ab}}{2a} \right)$$

with $\lambda = \log(1+\gamma)$, $a = a = (1-p)(1+\gamma)/\kappa$, $p/(\kappa(1+\gamma))$, *or*

$$\alpha = 1 - \frac{\log \left(\frac{\kappa - \sqrt{\kappa^2 - 4p(1-p)}}{2(1-p)} \right)}{\log(1+\gamma)} .$$

Economic consequences

- We have $\frac{d\alpha}{d\kappa} > 0$, therefore stronger fiscal pressure increases Pareto exponent and reduces inequalities.

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- We have $\frac{d\alpha}{d\gamma} < 0$, therefore a more active economy reduces Pareto exponent and increases inequalities.
- For $\kappa = 1$ (no dissipation by taxes) and $p = 1/2$ (stagnating economy), we have $\alpha = 0$, the distribution is constant.

A remarkable solution

In the wealth preserving model $\kappa = 1$, we have a remarkable solution.

By direct computation, the Pareto exponent is $\boxed{\alpha = 1}$.

This is the threshold for summability of the tail of the distribution.

Equivalent conditions

We consider the tail wealth:

$$W(f, x_0) = \int_{x_0}^{+\infty} f(x)\, dx < +\infty\ .$$

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2. *The Pareto exponent* α *is larger than* 1*,* $\alpha > 1$*.*

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1. *The tail wealth is summable,* $W(f, x_0) < +\infty$ *.*
2. *The Pareto exponent* α *is larger than* 1*,* $\alpha > 1$*.*
3. *The model is wealth dissipative, that is* $\kappa > 1$ *.*

Dynamical stability

The wealth operator is L^1-contracting:

Lemma

Let $F, G \in L^1(\mathbb{R})$, then

$$||\mathcal{W}_\kappa(F) - \mathcal{W}_\kappa(G)||_{L^1} \leq \kappa^{-1}||F - G||_{L^1} \ .$$

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Theorem

For any G that is a bounded perturbation of an invariant solution F_0 of the equilibrium equation, we have that $\mathcal{W}_\kappa^n(G) \to F_0$ for the L^1-norm at a geometric rate.

A refined model (with R. Douady)

We add he assumption of a minimal survival wealth $w_0 > 0$m so that $f(x) = 0$ for $0 < x < w_0$.

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The wealth evolution operator is more complex:

$$\mathcal{W}_\kappa(f) = \frac{p}{\kappa(1+\gamma)} f(x/(1+\gamma)) + \frac{(1-p)(1+\gamma)}{\kappa} f(x(1+\gamma))$$

Thank you

Thank you for your attention!

Solution of the functional equation

A function $F(x) = e^{\rho x}$ is a solution if $e^{\rho\lambda}$ satisfies the second degree equation:

$$a\left(e^{\rho\lambda}\right)^2 - \left(e^{\rho\lambda}\right) + b = 0 \ .$$

So $e^{\rho_0 x}$ and $e^{\rho_1 x}$ are the two exponential solutions.

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So $e^{\rho_0 x}$ and $e^{\rho_1 x}$ are the two exponential solutions.
Next, for a solution F, consider

$$H(x) = F(x+\lambda) - e^{\rho_0\lambda}F(x)\ .$$

Solution of the functional equation

Then the functional equation becomes

$$H(x) = \left(\frac{b}{a} \, e^{-\rho_0 \lambda} \right) H(x - \lambda) \, .$$

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And considering

$$\hat{H}(x) = \left(\frac{b}{a} \, e^{-\rho_0 \lambda} \right)^{-x/\lambda} H(x) \, ,$$

we have $\hat{H}(x) = \hat{H}(x - \lambda)$, i.e. there is a $\mathbb{Z}$-periodic function L such that

$$H(x) = \left(\frac{b}{a} \, e^{-\rho_0 \lambda} \right)^{x/\lambda} L(x/\lambda) \, .$$

Solution of the functional equation

Therefore we have

$$F(x+\lambda) - e^{\rho_0 \lambda} F(x) = \left(\frac{b}{a} \, e^{-\rho_0 \lambda} \right)^{x/\lambda} L(x/\lambda) \, .$$

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Therefore we have

$$F(x+\lambda) - e^{\rho_0 \lambda} F(x) = \left(\frac{b}{a} \, e^{-\rho_0 \lambda} \right)^{x/\lambda} L(x/\lambda) \; .$$

Now, put

$$\hat{F}(x) = e^{-\rho_0 x} F(x) \; .$$

Then we need to solve

$$\hat{F}(x+\lambda) - \hat{F}(x) = e^{-\rho_0 \lambda} \, \left(\frac{b}{a} \right)^{x/\lambda} \, e^{-2\rho_0 x} L(x/\lambda) \; .$$

Solution of the functional equation

If we write $G(x) = e^{\rho_0 \lambda} \hat{F}(x)$ and $c = -2\rho_0 + \lambda^{-1} \log(b/a)$, the last equation is

$$G(x+\lambda) - G(x) = e^{cx} L(x/\lambda) .$$

Solution of the functional equation

Lemma

For $c \in \mathbb{R}$, $\lambda > 0$, and L a $\mathbb{Z}$-periodic function, the solutions of the functional equation

$$G(x+\lambda) - G(x) = e^{cx} L(x/\lambda) \ , \tag{3}$$

are of the form $G(x) = G_0(x) + M(x/\lambda)$, where M is a $\mathbb{Z}$-periodic function, and for $c \neq 0$,

$$G_0(x) = \frac{e^{cx}}{e^{c\lambda} - 1} \, L(x/\lambda) \ ,$$

and for $c = 0$

$$G_0(x) = \lambda^{-1} x \, L(x/\lambda) \ .$$

Solution of the functional equation

From $\Delta > 0$ we get

Lemma

We have $c \neq 0$.

and the general solution of the Equilibrium Equation follows.