Open problems in hedgehog dynamics

Ricardo Pérez-Marco (CNRS, IMJ-PRG, Paris 7)

DinamIC Seminar, Imperial College

December 19, 2016
Lyapunov Stability

Definition

A fixed point $0$ of a homeomorphism $f$ is Lyapunov stable if for any neighborhood $0 \in U$ there is a neighborhood $V$ such that for all $n \geq 0$,

$$f^n(V) \subset U.$$
Birkhoff Theorem

Theorem (G.D. Birkhoff, 1922)

Let $f$ be a germ of planar homeomorphism of a neighborhood of $0 \in \mathbb{R}^2$ which is a Lyapunov unstable fixed point for $f^{-1}$. Then there exist a full non-trivial continuum $K_+, 0 \in K_+$, which is positive invariant by the dynamics of $f$,

$$f(K_+) \subset K_+.$$
Birkhoff Theorem

Theorem (G.D. Birkhoff, 1922)

If $f$ is a germ of planar homeomorphism of a neighborhood of $0 \in \mathbb{R}^2$ which is a Lyapunov unstable fixed point for $f^{-1}$. Then there exist a full non-trivial continuum $K_+, 0 \in K_+, which is positive invariant by the dynamics of $f$,

$$f(K_+) \subset K_+.$$
Birkhoff Theorem

Theorem (G.D. Birkhoff, 1922)

\[ f \text{ germ of planar homeomorphism of a neighborhood of } 0 \in \mathbb{R}^2 \text{ which is a Lyapunov unstable fixed point for } f^{-1}. \text{ Then there exist a full non-trivial continuum } K_+, 0 \in K_+, \text{ which is positive invariant by the dynamics of } f, \]

\[ f(K_+) \subset K_+. \]

We can replace \( \mathbb{R}^2 \) by \( \mathbb{R}^n \) or more generally a locally compact and locally connected space.
Birkhoff compacta

Corollary

If $0$ is Lyapunov unstable for both $f$ and $f^{-1}$ there are two Birkhoff continua $K_+$ and $K_-$ such that $0 \in K_+ \cap K_-$ and

$$f(K_+) \subset K_+,$$

$$f^{-1}(K_-) \subset K_-.$$ 

In general $K_+ \neq K_-$, for example, locally for an hyperbolic fixed point $K_+ \cap K_- = \{0\}$. 

R. Pérez-Marco  
CNRS, IMJ-PRG, Paris 7 

Open problems in hedgehog dynamics
Let $f(z) = \lambda z + \mathcal{O}(z^2)$ with $\lambda \neq 0$ be a germ of holomorphic diffeomorphism with a fixed point at 0.

**Theorem**

The fixed point 0 is Lyapunov stable if and only if $f$ is linearizable at 0, i.e. there is a change of variables $h(z) = z + \mathcal{O}(z^2)$ such that

$$h^{-1} \circ f \circ h = L_\lambda$$

where $L_\lambda(z) = \lambda z$. 
Siegel continuum

Theorem (RPM, Acta Math. 1997)

Let $f(z) = \lambda z + O(z^2)$ holomorphic, with 0 indifferent fixed point, $f(0) = 0$,

$$|\lambda| = |f'(0)| = 1$$

There exist a full non-trivial compact connected $K$, $0 \in K$, and totally invariant by the dynamics of $f$,

$$f(K) = f^{-1}(K) = K$$
Siegel continuum

Theorem (RPM, Acta Math. 1997)

Let $f(z) = \lambda z + O(z^2)$ holomorphic, with 0 indifferent fixed point, $f(0) = 0$,

$$|\lambda| = |f'(0)| = 1$$

There exist a full non-trivial compact connected $K$, $0 \in K$, and totally invariant by the dynamics of $f$,

$$f(K) = f^{-1}(K) = K$$
Siegel continuum

Theorem (RPM, Acta Math. 1997)

Let \( f(z) = \lambda z + O(z^2) \) holomorphic, with 0 indifferent fixed point, \( f(0) = 0 \),

\[ |\lambda| = |f'(0)| = 1 \]

There exist a full non-trivial compact connected \( K \), \( 0 \in K \), and totally invariant by the dynamics of \( f \),

\[ f(K) = f^{-1}(K) = K \]
Hedgehog

Definition (Hedgehog)

Let $\lambda = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R} - \mathbb{Q}$, if $K \neq \overline{K}$, then $K$ is a hedgehog.

Two types of hedgehogs:
Hedgehog

Definition (Hedgehog)

Let \( \lambda = e^{2\pi i \alpha} \), \( \alpha \in \mathbb{R} - \mathbb{Q} \), if \( K \neq \bar{K} \), then \( K \) is a hedgehog.

Two types of hedgehogs:

- \( K \) is a non-linearizable hedgehog when \( 0 \notin \bar{K} \).

