Open problems in hedgehog dynamics

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Lyapunov Stability

Definition

A fixed point 0 of a homeomorphism f is Lyapunov stable if for any neighborhood $0 \in U$ there is a neighborhood V such that for all $n \geq 0$,

$$f^n(V) \subset U .$$

Birkhoff Theorem

Theorem (G.D. Birkhoff, 1922)

f germ of planar homeomorphism of a neighborhood of $0 \in \mathbb{R}^2$ which is a Lyapunov unstable fixed point for f^{-1}. Then there exist a full non-trivial continuum K_+, $0 \in K_+$, which is positive invariant by the dynamics of f,

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We can replace $\mathbb{R}^2$ by $\mathbb{R}^n$ or more generally a locally compact and locally connected space.

Birkhoff compacta

Corollary

If 0 is Lyapunov unstable for both f and f^{-1} there are two Birkhoff continua K_+ and K_- such that $0 \in K_+ \cap K_-$ and

$$f(K_+) \subset K_+ \,,$$
$$f^{-1}(K_-) \subset K_- \,.$$

In general $K_+ \neq K_-$, for example, locally for an hyperbolic fixed point $K_+ \cap K_- = \{0\}$.

Holomorphic germs

Let $f(z) = \lambda z + \mathcal{O}(z^2)$ with $\lambda \neq 0$ be a germ of holomorphic diffeomorphism with a fixed point at 0.

Theorem

The fixed point 0 *is Lyapunov stable if and only if* f *is linearizable at* 0*, i.e. there is a change of variables* $h(z) = z + \mathcal{O}(z^2)$ *such that*

$$h^{-1} \circ f \circ h = L_\lambda$$

where $L_\lambda(z) = \lambda z$*.*

Siegel continuum

Theorem (RPM, Acta Math. 1997)

Let $f(z) = \lambda z + \mathcal{O}(z^2)$ *holomorphic, with* 0 *indifferent fixed point,* $f(0) = 0$,

$$|\lambda| = |f'(0)| = 1$$

There exist a full non-trivial compact connected K, $0 \in K$, *and totally invariant by the dynamics of* f,

$$f(K) = f^{-1}(K) = K$$

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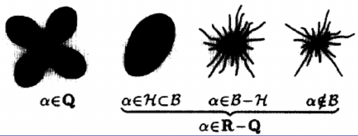

Hedgehog

Definition (Hedgehog)

Let $\lambda = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R} - \mathbb{Q}$, if $K \neq \overline{\mathring{K}}$, then K is a hedgehog.
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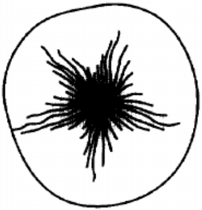
We focus on non-linearizable hedgehogs.



Circle maps

Theorem

Let $f(z) = \lambda z + \mathcal{O}(z^2)$ holomorphic, $\lambda = e^{2\pi i\alpha}$. and K hedgehog for f. Then the action on prime-ends of $\mathbb{C} - K$ is an analytic circle diffeomorphism g with rotation number

$$\rho(g) = \alpha \ .$$

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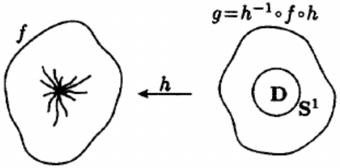

Naishul's theorem

Theorem

Let f_1 and f_2 be topologically conjugated with indifferent fixed points at 0. *Then*

$$f_1'(0) = f_2'(0)$$

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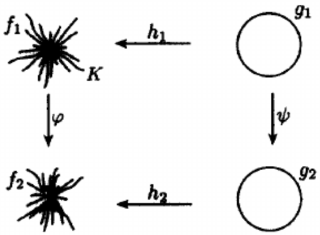

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Arnold linearization thm $\Longrightarrow$ Siegel linearization thm.

Interior

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The interior of K is empty

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$\Longrightarrow$ No wandering components inside hedgehogs.

Local connectivity

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Theorem (K. Biswas, 2003)

There are hedgehogs containing smooth combs.

A comb is Cantor $\times$ Interval.

Prime ends

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$\Longrightarrow$ Complex topology

Filtration

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A hedgehog is filtered by a unique family of sub-hedgehogs.

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For any hedgehog K, there is a unique (up to parametrization) continuous family for the Haussdorf topology $(K_t)_{t\in[0,1]}$ with $K_0 = \{0\}$, $K_1 = K$, and K_t hedgehog.

$\Longrightarrow$ Rich structure

Measure

Theorem (K. Biswas)

There are hedgehogs with Hausdorff dimension 1 *and also hedgehogs with positive area.*

Harmonic Ergodicity

Theorem

Let μ_H the harmonic measure of $\mathbb{C} - K$.
The dynamics of f on K is μ_H-ergodic.
The orbit of μ_H-almost every $z \in K$ is dense in K.

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Corollary

The orbit of μ_H-almost every $z \in K$ is dense in K.

$\implies \mu_H(K_t) = 0$ for $t < 1$, i.e. the strata K_t are hidden inside K

Uniform Recurrence

Let (p_n/q_n) be the sequence of convergents of $\alpha \in \mathbb{R} - \mathbb{Q}$.
For the rotation R_α of angle α, we have uniformly

$$R_\alpha^{q_n} \to Id$$

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Theorem (A. Avila, D. Cheraghi, 2013)

The conjecture is true for hedgehogs of a quadratic polynomial $P_\alpha(z) = e^{2\pi\alpha} z + z^2$ with α of "high type".

Symmetries

Conjecture

Any element of the topological centralizer

$$\mathrm{Cent}(f_{/K}) = \{g \in \mathrm{Homeo(K,0)};\, g \circ f_{/K} = f_{/K} \circ g\}$$

has a well defined rotation number $\rho(g)$.

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Definition

$$G(f, K) = \{\rho(g);\, g \in \mathrm{Cent}(f_{/K})\}$$

Unbreakable symmetries

Theorem

The group

$$G(\alpha) = \bigcap_{f;\rho(f)=\alpha} G(f, K)$$

is non-trivial and uncountable.

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Question

Determine the group

$$G(\alpha) = \bigcap_{f;\rho(f)=\alpha} G(f, K) \ .$$

Rigidity

Conjecture

If

$$G(f_1, K_1) = G(f_2, K_2)$$

then

$$K_1 \approx K_2$$

and the dynamics of f_1 on K_1 and f_2 on K_2 are topologically conjugated.

Drawing hedgehogs

Problem (J. Milnor, 1997)

Make a computer picture of a hedgehog.

Higher dimension

Question

Do there exist hedgehogs in higher dimension?