Double spend races

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(joint work with Cyril Grunspan (ESILV, Paris))

Finance Seminar (Stony Brook)

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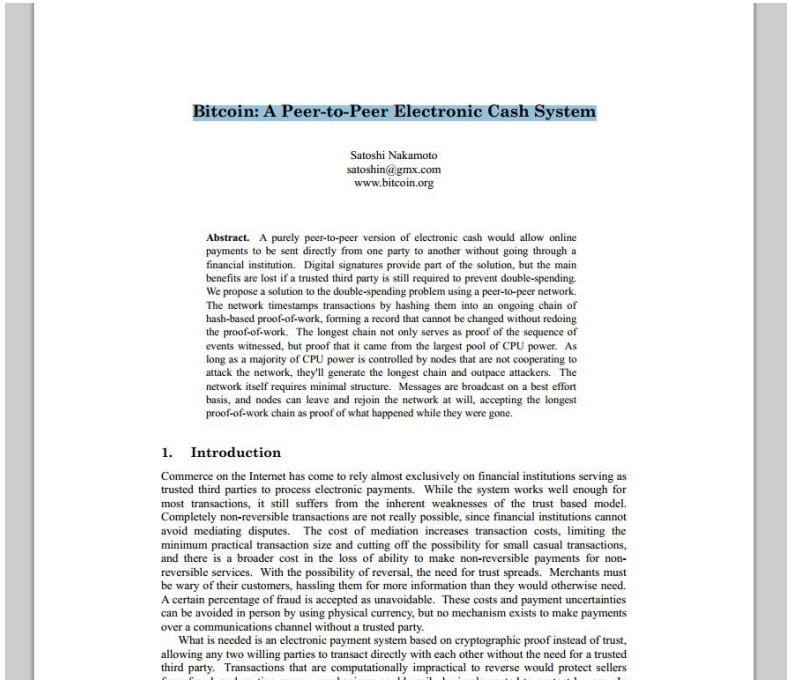
"Double spend races" (with C. Grunspan), ArXiv 1702.02867, 2017.

Double spend races

Bitcoin paper

S. Nakamoto, November 1st 2008,

“Bitcoin: A peer-to-peer electronic cash system”



Bitcoin: A Peer-to-Peer Electronic Cash System

Satoshi Nakamoto
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www.bitcoin.org

Abstract. A purely peer-to-peer version of electronic cash would allow online payments to be sent directly from one party to another without going through a financial institution. Digital signatures provide part of the solution, but the main benefits are lost if a trusted third party is still required to prevent double-spending. We propose a solution to the double-spending problem using a peer-to-peer network. The network timestamps transactions by hashing them into an ongoing chain of hash-based proof-of-work, forming a record that cannot be changed without redoing the proof-of-work. The longest chain not only serves as proof of the sequence of events witnessed, but proof that it came from the largest pool of CPU power. As long as a majority of CPU power is controlled by nodes that are not cooperating to attack the network, they'll generate the longest chain and outpace attackers. The network itself requires minimal structure. Messages are broadcast on a best effort basis, and nodes can leave and rejoin the network at will, accepting the longest proof-of-work chain as proof of what happened while they were gone.

1. Introduction

Commerce on the Internet has come to rely almost exclusively on financial institutions serving as trusted third parties to process electronic payments. While the system works well enough for most transactions, it still suffers from the inherent weaknesses of the trust based model. Completely non-reversible transactions are not really possible, since financial institutions cannot avoid mediating disputes. The cost of mediation increases transaction costs, limiting the minimum practical transaction size and cutting off the possibility for small casual transactions, and there is a broader cost in the loss of ability to make non-reversible payments for non-reversible services. With the possibility of reversal, the need for trust spreads. Merchants must be wary of their customers, hassling them for more information than they would otherwise need. A certain percentage of fraud is accepted as unavoidable. These costs and payment uncertainties can be avoided in person by using physical currency, but no mechanism exists to make payments over a communications channel without a trusted party.

