Bitcoin Selfish Mining and Dyck Words (with C. Grunspan)

Ricardo Pérez-Marco (CNRS, IMJ-PRG, Univ. Paris 7)

Breaking Bitcoin 2019

Amsterdam

June 7, 2019
Block withholding strategies

1. Profitability
2. Block withholding mining strategies
3. Markov model.
4. Martingale analysis
5. Combinatorics approach
6. Other block withholding strategies
7. Selfish Mining in Ethereum
Bibliography

Bibliography

Bibliography

Bibliography

Bibliography

Results
Profit and Loss per unit of time

\[ P & L = R - C \]

where \( R \) is the revenue and \( C \) is the cost.

Time considerations:
What matters for a business is the profit and loss per unit of time:
\[ P & L_T \]

And the asymptotic profit and loss per unit of time:
\[ P & L_T \to \infty = \lim_{T \to +\infty} R(T) - C(T) \]
Profit and Loss per unit of time

- **Profit and Loss:**

  \[ P & L = R - C \]

  where \( R \) is the revenue and \( C \) is the cost.
Profit and Loss per unit of time

- **Profit and Loss:**

  \[ P&L = R - C \]

  where \( R \) is the revenue and \( C \) is the cost.

- **Time considerations:**
Profit and Loss per unit of time

• Profit and Loss:

\[ P&L = R - C \]

where \( R \) is the revenue and \( C \) is the cost.

• Time considerations:

What matters for a business is the \( P&L \) per unit of time: \( P&LT \).
Profit and Loss per unit of time

- **Profit and Loss:**

  
  \[ P\&L = R - C \]

  where \( R \) is the revenue and \( C \) is the cost.

- **Time considerations:**

  What matters for a business is the \( P\&L \) per unit of time: \( P\&LT \).

  And the asymptotic \( P\&L \) per unit of time

  \[ P\&LT_{\infty} = \lim_{T \to +\infty} \frac{R(T) - C(T)}{T} \]
Repetition games
Repetition games

- Repeat over and over (cycles) a profitable strategy.
Repetition games

- Repeat over and over (cycles) a profitable strategy.

We consider three random variables:

\[ R \text{ revenue per cycle, } C \text{ cost per cycle, } T \text{ time per cycle.} \]

**Theorem (Profitability of a Repetition Game)**

\[ P \lim_{T \to \infty} = E[R] - E[C]. \]

**Proof.** Strong law of large numbers.
Repetition games

- Repeat over and over (cycles) a profitable strategy.

We consider three random variables:

- $R$ revenue per cycle.

\[ \text{Theorem (Profitability of a Repetition Game)} \]
\[ P \lt \lim_{\infty} = E[R] - E[C] \cdot E[T]. \]

\[ \text{Proof.} \]
\[ \text{Strong law of large numbers.} \]
Repetition games

- Repeat over and over (cycles) a profitable strategy.

We consider three random variables:

- $R$ revenue per cycle.
Repetition games

- Repeat over and over \textit{(cycles)} a profitable strategy.

We consider three random variables:

- \( R \) revenue per cycle.
- \( C \) cost per cycle.
Repetition games

- Repeat over and over (cycles) a profitable strategy.

We consider three random variables:

- $R$ revenue per cycle.
- $C$ cost per cycle.
Repetition games

- Repeat over and over (cycles) a profitable strategy.

We consider three random variables:

- $R$ revenue per cycle.
- $C$ cost per cycle.
- $T$ time per cycle.
Repetition games

- Repeat over and over (cycles) a profitable strategy.

We consider three random variables:

- $R$ revenue per cycle.
- $C$ cost per cycle.
- $T$ time per cycle.

Theorem (Profitability of a Repetition Game)

$$P_{LT} = E[R] - E[C] / E[T].$$

Proof. Strong law of large numbers.
Repetition games

- Repeat over and over \((\text{cycles})\) a profitable strategy.

We consider three random variables:

- \(R\) revenue per cycle.
- \(C\) cost per cycle.
- \(T\) time per cycle.

**Theorem (Profitability of a Repetition Game)**

\[
P \& LT_\infty = \frac{\mathbb{E}[R] - \mathbb{E}[C]}{\mathbb{E}[T]}.
\]
Repetition games

- Repeat over and over (cycles) a profitable strategy.

We consider three random variables:

- $R$ revenue per cycle.
- $C$ cost per cycle.
- $T$ time per cycle.