R. P´erez-Marco CNRS, IMJ-PRG, Paris 7

Open problems in hedgehog dynamics
Hedgehog

Definition (Hedgehog)

\[
\lambda = e^{2\pi i \alpha}, \quad \alpha \in \mathbb{R} - \mathbb{Q}, \quad \text{if} \quad K \neq \bar{K}, \quad \text{then} \quad K \text{ is a hedgehog.}
\]

Two types of hedgehogs:

- **K** is a non-linearizable hedgehog when \(0 \notin \bar{K}\).
- **K** is a linearizable hedgehog when \(0 \in \bar{K}\).
Hedgehog

Definition (Hedgehog)

Let $\lambda = e^{2\pi i \alpha}$, $\alpha \in \mathbb{R} - \mathbb{Q}$, if $K \neq \bar{K}$, then $K$ is a hedgehog.

Two types of hedgehogs:

- $K$ is a non-linearizable hedgehog when $0 \notin \bar{K}$.
- $K$ is a linearizable hedgehog when $0 \in \bar{K}$.

Open problems in hedgehog dynamics
Hedgehog

Definition (Hedgehog)

Let $\lambda = e^{2\pi i \alpha}$, $\alpha \in \mathbb{R} - \mathbb{Q}$, if $K \neq \bar{K}$, then $K$ is a hedgehog. Two types of hedgehogs:

- $K$ is a non-linearizable hedgehog when $0 \notin \bar{K}$.
- $K$ is a linearizable hedgehog when $0 \in \bar{K}$.

We focus on non-linearizable hedgehogs.
Circle maps

**Theorem**

Let \( f(z) = \lambda z + O(z^2) \) holomorphic, \( \lambda = e^{2\pi i \alpha} \). and \( K \) hedgehog for \( f \). Then the action on prime-ends of \( \mathbb{C} - K \) is an analytic circle diffeomorphism \( g \) with rotation number

\[ \rho(g) = \alpha. \]
Circle maps

Theorem

Let \( f(z) = \lambda z + O(z^2) \) holomorphic, \( \lambda = e^{2\pi i \alpha} \). and \( K \) hedgehog for \( f \). Then the action on prime-ends of \( \mathbb{C} - K \) is an analytic circle diffeomorphism \( g \) with rotation number

\[
\rho(g) = \alpha.
\]
Circle maps

Theorem

Let \( f(z) = \lambda z + o(z^2) \) holomorphic, \( \lambda = e^{2\pi i \alpha} \). and \( K \) hedgehog for \( f \). Then the action on prime-ends of \( \mathbb{C} - K \) is an analytic circle diffeomorphism \( g \) with rotation number

\[
\rho(g) = \alpha.
\]
Naishul’s theorem

Theorem

Let $f_1$ and $f_2$ be topologically conjugated with indifferent fixed points at 0. Then

$$f_1'(0) = f_2'(0)$$
Naishul’s theorem

**Theorem**

Let $f_1$ and $f_2$ be topologically conjugated with indifferent fixed points at 0. Then

$$f_1'(0) = f_2'(0)$$
### Naishul’s theorem

**Theorem**

Let $f_1$ and $f_2$ be topologically conjugated with indifferent fixed points at 0. Then

$$f'_1(0) = f'_2(0)$$

Follows from $\rho(g_1) = \rho(g_2)$ (Poincaré)
Naishul’s theorem

**Theorem**

Let $f_1$ and $f_2$ be topologically conjugated with indifferent fixed points at 0. Then

$$f'_1(0) = f'_2(0)$$

Follows from $\rho(g_1) = \rho(g_2)$ (Poincaré)
Linearization

Theorem

If $f$ is non-linearizable then $g$ is non-linearizable.
Linearization

**Theorem**

*If $f$ is non-linearizable then $g$ is non-linearizable.*
Linearization

Theorem

If \( f \) is non-linearizable then \( g \) is non-linearizable.

Proof: If \( g \) is linearizable then 0 is Lyapunov stable.
Linearization

**Theorem**

*If $f$ is non-linearizable then $g$ is non-linearizable.*

Proof: If $g$ is linearizable then 0 is Lyapunov stable.

$\implies$ Dictionary between linearization problems.
Linearization

**Theorem**

If $f$ is non-linearizable then $g$ is non-linearizable.

Proof: If $g$ is linearizable then $0$ is Lyapunov stable.

$\longrightarrow$ Dictionary between linearization problems.