What is needed is an electronic payment system based on cryptographic proof instead of trust, allowing any two willing parties to transact directly with each other without the need for a trusted third party. Transactions that are computationally impractical to reverse would protect sellers

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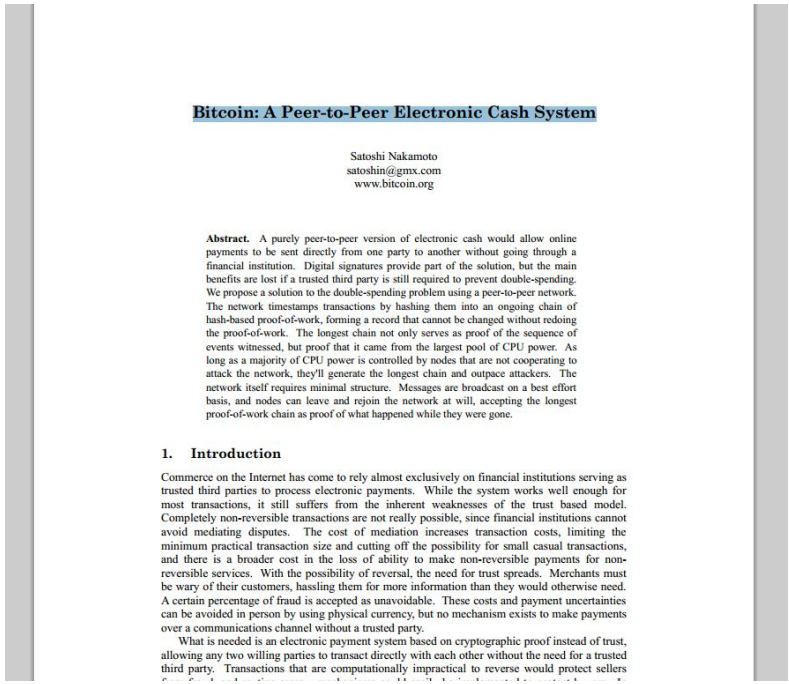

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“Bitcoin and Decentralized Trust Protocols”, Newsletter European Math. Soc., 100, June 2016. ArXiv 1601.05254.

Transparency Theorem

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Theorem

Transparency Theorem: *An electronic decentralized currency must rely on a public ledger.*

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- The blocks are generated by “miners” that validate current transactions.

The Trust Machine

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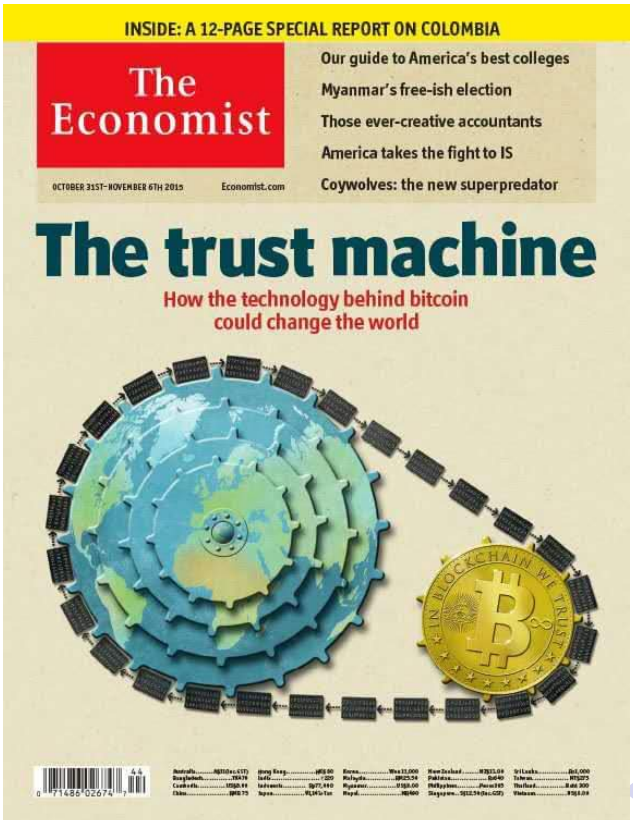
- The mechanism of consensus: “The trust machine”.



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- The miner that solves it receives an award in newly created bitcoins.

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- The attacker must control a substantial fraction of the mining power of the network.

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The recipient waits until the transaction has been added to a block and z blocks have been linked after it. He doesn't know the exact amount of progress the attacker has made, but assuming the honest blocks took the average expected time per block, the attacker's potential progress will be a Poisson distribution with expected value:

$$\lambda = z\frac{q}{p}$$

To get the probability the attacker could still catch up now, we multiply the Poisson density for each amount of progress he could have made by the probability he could catch up from that point:

$$\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \cdot \begin{cases} (q/p)^{(z-k)} & \text{if } k \le z \\ 1 & \text{if } k > z \end{cases}$$

Rearranging to avoid summing the infinite tail of the distribution...

$$1 - \sum_{k=0}^{z} \frac{\lambda^k e^{-\lambda}}{k!} \left(1 - (q/p)^{(z-k)}\right)$$

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- Nakamoto empirically observes that it decreases exponentially with the number of validations.
- This is the crucial point giving security to the Bitcoin protocol!