Theorem (Profitability of a Repetition Game)

$$P&LT_{\infty} = \frac{\mathbb{E}[R] - \mathbb{E}[C]}{\mathbb{E}[T]}.$$
Repetition games

- Repeat over and over *(cycles)* a profitable strategy.

We consider three random variables:

- $R$ revenue per cycle.
- $C$ cost per cycle.
- $T$ time per cycle.

**Theorem (Profitability of a Repetition Game)**

$$\text{P&L} \tau = \frac{\mathbb{E}[R] - \mathbb{E}[C]}{\mathbb{E}[T]}.$$ 

**Proof.** Strong law of large numbers.
Comparing repetition games with equal costs
Comparing repetition games with equal costs

Definition (Revenue and Cost Ratio)

The **Revenue Ratio**, resp. **Cost Ratio**, of a Repetition Strategy $\xi$ is

\[
\Gamma(\xi) = \frac{\mathbb{E}[R]}{\mathbb{E}[T]} \quad \text{resp.} \quad \Upsilon(\xi) = \frac{\mathbb{E}[C]}{\mathbb{E}[T]}
\]
Comparing repetition games with equal costs

Definition (Revenue and Cost Ratio)

The **Revenue Ratio**, resp. **Cost Ratio**, of a Repetition Strategy $\xi$ is

$$\Gamma(\xi) = \frac{\mathbb{E}[R]}{\mathbb{E}[T]} \quad \text{resp.} \quad \Upsilon(\xi) = \frac{\mathbb{E}[C]}{\mathbb{E}[T]}$$
Comparing repetition games with equal costs

Definition (Revenue and Cost Ratio)

The Revenue Ratio, resp. Cost Ratio, of a Repetition Strategy \( \xi \) is

\[
\Gamma(\xi) = \frac{\mathbb{E}[R]}{\mathbb{E}[T]} \quad \text{resp.} \quad \Upsilon(\xi) = \frac{\mathbb{E}[C]}{\mathbb{E}[T]}
\]

Theorem (Comparing profitabilities of equal cost strategies)

We consider two integrable strategies with \( \Upsilon(\xi) = \Upsilon(\xi') \). Then \( \xi \) is more profitable than strategy \( \xi' \) in the long run if and only if

\[
\Gamma(\xi') \leq \Gamma(\xi).
\]
### Block withholding mining

- Protocol rules dictate miners to release and propagate blocks as soon as they are mined.
Block withholding mining

- Protocol rules dictate miners to release and propagate blocks as soon as they are mined.

- Block withholding strategy: Timely release blocks in order to invalidate honest blocks.
Block withholding mining

- Protocol rules dictate miners to release and propagate blocks as soon as they are mined.

- Block withholding strategy: Timely release blocks in order to invalidate honest blocks.

- Other block withholding strategies: Selfish Mining (SM), Lead Stubborn Mining (LSM), Equal Fork Stubborn Mining (EFSM), Trail Mining (TSM).
Block withholding mining

- Protocol rules dictate miners to release and propagate blocks as soon as they are mined.

- Block withholding strategy: Timely release blocks in order to invalidate honest blocks.

- Other block withholding strategies: Selfish Mining (SM), Lead Stubborn Mining (LSM), Equal Fork Stubborn Mining (EFSM), Trail Mining (TSM).

They differ by the algorithm by which we release blocks. This depends on the chain state (length of official blockchain, length of secret fork, etc)
Block withholding mining

- Protocol rules dictate miners to release and propagate blocks as soon as they are mined.

- Block withholding strategy: Timely release blocks in order to invalidate honest blocks.

- Other block withholding strategies: Selfish Mining (SM), Lead Stubborn Mining (LSM), Equal Fork Stubborn Mining (EFSM), Trail Mining (TSM).

They differ by the algorithm by which we release blocks. This depends on the chain state (length of official blockchain, length of secret fork, etc)

- Key observation:

  Block withholding and honest strategies have exactly the same cost
Important observations

- Part of the revenue comes from the block rewards determined by the protocol. The revenue in fiat currency depends on the market exchange rate of the coin, but the comparison of the profitability is independent of the exchange rate.
- The cost depends on external factors like energy prize, hardware costs, etc. These costs are independent of the strategy as long as the strategy is a full-time intensive computation.
Important observations

- Part of the Revenue comes the block rewards determined by the protocol. The revenue in fiat currency depends on the market exchange rate of the coin, but the comparison of the profitability is independent of the exchange rate.
Important observations

- Part of the Revenue comes the block rewards determined by the protocol. The revenue in fiat currency depends on the market exchange rate of the coin, but the comparison of the profitability is independent of the exchange rate.