**Theorem**

*Arnold linearization thm* $\longrightarrow$ *Siegel linearization thm.*
The interior of $K$ is empty

$\mathring{K} = \emptyset$
The interior of $K$ is empty

$\hat{K} = \emptyset$
Interior

Theorem

The interior of $K$ is empty

$\hat{K} = \emptyset$

$\implies$ No wandering components inside hedgehogs.
Local connectivity

**Theorem**

*The only point where a hedgehog is locally connected is the fixed point.*
Local connectivity

Theorem

The only point where a hedgehog is locally connected is the fixed point.
Local connectivity

Theorem

*The only point where a hedgehog is locally connected is the fixed point.*

Theorem (K. Biswas, 2003)

*There are hedgehogs containing smooth combs.*

A comb is Cantor $\times$ Interval.
Theorem

The impression of all prime-ends of $\mathbb{C} - K$ is non-trivial and contains the fixed points.
The impression of all prime-ends of $\mathbb{C} - K$ is non-trivial and contains the fixed points.
Prime ends

Theorem

The impression of all prime-ends of $\mathbb{C} - K$ is non-trivial and contains the fixed points.

$\implies$ Complex topology
Filtration

Theorem

A hedgehog is filtered by a unique family of sub-hedgehogs.
Filtration

Theorem

A hedgehog is filtered by a unique family of sub-hedgehogs.
Filtration

Theorem

*A hedgehog is filtered by a unique family of sub-hedgehogs.*

For any hedgehog $K$, there is a unique (up to parametrization) continuous family for the Haussdorf topology $(K_t)_{t \in [0,1]}$ with $K_0 = \{0\}$, $K_1 = K$, and $K_t$ hedgehog.

$\implies$ Rich structure
Measure

Theorem (K. Biswas)

There are hedgehogs with Hausdorff dimension 1 and also hedgehogs with positive area.
Harmonic Ergodicity

<table>
<thead>
<tr>
<th>Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>Let $\mu_H$ the harmonic measure of $\mathbb{C} - K$. The dynamics of $f$ on $K$ is $\mu_H$-ergodic. The orbit of $\mu_H$-almost every $z \in K$ is dense in $K$.</em>**</td>
</tr>
</tbody>
</table>

---

R. Pérez-Marco

Open problems in hedgehog dynamics
Harmonic Ergodicity

**Theorem**

Let $\mu_H$ the harmonic measure of $\mathbb{C} - K$.

The dynamics of $f$ on $K$ is $\mu_H$-ergodic.

The orbit of $\mu_H$-almost every $z \in K$ is dense in $K$.
Harmonic Ergodicity

**Theorem**

Let $\mu_H$ the harmonic measure of $\mathbb{C} - K$.  

*The dynamics of $f$ on $K$ is $\mu_H$-ergodic.*  

*The orbit of $\mu_H$-almost every $z \in K$ is dense in $K$.*

Proof: By using Lebesgue ergodicity of analytic circle diffeomorphism with irrational rotation number (Katok, Herman).

---

R. Pérez-Marco  
CNRS, IMJ-PRG, Paris 7  
Open problems in hedgehog dynamics
Harmonic Ergodicity

**Theorem**

Let $\mu_H$ the harmonic measure of $\mathbb{C} - K$.

The dynamics of $f$ on $K$ is $\mu_H$-ergodic.

The orbit of $\mu_H$-almost every $z \in K$ is dense in $K$.

**Proof:** By using Lebesgue ergodicity of analytic circle diffeomorphism with irrational rotation number (Katok, Herman).

**Corollary**

The orbit of $\mu_H$-almost every $z \in K$ is dense in $K$.

$\implies \mu_H(K_t) = 0$ for $t < 1$, i.e. the strata $K_t$ are hidden inside $K$. 

R. Pérez-Marco  
CNRS, IMJ-PRG, Paris 7
Uniform Recurrence

Let \((p_n/q_n)\) be the sequence of convergents of \(\alpha \in \mathbb{R} - \mathbb{Q}\). For the rotation \(R_\alpha\) of angle \(\alpha\), we have uniformly

\[
R_{\alpha}^{q_n} \to \text{Id}
\]
Uniform Recurrence

Let \((p_n/q_n)\) be the sequence of convergents of \(\alpha \in \mathbb{R} - \mathbb{Q}\). For the rotation \(R_\alpha\) of angle \(\alpha\), we have uniformly

\[ R_\alpha^{q_n} \rightarrow Id \]

Theorem (Uniform Recurrence)

*We have uniformly on \(K\)

\[ f_{/K}^{q_n} \rightarrow Id_K \]
Uniform Recurrence

Let \( (p_n/q_n) \) be the sequence of convergents of \( \alpha \in \mathbb{R} - \mathbb{Q} \).
For the rotation \( R_\alpha \) of angle \( \alpha \), we have uniformly
\[
R^{q_n}_\alpha \to \text{Id}
\]

Theorem (Uniform Recurrence)

We have uniformly on \( K \)
\[
f^{q_n}_{/K} \to \text{Id}_K
\]
Uniform Recurrence