Exact computation

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Theorem (C. Grunspan, R. Pérez-Marco, 2017)

The probability of success of the attacker is

$$P(z) = I_{4pq}(z, 1/2) \ ,$$

where $I_x(a,b)$ is the Regularized Incomplete Beta Function

$$I_x(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^x t^{a-1}(1-t)^{b-1}\, dt \ .$$

An important corollary

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Main security result for the Bitcoin Protocol.

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Theorem

When $z \to +\infty$ *we have , with* $s = 4pq < 1$,

$$P(z) \sim \frac{s^z}{\sqrt{\pi(1-s)z}},$$

thus the probability $P(z)$ *decreases exponentially to* 0 *with* z

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q	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
z	6	9	13	20	32	58	133	539
z_{SN}	5	8	11	15	24	41	81	340

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- τ is known to the recipient of the first Tx, so he can refine the estimate of the probability of success of the attacker.
- We consider the dimensionless parameter that measures the deviation from average validation time

$$\kappa = \frac{\tau}{z t_0} \ ,$$

where t_0 is the average time of validation of blocks (10 minutes), and $P(z, \kappa)$ is the probability we study.

- $\kappa = 1$ corresponds to the situation studied by Nakamoto

$$P_{SN}(z) = P(z, 1) \ .$$

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- We can also recover the exact probability $P(z)$ as a weighted average,

$$P(z) = \int_0^{+\infty} P(z, \kappa)\, d\rho_z(\kappa) \ ,$$

with the density function

$$d\rho_z(\kappa) = \frac{z^z}{(z-1)!} \kappa^{z-1} e^{-z\kappa}\, d\kappa \ .$$

A refined theorem

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Theorem

We have

$$P(z,\kappa) = 1 - Q(z,\kappa z q/p) + \left(\frac{q}{p}\right)^z e^{\kappa z \frac{p-q}{p}} Q(z,\kappa z)\ .$$

Where Q is the Incomplete Gamma Function,

$$Q(s,x) = \frac{\Gamma(s,x)}{\Gamma(x)}\ ,$$

where

$$\Gamma(s,x) = \int_x^{+\infty} t^{s-1} e^{-t}\, dt$$

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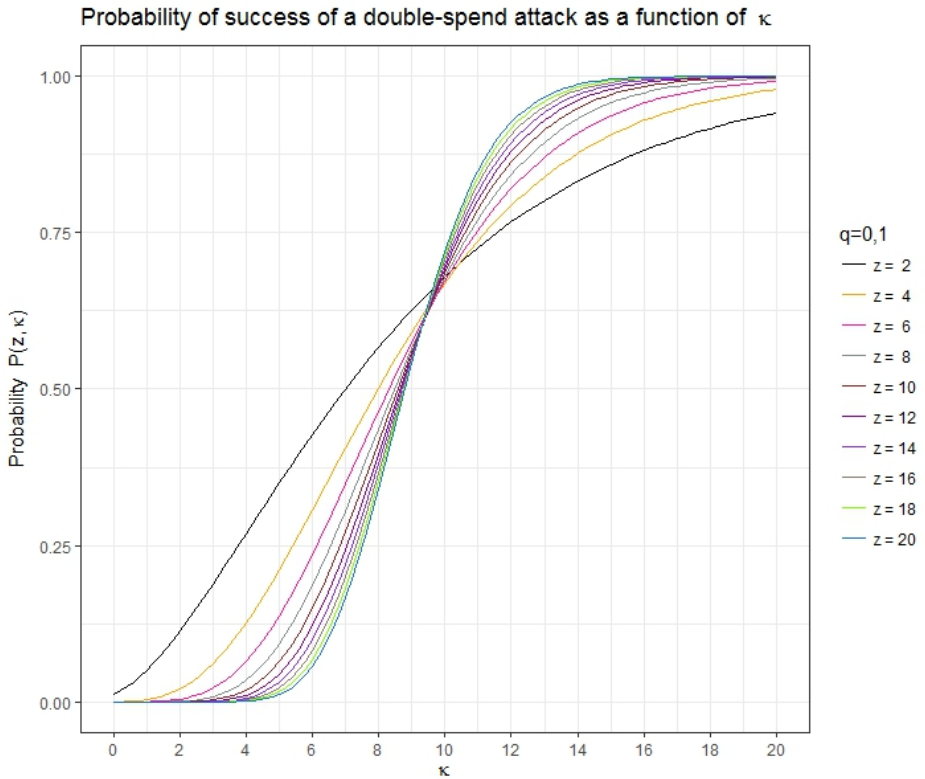