- The Cost depends on external factors like energy prize, hardware costs, etc. These costs are independent of the strategy as long as the strategy is a full time intensive computation.
Profitability of block withholding strategies

A block withholding strategy $\xi$ is profitable in the long run iff $\Gamma(HM) \leq \Gamma(\xi)$.

**Theorem (Stability Theorem, G-PM, 2018)**

Without a difficulty adjustment, no block withholding strategy $\xi$ can be profitable and we always have $\Gamma(\xi) \leq \Gamma(HM)$.

**Proof.** Martingale techniques, and Doob's Stopping Time Thm.

R. Pérez-Marco

Bitcoin Selfish Mining and Dyck Words (with C. Grunspan)
Profitability of block withholding strategies

- A block withholding strategy $\xi$ is profitable in the long run iff

$$\Gamma(HM) \leq \Gamma(\xi)$$
A block withholding strategy $\xi$ is profitable in the long run iff

$$\Gamma(HM) \leq \Gamma(\xi)$$

**Theorem (Stability Theorem, G-PM, 2018)**

*Without a difficulty adjustment, no block withholding strategy $\xi$ can be profitable and we always have*

$$\Gamma(\xi) \leq \Gamma(HM)$$
Profitability of block withholding strategies

- A block withholding strategy $\xi$ is profitable in the long run iff
  \[ \Gamma(HM) \leq \Gamma(\xi) \]

Theorem (Stability Theorem, G-PM, 2018)

*Without a difficulty adjustment, no block withholding strategy $\xi$ can be profitable and we always have*

\[ \Gamma(\xi) \leq \Gamma(HM) \]
Profitability of block withholding strategies

- A block withholding strategy $\xi$ is profitable in the long run iff

$$\Gamma(HM) \leq \Gamma(\xi)$$

Theorem (Stability Theorem, G-PM, 2018)

*Without a difficulty adjustment, no block withholding strategy $\xi$ can be profitable and we always have

$$\Gamma(\xi) \leq \Gamma(HM)$$

Proof. Martingale techniques, and Doob’s Stopping Time Thm.*
Profitability of block withholding strategies

- A block withholding strategy $\xi$ is profitable in the long run iff

$$\Gamma(HM) \leq \Gamma(\xi)$$

**Theorem (Stability Theorem, G-PM, 2018)**

*Without a difficulty adjustment, no block withholding strategy $\xi$ can be profitable and we always have*

$$\Gamma(\xi) \leq \Gamma(HM)$$

**Proof.** Martingale techniques, and Doob’s Stopping Time Thm.
An attack on the difficulty adjustment
An attack on the difficulty adjustment

Corollary

A block withholding strategy is an attack on the Difficulty Adjustment Formula.
An attack on the difficulty adjustment

Corollary

A block withholding strategy is an attack on the Difficulty Adjustment Formula.
An attack on the difficulty adjustment

Corollary

A block withholding strategy is an attack on the Difficulty Adjustment Formula.

- Main problem: The DA formula does not account properly orphaned blocks.
An attack on the difficulty adjustment

Corollary

A block withholding strategy is an attack on the Difficulty Adjustment Formula.

- Main problem: The DA formula does not account properly orphaned blocks.
- Solution: Modification of the DA formula to take orphan blocks into account.
An attack on the difficulty adjustment

Corollary

A block withholding strategy is an attack on the Difficulty Adjustment Formula.

- Main problem: The DA formula does not account properly orphaned blocks.
- Solution: Modification of the DA formula to take orphan blocks into account.
An attack on the difficulty adjustment

Corollary

A block withholding strategy is an attack on the Difficulty Adjustment Formula.

- Main problem: The DA formula does not account properly orphaned blocks.
- Solution: Modification of the DA formula to take orphan blocks into account.
The Markov model

- Main parameters $0 < q < 1/2$ relative hashrate of the attacker, $0 \leq \gamma \leq 1$ connectivity of the attacker.
The Markov model

- Main parameters $0 < q < 1/2$ relative hashrate of the attacker, $0 \leq \gamma \leq 1$ connectivity of the attacker.

