A simple dynamical model leading to Pareto distribution and stability

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Pareto distribution and stability

1. Pareto wealth distribution
2. The model
3. Invariant distributions
4. Pareto exponent
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6. A refined model
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Pareto distribution

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$$\lim_{x \to +\infty} - \frac{\log f(x)}{\log x} = \alpha$$
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Pareto’s distribution is universal and appears in numerous contexts:

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- Etc, etc,...
An idea hinting at the universality:

**Pareto distribution is typical from competitive systems**

where the reward is proportional to the actual wealth.
Dynamical Stability

Pareto did conjecture the **Dynamical Stability** of the distribution:

"Si, par exemple, on enlevait tout leur revenu aux citoyens les plus riches, en supprimant la queue de la figure des revenus, celle-ci ne conserverait pas cette forme, mais tôt ou tard elle se rétablirait suivant une forme semblable à la première."
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"If, for instance, we confiscate all income to the richest citizens, thus erasing the tail of income distribution, this shape will not persist and sooner or later it will evolve to a similar shape of the original."
The game

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At each round, financial decisions is a betting game, and taxes a fixed drain a fixed amount.

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At each round, finantial decisions is a betting game, and taxes a fixed drain a fixed amount.

- Each finantial decision bets a fraction \(\gamma\) of wealth.
- Wins with probability \(0 < p < 1\), then wealth is multiplied by \(1 + \gamma\),
- Looses with probability \(0 < q = 1 - p < 1\), then wealth is divided by \(1 + \gamma\).
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- Wins with probability \(0 < p < 1\), then wealth is multiplied by \(1 + \gamma\),
- Looses with probability \(0 < q = 1 - p < 1\), then wealth is divided by \(1 + \gamma\).
- Global taxes drains individual wealth by a factor \(\kappa > 1\).
Wealth operator

The distribution of wealth $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous, positive, asymptotically decreasing to 0 at $+\infty$, such that $f(x)dx$ is the number of individuals with wealth in the interval $[x, x + dx]$. 
Wealth operator

The distribution of wealth \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a continuous, positive, asymptotically decreasing to 0 at \( +\infty \), such that \( f(x)dx \) is the number of individuals with wealth in the interval \([x, x + dx]\).

In one round, the wealth distribution is transformed by the wealth operator:

\[
\mathcal{W}_\kappa(f) = \frac{p}{\kappa(1+\gamma)} f\left(\frac{x}{1+\gamma}\right) + \frac{(1-p)(1+\gamma)}{\kappa} f\left(x(1+\gamma)\right)
\]
Wealth preserving model

We have

$$\|\mathcal{W}(f)\|_{L^1} = \frac{1}{\kappa} \|f\|_{L^1},$$
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The condition \(\kappa > 1\) is equivalent to \(p > 1/2\).
Equilibrium equation

The invariant distributions satisfy the Equilibrium Equation:

\[ f(x) = \frac{p}{\kappa(1 + \gamma)} f\left(\frac{x}{1 + \gamma}\right) + \frac{(1 - p)(1 + \gamma)}{\kappa} f\left(x(1 + \gamma)\right) \] (1)
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\]

Changing variables \(x \mapsto e^x\), we consider \(F(x) = f(e^x)\). Then the equilibrium equation becomes the following functional equation:

\[
a F(x + \lambda) - F(x) + b F(x - \lambda) = 0 , \quad (2)
\]

where \(a, b, \lambda > 0\) and \(a = (1 - p)(1 + \gamma)/\kappa, b = p/(\kappa(1 + \gamma))\), \(\lambda = \log(1 + \gamma)\).
Solution

Theorem

The general solution of the Equilibrium Equation (2),

\[ a \, F(x + \lambda) - F(x) + b \, F(x - \lambda) = 0, \]

(with \( a, b, \lambda \) as before) is

\[ F(x) = e^{\rho_0 x} L_0(x/\lambda) + e^{\rho_1 x} L_1(x/\lambda) \]

where \( L_0 \) and \( L_1 \) are \( \mathbb{Z} \)-periodic functions, and \( e^{\lambda \rho_0} \) and \( e^{\lambda \rho_1} \) are the two real solutions of the second degree equation

\[ aX^2 - X + b = 0, \]

where \( \rho_0 < 0 < \rho_1 \).
Admissible solutions

The discriminant is positive since $\kappa > 1$ and

$$\Delta = 1 - 4ab = 1 - 4\frac{p(1-p)}{\kappa^2} > 1 - 4p(1-p) > 0.$$
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Two real roots $0 < X_0 < 1 < X_1$, so $\rho_0 < 0 < \rho_1$. The only admissible solutions with $\lim_{x \to +\infty} f(x) = 0$ are $F(x) = e^{\rho_0 x} L_0(x/\lambda)$ or

$$f(x) = x^{\rho_0} L_0(\log x/\lambda).$$
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Therefore they all have Pareto asymptotic:

$$\lim_{x \to +\infty} -\frac{\log f(x)}{\log x} = -\rho_0 = \alpha.$$
Exponent formula

We can compute in closed form the Pareto exponent:

**Corollary**

The Pareto exponent is given by

\[ \alpha = -\rho_0 = -\lambda^{-1} \log \left( \frac{1 - \sqrt{1 - 4ab}}{2a} \right) \]

with \( \lambda = \log(1 + \gamma) \), \( a = a = \frac{(1 - p)(1 + \gamma)}{\kappa}, \frac{p}{\kappa(1 + \gamma)} \), or

\[ \alpha = 1 - \frac{\log \left( \frac{\kappa - \sqrt{\kappa^2 - 4p(1-p)}}{2(1-p)} \right)}{\log(1 + \gamma)} . \]