Let \((p_n/q_n)\) be the sequence of convergents of \(\alpha \in \mathbb{R} - \mathbb{Q}\). For the rotation \(R_\alpha\) of angle \(\alpha\), we have uniformly

\[R_\alpha^{q_n} \to \text{Id}\]

Theorem (Uniform Recurrence)

\textit{We have uniformly on } K \textit{ }

\[f_{/K}^{q_n} \to \text{Id}_K\]

Corollary

\textit{The topological centralizer of } f_{/K} \textit{ is uncountable.}
Uniform Recurrence

Let \((p_n/q_n)\) be the sequence of convergents of \(\alpha \in \mathbb{R} - \mathbb{Q}\).
For the rotation \(R_\alpha\) of angle \(\alpha\), we have uniformly

\[ R_\alpha^{q_n} \to \text{Id} \]

**Theorem (Uniform Recurrence)**

*We have uniformly on \(K*

\[ f_{/K}^{q_n} \to \text{Id}_K \]

**Corollary**

*The topological centralizer of \(f_{/K}\) is uncountable.*
Uniform Recurrence

Let \((p_n/q_n)\) be the sequence of convergents of \(\alpha \in \mathbb{R} - \mathbb{Q}\). For the rotation \(R_\alpha\) of angle \(\alpha\), we have uniformly

\[ R_{q_n} \rightarrow \text{Id} \]

Theorem (Uniform Recurrence)

We have uniformly on \(K\)

\[ f_{/K}^{q_n} \rightarrow \text{Id}_K \]

Corollary

The topological centralizer of \(f_{/K}\) is uncountable.
Unique Ergodicity

Conjecture (RPM, 1995)

*The dynamics of \( f \) in \( K \) is uniquely ergodic.*
Unique Ergodicity

Conjecture (RPM, 1995)

The dynamics of $f$ in $K$ is uniquely ergodic.
Unique Ergodicity

Conjecture (RPM, 1995)

*The dynamics of f in K is uniquely ergodic.*

That is, the only invariant probability measure is the Dirac at 0.
Unique Ergodicity

Conjecture (RPM, 1995)

The dynamics of $f$ in $K$ is uniquely ergodic.

That is, the only invariant probability measure is the Dirac at 0.

Theorem (A. Avila, D. Cheraghi, 2013)

The conjecture is true for hedgehogs of a quadratic polynomial $P_{\alpha}(z) = e^{2\pi \alpha} z + z^2$ with $\alpha$ of “high type”.

R. Pérez-Marco

Open problems in hedgehog dynamics
Symmetries

Conjecture

Any element of the topological centralizer

\[ \text{Cent}(f/K) = \{ g \in \text{Homeo}(K,0); g \circ f/K = f/K \circ g \} \]

has a well defined rotation number \( \rho(g) \).
Symmetries

Conjecture

Any element of the topological centralizer

\[ \text{Cent}(f_K) = \{ g \in \text{Homeo}(K,0) ; g \circ f_K = f_K \circ g \} \]

has a well defined rotation number \( \rho(g) \).

Definition

\[ G(f, K) = \{ \rho(g) ; g \in \text{Cent}(f_K) \} \]
Unbreakable symmetries

Theorem

The group

\[ G(\alpha) = \bigcap_{f; \rho(f) = \alpha} G(f, K) \]

is non-trivial and uncountable.
Unbreakable symmetries

**Theorem**

The group

\[ G(\alpha) = \bigcap_{f; \rho(f) = \alpha} G(f, K) \]

is non-trivial and uncountable.
Unbreakable symmetries

**Theorem**

The group

\[ G(\alpha) = \bigcap_{f; \rho(f) = \alpha} G(f, K) \]

is non-trivial and uncountable.

**Question**

Determine the group

\[ G(\alpha) = \bigcap_{f; \rho(f) = \alpha} G(f, K) . \]
Rigidity

Conjecture

If

\[ G(f_1, K_1) = G(f_2, K_2) \]

then

\[ K_1 \approx K_2 \]

and the dynamics of \( f_1 \) on \( K_1 \) and \( f_2 \) on \( K_2 \) are topologically conjugated.
Drawing hedgehogs

Problem (J. Milnor, 1997)

*Make a computer picture of a hedgehog.*
Higher dimension

Question

Do there exist hedgehogs in higher dimension?
<table>
<thead>
<tr>
<th>Hedgehogs</th>
<th>Applications</th>
<th>Structure</th>
<th>Dynamics</th>
<th>Open Problems</th>
</tr>
</thead>
</table>

Open problems in hedgehog dynamics
Open problems in hedgehog dynamics
Open problems in hedgehog dynamics
Open problems in hedgehog dynamics
Open problems in hedgehog dynamics
Open problems in hedgehog dynamics