- Chain state essentially defined by length of the official blockchain and of the secret fork.
The Markov model

- Main parameters $0 < q < 1/2$ relative hashrate of the attacker, $0 \leq \gamma \leq 1$ connectivity of the attacker.

- Chain state essentially defined by length of the official blockchain and of the secret fork.

- The relative proportion of blocks mined by the block withholders gives the long run profitability of the strategy, but only after the difficulty adjustment.

R. Pérez-Marco
Bitcoin Selfish Mining and Dyck Words (with C. Grunspan)
The Markov model

• Main parameters $0 < q < 1/2$ relative hashrate of the attacker, $0 \leq \gamma \leq 1$ connectivity of the attacker.

• Chain state essentially defined by length of the official blockchain and of the secret fork.

• The relative proportion of blocks mined by the block withholders gives the long run profitability of the strategy, but only after the difficulty adjustment.

• The Markov chain model cannot provide any information about the time it takes to jump from one state to another: Hence it is impossible to answer some basic questions.
The Markov model

- Main parameters $0 < q < 1/2$ relative hashrate of the attacker, $0 \leq \gamma \leq 1$ connectivity of the attacker.

- Chain state essentially defined by length of the official blockchain and of the secret fork.

- The relative proportion of blocks mined by the block withholders gives the long run profitability of the strategy, but only after the difficulty adjustment.

- The Markov chain model cannot provide any information about the time it takes to jump from one state to another: Hence it is impossible to answer some basic questions.

- **Example of practical basic question**: How long it takes to the selfish miner to enter profitability?
The Markov model

- Main parameters $0 < q < 1/2$ relative hashrate of the attacker, $0 \leq \gamma \leq 1$ connectivity of the attacker.

- Chain state essentially defined by length of the official blockchain and of the secret fork.

- The relative proportion of blocks mined by the block withholders gives the long run profitability of the strategy, but only after the difficulty adjustment.

- The Markov chain model cannot provide any information about the time it takes to jump from one state to another: Hence it is impossible to answer some basic questions.

- Example of practical basic question: How long it takes to the selfish miner to enter profitability?
Poisson races

Theorem (Poisson Races)

$N$ and $N'$ two independent Poisson processes with parameters $\alpha'$ and $\alpha$ with $\alpha' < \alpha$ and $N(0) = N'(0) = 0$.

Then, the stopping time $\tau = \inf\{t > 0; N(t) = N'(t) + 1\}$ is finite a.s. and integrable. Moreover, we have

$E[\tau] = \frac{1}{\alpha - \alpha'}$,  
$E[N'(\tau)] = \frac{\alpha}{\alpha - \alpha'}$,  
$E[N(\tau)] = \frac{\alpha'}{\alpha - \alpha'}$.  

R. Pérez-Marco

Bitcoin Selfish Mining and Dyck Words (with C. Grunspan)
Poisson races

- $N'(t)$, resp. $N(t)$, numbers of validated blocks at time $t$ by the selfish, resp. honest, miners are Poisson processes with resp. parameters $\alpha'$ and $\alpha$, $\alpha' < \alpha$. 
Poisson races

- \( N'(t) \), resp. \( N(t) \), numbers of validated blocks at time \( t \) by the selfish, resp. honest, miners are Poisson processes with resp. parameters \( \alpha' \) and \( \alpha \), \( \alpha' < \alpha \).

\[
\mathbb{P}[N(t) = n] = \frac{(\alpha t)^n}{n!} e^{-\alpha t}, \quad \mathbb{P}[N'(t) = n] = \frac{(\alpha' t)^n}{n!} e^{-\alpha' t}
\]
Poisson races

- $N'(t)$, resp. $N(t)$, numbers of validated blocks at time $t$ by the selfish, resp. honest, miners are Poisson processes with resp. parameters $\alpha'$ and $\alpha$, $\alpha' < \alpha$.

\[ P[N(t) = n] = \frac{(\alpha t)^n}{n!} e^{-\alpha t}, \quad P[N'(t) = n] = \frac{(\alpha' t)^n}{n!} e^{-\alpha't} \]

Theorem (Poisson Races)

$N$ and $N'$ two independent Poisson processes with parameters $\alpha'$ and $\alpha$ with $\alpha' < \alpha$ and $N(0) = N'(0) = 0$. 