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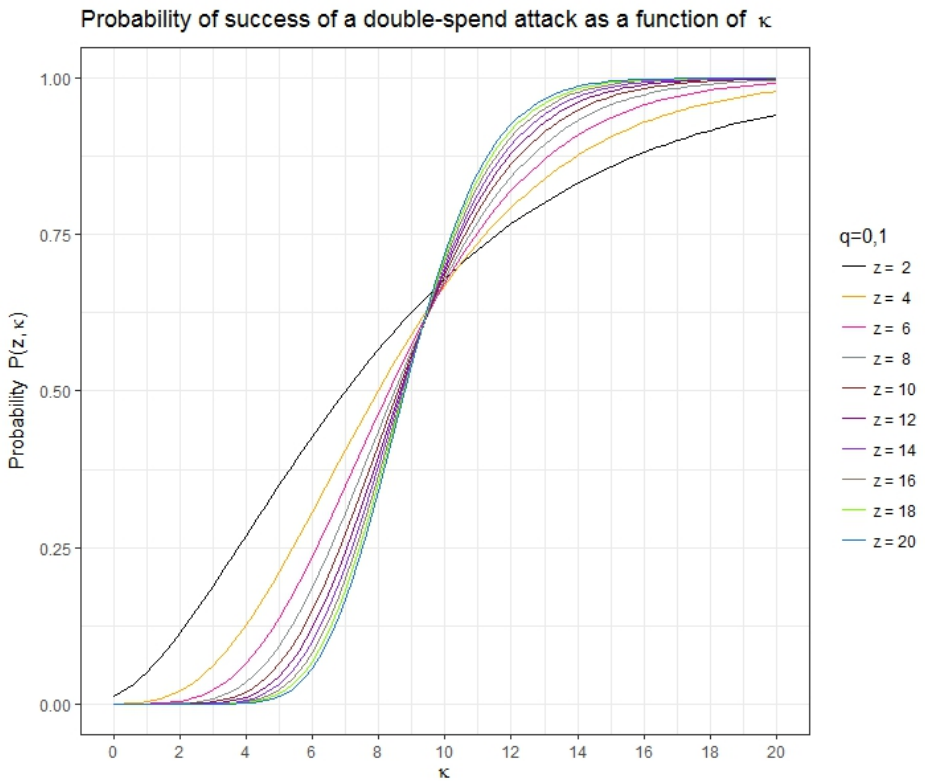

- Double entry "Satoshi risk tables" with parameter (q, κ), ArXiv 1702.04421, 2017.

Mathematics of mining

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• Mining consists in performing independent hashes changing a nounce, thus the random variable $\boldsymbol{T}$ it takes to mine a block follows an exponential distribution,

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• If $(\boldsymbol{T}_1, \ldots, \boldsymbol{T}_n)$ are independent identically distributed exponential random variables of parameter α, then, by convolution, $\boldsymbol{S}_n = \boldsymbol{T}_1 + \ldots + \boldsymbol{T}_n$ is a random variable with a gamma density with parameter (n, α).

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• The random process $\boldsymbol{N}(t)$ of the number of mined blocks at time t is a Poisson process and $\boldsymbol{N}(t)$ has a Poisson distribution with expectation αt.

Elementary lemma

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Similar to Gambler's Ruin problem:

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Lemma (S. Nakamoto, 2009)

Let q_n be the probability of the event E_n, "catching up from n blocks behind". We have

$$q_n = (q/p)^n \, .$$

- Let $\boldsymbol{T}$ and $\boldsymbol{S}_n$, resp. $\boldsymbol{T}'$ and $\boldsymbol{S}'_n$, be the random variables associated to the group of honest, resp. attacker, miners.

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- We have

$$p = \frac{\alpha}{\alpha + \alpha'} \ ,$$
$$q = \frac{\alpha'}{\alpha + \alpha'} \ .$$

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Proposition (C. Grunspan, R. Pérez-Marco, 2017)

The random variable $\boldsymbol{X}_n$ has a negative binomial distribution with parameters (n, p), i.e. for $k \geq 0$,

$$\mathbb{P}[\boldsymbol{X}_n = k] = p^n q^k \binom{k+n-1}{k}.$$

Probability of success of attacker

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Proposition (Probability of success of the attacker)

The probability of success by the attackers after z blocks have been mined by the honest miners is

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• A manipulation with special functions gives the closed form formula of the main Theorem.

Thank you for your attention!