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Economic consequences

- We have $\frac{d\alpha}{d\kappa} > 0$, therefore stronger fiscal pressure increases Pareto exponent and reduces inequalities.
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- We have $\frac{d\alpha}{d\kappa} > 0$, therefore stronger fiscal pressure increases Pareto exponent and reduces inequalities.
- We have $\frac{d\alpha}{d\gamma} < 0$, therefore a more active economy reduces Pareto exponent and increases inequalities.
- For $\kappa = 1$ (no dissipation by taxes) and $p = 1/2$ (stagnating economy), we have $\alpha = 0$, the distribution is constant.
A remarkable solution

In the wealth preserving model \( \kappa = 1 \), we have a remarkable solution.

By direct computation, the Pareto exponent is \( \alpha = 1 \).

This is the threshold for summability of the tail of the distribution.
We consider the tail wealth:

\[ W(f, x_0) = \int_{x_0}^{+\infty} f(x) \, dx < +\infty. \]

**Theorem**

For a stable solution, the following conditions are equivalent:

1. The tail wealth is summable, \( W(f, x_0) < +\infty \).
2. The Pareto exponent \( \alpha \) is larger than 1, \( \alpha > 1 \).
3. The model is wealth dissipative, that is \( \kappa > 1 \).
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Dynamical stability

The wealth operator is $L^1$-contracting:

**Lemma**

Let $F, G \in L^1(\mathbb{R})$, then

$$\|\mathcal{W}_\kappa(F) - \mathcal{W}_\kappa(G)\|_{L^1} \leq \kappa^{-1} \|F - G\|_{L^1}.$$
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**Theorem**

For any $G$ that is a bounded perturbation of an invariant solution $F_0$ of the equilibrium equation, we have that $\mathcal{W}^n_\kappa(G) \to F_0$ for the $L^1$-norm at a geometric rate.
A refined model (with R. Douady)

We add the assumption of a minimal survival wealth $w_0 > 0$ so that $f(x) = 0$ for $0 < x < w_0$. 
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We add the assumption of a minimal survival wealth \( w_0 > 0 \) so that \( f(x) = 0 \) for \( 0 < x < w_0 \).

The wealth evolution operator is more complex:

\[
\mathcal{W}_\kappa(f) = \frac{p}{\kappa(1 + \gamma)} f\left(\frac{x}{1 + \gamma}\right) + \frac{(1 - p)(1 + \gamma)}{\kappa} f\left(x(1 + \gamma)\right)
\]
Thank you for your attention!
Solution of the functional equation

A function $F(x) = e^{\rho x}$ is a solution if $e^{\rho \lambda}$ satisfies the second degree equation:

$$a \left( e^{\rho \lambda} \right)^2 - \left( e^{\rho \lambda} \right) + b = 0.$$ 

So $e^{\rho_0 x}$ and $e^{\rho_1 x}$ are the two exponential solutions.
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Next, for a solution $F$, consider

$$H(x) = F(x + \lambda) - e^{\rho_0 \lambda} F(x).$$
Solution of the functional equation

Then the functional equation becomes

\[ H(x) = \left( \frac{b}{a} e^{-\rho_0 \lambda} \right) H(x - \lambda). \]
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And considering

\[ \hat{H}(x) = \left( \frac{b}{a} e^{-\rho_0 \lambda} \right)^{-x/\lambda} H(x), \]

we have \( \hat{H}(x) = \hat{H}(x - \lambda) \), i.e. there is a \( \mathbb{Z} \)-periodic function \( L \) such that

\[ H(x) = \left( \frac{b}{a} e^{-\rho_0 \lambda} \right)^{x/\lambda} L(x/\lambda). \]
Solution of the functional equation

Therefore we have

\[ F(x + \lambda) - e^{\rho_0 \lambda} F(x) = \left( \frac{b}{a} e^{-\rho_0 \lambda} \right)^{x/\lambda} L(x/\lambda). \]
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Therefore we have

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Now, put

\[ \hat{F}(x) = e^{-\rho_0 x} F(x) . \]

Then we need to solve

\[ \hat{F}(x + \lambda) - \hat{F}(x) = e^{-\rho_0 \lambda} \left( \frac{b}{a} \right)^{x/\lambda} e^{-2\rho_0 x} L(x/\lambda) . \]
Solution of the functional equation

If we write $G(x) = e^{\rho_0 \lambda} \hat{F}(x)$ and $c = -2\rho_0 + \lambda^{-1} \log(b/a)$, the last equation is

$$G(x + \lambda) - G(x) = e^{cx} L(x/\lambda).$$
For $c \in \mathbb{R}$, $\lambda > 0$, and $L$ a $\mathbb{Z}$-periodic function, the solutions of the functional equation

$$G(x + \lambda) - G(x) = e^{cx} L(x/\lambda) ,$$

are of the form $G(x) = G_0(x) + M(x/\lambda)$, where $M$ is a $\mathbb{Z}$-periodic function, and for $c \neq 0$,

$$G_0(x) = \frac{e^{cx}}{e^{c\lambda} - 1} L(x/\lambda) ,$$

and for $c = 0$

$$G_0(x) = \lambda^{-1} x L(x/\lambda) .$$
Solution of the functional equation

From $\Delta > 0$ we get

Lemma

*We have $c \neq 0$.*

and the general solution of the Equilibrium Equation follows.