R. Pérez-Marco

Bitcoin Selfish Mining and Dyck Words (with C. Grunspan)
Poisson races

- $N'(t)$, resp. $N(t)$, numbers of validated blocks at time $t$ by the selfish, resp. honest, miners are Poisson processes with resp. parameters $\alpha'$ and $\alpha$, $\alpha' < \alpha$.

$$\mathbb{P}[N(t) = n] = \frac{(\alpha t)^n}{n!} e^{-\alpha t}, \quad \mathbb{P}[N'(t) = n] = \frac{(\alpha' t)^n}{n!} e^{-\alpha' t}$$

**Theorem (Poisson Races)**

$N$ and $N'$ two independent Poisson processes with parameters $\alpha'$ and $\alpha$ with $\alpha' < \alpha$ and $N(0) = N'(0) = 0$. 
Poisson races

- \( N'(t) \), resp. \( N(t) \), numbers of validated blocks at time \( t \) by the selfish, resp. honest, miners are Poisson processes with resp. parameters \( \alpha' \) and \( \alpha \), \( \alpha' < \alpha \).

\[
\mathbb{P}[N(t) = n] = \frac{(\alpha t)^n}{n!} e^{-\alpha t}, \quad \mathbb{P}[N'(t) = n] = \frac{(\alpha' t)^n}{n!} e^{-\alpha' t}
\]

Theorem (Poisson Races)

\( N \) and \( N' \) two independent Poisson processes with parameters \( \alpha' \) and \( \alpha \) with \( \alpha' < \alpha \) and \( N(0) = N'(0) = 0 \). Then, the stopping time

\[
\tau = \inf\{t > 0; N(t) = N'(t) + 1\}
\]

is finite a.s. and integrable.
Poisson races

- \( N'(t) \), resp. \( N(t) \), numbers of validated blocks at time \( t \) by the selfish, resp. honest, miners are Poisson processes with resp. parameters \( \alpha' \) and \( \alpha \), \( \alpha' < \alpha \).

\[
P[N(t) = n] = \frac{(\alpha t)^n}{n!} e^{-\alpha t}, \quad P[N'(t) = n] = \frac{(\alpha' t)^n}{n!} e^{-\alpha' t}
\]

**Theorem (Poisson Races)**

\( N \) and \( N' \) two independent Poisson processes with parameters \( \alpha' \) and \( \alpha \) with \( \alpha' < \alpha \) and \( N(0) = N'(0) = 0 \). Then, the stopping time

\[
\tau = \inf\{ t > 0; N(t) = N'(t) + 1 \}
\]

is finite a.s. and integrable. Moreover, we have

\[
E[\tau] = \frac{1}{\alpha - \alpha'}, \quad E[N'(\tau)] = \frac{\alpha'}{\alpha - \alpha'}, \quad E[N(\tau)] = \frac{\alpha}{\alpha - \alpha'}.
\]
Direct computation of long term profitability

• Example: For $q = 0.1$ and $\gamma = 0.9$ it takes 10 weeks for a selfish mining to become profitable.
Direct computation of long term profitability

- Example: For $q = 0.1$ and $\gamma = 0.9$ it takes \textbf{10 weeks} for a selfish mining to become profitable.

- It is \textbf{impossible} to obtain this type of result using the Markov model approach.
Direct computation of long term profitability

• Example: For $q = 0.1$ and $\gamma = 0.9$ it takes **10 weeks** for a selfish mining to become profitable.

• It is **impossible** to obtain this type of result using the Markov model approach.

• We present a new, very direct, and elementary approach to compute the Revenue Ratio **without using Martingales, nor Markov Chains**.
Direct computation of long term profitability

- Example: For \( q = 0.1 \) and \( \gamma = 0.9 \) it takes \textbf{10 weeks} for a selfish mining to become profitable.

- It is \textbf{impossible} to obtain this type of result using the Markov model approach.

- We present a new, very direct, and elementary approach to compute the Revenue Ratio \textbf{without using Martingales, nor Markov Chains}.

- It uses \textbf{Dyck words, Catalan numbers, and Catalan distributions}. 
Catalan numbers (Euler-Segner numbers)
Catalan numbers (Euler-Segner numbers)

- The sequence of **Catalan numbers** is

  \[ C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}. \]

  so \( C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, C_6 = 132, \ldots \)
Catalan numbers (Euler-Segner numbers)

- The sequence of Catalan numbers is

\[ C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}. \]

so \( C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, C_6 = 132, \ldots \)

- The generating series of Catalan numbers is

\[ C(x) = \sum_{n=0}^{+\infty} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x} \] (Binet, 1838)
Catalan numbers (Euler-Segner numbers)

- The sequence of Catalan numbers is

\[ C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}. \]

so \( C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, C_6 = 132, \ldots \)

- The generating series of Catalan numbers is

\[ C(x) = \sum_{n=0}^{+\infty} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x} \quad \text{(Binet, 1838)} \]

- Catalan distributions: For \( 0 < q < p < 1 \) with \( p + q = 1 \):

\[ \sum_{n \geq 0} p(pq)^n C_n = 1 \]

\[ \sum_{n \geq 0} np(pq)^n C_n = \frac{q}{p - q} \]
Other definitions of Catalan numbers

• Number of triangulations of a convex polygon with \( n + 2 \) sides.

• Number of monotonic lattice paths on a \( n \times n \) square grid from one corner to the opposite without crossing the diagonal.
Other definitions of Catalan numbers

- Number of triangulations of a convex polygon with $n + 2$ sides.
Other definitions of Catalan numbers

- Number of **triangulations** of a convex polygon with \( n + 2 \) sides.

- Number of **monotonic lattice paths** on a \( n \times n \) square grid from one corner to the opposite without crossing the diagonal.
• Number of full binary trees with \( n+1 \) leaves.
• Number of Dyck words of length \( 2n \).

A Dyck word built on \( \{S, H\} \) is a string consisting of \( n \) \( S \)'s and \( n \) \( H \)'s such that no initial segment of the string has more \( H \)'s than \( S \)'s.

\[ SSSHHH, \quad SHSSHH, \quad SHSHSH, \quad SSHSHH \]

Let \( D \) be the space of all Dyck words. Catalan distributions define probability measures on \( D \).
• Number of **full binary trees** with \( n + 1 \) leaves.
• Number of **full binary trees** with \( n + 1 \) leaves.

• Number of **Dyck word** of length \( 2n \).
• Number of **full binary trees** with \( n + 1 \) leaves.

![Diagram of binary trees](image)

• Number of **Dyck word** of length \( 2n \).

A Dyck word built on \( \{ S, H \} \) is a string consisting of \( n \) \( S \)'s and \( n \) \( H \)'s such that no initial segment of the string has more \( H \)'s than \( S \)'s.

\[
SSSHHH, SHSSHH, SHSHSH, SSHHSH, SSHSHH
\]
• Number of full binary trees with $n + 1$ leaves.

A Dyck word built on \{S, H\} is a string consisting of $n$ S’s and $n$ H’s such that no initial segment of the string has more H’s than S’s.

\textbf{SSSHHH, SHSSHH, SHSHSH, SSHHSH, SSHSHH}

• Let $\mathcal{D}$ be the space of all Dyck words. Catalan distributions define probability measures on $\mathcal{D}$. 
Selfish Mining and Dyck words

• Attack cycles are described with the chronological sequence of discoveries S and H.
Selfish Mining and Dyck words

- Attack cycles are described with the chronological sequence of discoveries S and H.

**Theorem (Selfish Mining and Dyck words)**

*The attack cycles of the SM strategy are* $H, SHH, SHS$ and $SSwH$ *where* $w \in \mathcal{D}$.
Selfish Mining and Dyck words

- Attack cycles are described with the chronological sequence of discoveries S and H.

**Theorem (Selfish Mining and Dyck words)**

The attack cycles of the SM strategy are $H, SHH, SHS$ and $SSwH$ where $w \in D$.
Selfish Mining and Dyck words

- Attack cycles are described with the chronological sequence of discoveries S and H.

Theorem (Selfish Mining and Dyck words)

The attack cycles of the SM strategy are $H, SHH, SHS$ and $SSwH$ where $w \in D$.

- For example, $SSSHSHH$ describes an attack cycle of probability $P(SSSHSHH) = q^4 p^3$
Selfish Mining and Dyck words

- Attack cycles are described with the chronological sequence of discoveries $S$ and $H$.

**Theorem (Selfish Mining and Dyck words)**

*The attack cycles of the SM strategy are $H$, $SHH$, $SHS$ and $SSwH$ where $w \in D$. *

- For example, $SSSHSHH$ describes an attack cycle of probability $P(SSSHSHH) = q^4p^3$

**Corollary**

*Let $L$ be the number of official blocks added to the blockchain after an attack cycle. Then, $P[L = 1] = p$, $P[L = 2] = p + pq^2$ and*

$$P[L = n] = pq^2(pq)^n C_{n-2}$$
Selfish Mining

- In the same way we get the distribution of $Z$, number of blocks by the attacker added to the blockchain after a cycle.
Selfish Mining

• In the same way we get the distribution of $Z$, number of blocks by the attacker added to the blockchain after a cycle.

• We express a dimensionless Revenue Ratio in $b/\tau_0$ units where $b =$ is the coinbase and $\tau_0 =$ Bitcoin interblock time.
Selfish Mining

- In the same way we get the distribution of $Z$, number of blocks by the attacker added to the blockchain after a cycle.

- We express a dimensionless Revenue Ratio in $b/\tau_0$ units where $b = \text{is the coinbase and } \tau_0 = \text{Bitcoin interblock time.}$

**Theorem (Revenue Ratio of SM strategy)**

The Revenue Ratio of SM is

$$\Gamma_B = \frac{[\,(p - q)(1 + pq) + pq]\, q - (p - q)(1 - \gamma)p^2 q}{pq^2 + p - q}$$
Selfish Mining

- In the same way we get the distribution of $Z$, number of blocks by the attacker added to the blockchain after a cycle.
- We express a dimensionless Revenue Ratio in $b/\tau_0$ units where $b = \text{is the coinbase and } \tau_0 = \text{Bitcoin interblock time.}$

**Theorem (Revenue Ratio of SM strategy)**

The Revenue Ratio of SM is

$$\Gamma_B = \frac{[(p - q)(1 + pq) + pq]q - (p - q)(1 - \gamma)p^2q}{pq^2 + p - q}$$
Selfish Mining

- In the same way we get the distribution of $Z$, number of blocks by the attacker added to the blockchain after a cycle.
- We express a dimensionless Revenue Ratio in $b/\tau_0$ units where $b$ is the coinbase and $\tau_0 = \text{Bitcoin interblock time}$.

**Theorem (Revenue Ratio of SM strategy)**

The Revenue Ratio of SM is

$$\Gamma_B = \frac{[(p - q)(1 + pq) + pq]q - (p - q)(1 - \gamma)p^2q}{pq^2 + p - q}$$

**Sketch of proof.**

We have $\mathbb{E}[T] = \mathbb{E}[L] \tau_0$, $\mathbb{E}[Z] = \mathbb{E}[L] - (p + (2 - \gamma)p^2q)$ and $\mathbb{E}[L] = 1 + \frac{p^2q}{p-q}$. 

R. Pérez-Marco

Bitcoin Selfish Mining and Dyck Words (with C. Grunspan)
Other classical block withholding strategies

Theorem (Revenue Ratio of LSM strategy)

\[ \Gamma_{LSM} = q \left( p + pq - q^2 \right) \left( \frac{p + pq - q}{p - q} \right) \gamma \cdot \left( 1 - p \right) C \left( (1 - \gamma) pq \right) \]

Theorem (Revenue Ratio of EFSM strategy)

\[ \Gamma_{EFSM} = q \left( p - (1 - \gamma) (p - q) \gamma p \left( 1 - p C \left( (1 - \gamma) pq \right) \right) \right) \]
Other classical block withholding strategies

- LSM and EFSM: strategies invented by Miller & al.
Other classical block withholding strategies

- LSM and EFSM: strategies invented by Miller & al.

Theorem (Revenue Ratio of LSM strategy)

The Revenue Ratio of LSM is

\[ \Gamma_{LSM} = \frac{q(p + pq - q^2)}{p + pq - q} - \frac{pq(p - q)(1 - \gamma)}{\gamma} \cdot \frac{1 - p(1 - \gamma)C((1 - \gamma)pq)}{p + pq - q} \]
Other classical block withholding strategies

- LSM and EFSM: strategies invented by Miller & al.

Theorem (Revenue Ratio of LSM strategy)

The Revenue Ratio of LSM is

\[ \Gamma_{LSM} = \frac{q(p + pq - q^2)}{p + pq - q} - \frac{pq(p - q)(1 - \gamma)}{1 - \gamma \gamma} \cdot \frac{1 - p(1 - \gamma)C((1 - \gamma)pq)}{p + pq - q} \]
Other classical block withholding strategies

- LSM and EFSM: strategies invented by Miller & al.

**Theorem (Revenue Ratio of LSM strategy)**

The Revenue Ratio of LSM is

\[ \Gamma_{LSM} = \frac{q(p + pq - q^2)}{p + pq - q} - \frac{pq(p - q)(1 - \gamma)}{\gamma} \cdot \frac{1 - p(1 - \gamma)C((1 - \gamma)pq)}{p + pq - q} \]

**Theorem (Revenue Ratio of EFSM strategy)**

The Revenue Ratio of EFSM is

\[ \Gamma_{EFSM} = \frac{q}{p} - \frac{(1 - \gamma)(p - q)}{\gamma p} \left( 1 - pC((1 - \gamma)pq) \right) \]
Comparison of strategies
Comparison of strategies

We plot the region \((q, \gamma) \in [0, 0.5] \times [0, 1]\) showing which strategy is more profitable.

Figure: Most profitables strategies.
Ethereum mining rewards

- Different reward system and difficulty adjustment algorithm.
Ethereum mining rewards

- Different reward system and difficulty adjustment algorithm.
- Takes uncles and nephews into account.
Ethereum mining rewards

- Different reward system and difficulty adjustment algorithm.
- Takes uncles and nephews into account.
- Only one SM strategy in Bitcoin but several in Ethereum
Ethereum mining rewards

- Different reward system and difficulty adjustment algorithm.
- Takes uncles and nephews into account.
- Only one SM strategy in Bitcoin but several in Ethereum
- SM1: Maximum belligerence and signals all uncles. **Goal: Maximize the revenue.**
Ethereum mining rewards

- Different reward system and difficulty adjustment algorithm.
- Takes uncles and nephews into account.
- Only one SM strategy in Bitcoin but several in Ethereum
  - SM1: Maximum belligerence and signals all uncles. Goal: Maximize the revenue.
  - SM2A: Minimum belligerence and signals all uncles.

R. Pérez-Marco
Bitcoin Selfish Mining and Dyck Words (with C. Grunspan)
Ethereum mining rewards

- Different reward system and difficulty adjustment algorithm.
- Takes uncles and nephews into account.
- Only one SM strategy in Bitcoin but several in Ethereum
  - SM1: Maximum belligerence and signals all uncles. **Goal: Maximize the revenue.**
  - SM2A: Minimum belligerence and signals all uncles.
  - SM2B: Minimum belligerence and signals no uncle. **Goal: Minimize the difficulty parameter.**
Ethereum mining rewards

- Different reward system and difficulty adjustment algorithm.
- Takes uncles and nephews into account.
- Only one SM strategy in Bitcoin but several in Ethereum
  - SM1: Maximum belligerence and signals all uncles. **Goal: Maximize the revenue.**
  - SM2A: Minimum belligerence and signals all uncles.
  - SM2B: Minimum belligerence and signals no uncle. **Goal: Minimize the difficulty parameter.**
- **Best strategy** (for $q > 25\%$): Avoid competitions and ignore uncles.
Selfish Mining in Ethereum

Theorem (Comparison of SM strategies in Ethereum)

We have $\Gamma(SM1) < \Gamma(SM2A)$. For $q > 30.1\% \Gamma(SM2A) < \Gamma(SM2B)$
Selfish Mining in Ethereum

Theorem (Comparison of SM strategies in Ethereum)

We have $\Gamma(SM1) < \Gamma(SM2A)$. For $q > 30.1\%$ $\Gamma(SM2A) < \Gamma(SM2B)$.
Theorem (Comparison of SM strategies in Ethereum)

We have $\Gamma(SM1) < \Gamma(SM2A)$. For $q > 30.1\%$, $\Gamma(SM2A) < \Gamma(SM2B)$.
Conclusions

- The **Revenue Ratio** is the correct objective function to compare profitabilities.
Conclusions

- The **Revenue Ratio** is the correct objective function to compare profitabilities.
- Only the **Martingale approach** gives the full picture.
Conclusions

• The **Revenue Ratio** is the correct objective function to compare profitabilities.

• Only the **Martingale approach** gives the full picture.

• For computing only long term profitabilities, the **Markov approach** can be replaced by the more direct and simple **combinatorial approach with Dyck words**.
Conclusions

- The **Revenue Ratio** is the correct objective function to compare profitability.
- Only the **Martingale approach** gives the full picture.
- For computing only long term profitabilities, the **Markov approach** can be replaced by the more direct and simple **combinatorial approach with Dyck words**.
- Ethereum Selfish Mining confirms that **SM is an attack to the DA**.
Thank you for your attention!