Journée à la mémoire de Jean-Christophe Yoccoz

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Interesting case $\alpha \in \mathbb{R} - \mathbb{Q}$. With $||x|| = d(x, \mathbb{Z})$, $q_0 = 1$,

$$q_{n+1} = \text{Min}\{q \geq 1; ||q\alpha|| < ||q_n\alpha||\} \ .$$

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$$\sum_{n=0}^{+\infty} \frac{\log q_{n+1}}{q_n} < +\infty$$

What Julia and Fatou missed

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Yoccoz normalization of the quadratic polynomial:

$$P_\lambda(z) = \lambda(z - z^2)$$

Critical point $c = 1/2$. Linearization H_λ entire for $|\lambda| < 1$.

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By a classical theorem of Fatou (!) P_λ is linearizable for a.a. α!

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The Brjuno condition is optimal.

Yoccoz finds a new geometric proof of linearisation that is reversible and yields, when $\alpha \notin \mathcal{B}$, non-linearizable examples with small cycles.

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For α satisfying a diophantine condition, any analytic circle diffeomorphism with α rotation number and close enough to R_α is linearizable.

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Yoccoz finds the optimal arithmetic condition $(\mathcal{H})$ for global linearization of analytic circle diffeomorphisms without assuming proximity to the rotation.

Global linearisation

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Theorem (Herman 1979, Arnold Conjecture)

There is a full measure set A of rotation numbers for which all analytic circle diffeomorphisms are analytically linearizable.

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Theorem (Yoccoz 1987)

Explicit determination of the optimal condition $\mathcal{H}$.

Analytic vector fields

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Let X_α be an analytic vector field in $(x, y) \in \mathbb{C}^2$ with a singularity at 0 in the Siegel domain ($\alpha \in \mathbb{R}$):

$$\dot{x} = x + \mathcal{O}(2)$$
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Theorem (PM, Yoccoz 1990)

One-to-one equivalence of the holomorphic conjugacy classes of germs with rotation number α and the holomorphic conjugacy classes of vector fields (X_α).

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Corollary

Brjuno condition is optimal for the linearization of vector fields.

Cours de Systèmes Dynamiques 1987-1988

INTRODUCTION AUX SYSTEMES DYNAMIQUES

J.–C. YOCCOZ

(1er semestre)

Résumé :

A partir de l'exemple fondamental de la dynamique des difféomorphismes du plan préservant les aires au voisinage d'un point fixe elliptique, on introduira quelques–uns des concepts et points de vues les plus utiles dans l'étude des systèmes dynamiques : rudiments de mécanique hamiltonienne, éléments de théorie ergodique, formes normales et étude dynamique des difféomorphismes au voisinage d'un point fixe, exemples de problèmes liés aux "petits dénominateurs", formes fortes et faibles d'hyperbolicité.

Connaissances préalables :

Calcul différentiel dans $\mathbb{R}^n$ (sous–variétés, équations différentielles), théorie de la mesure.

Bibliographie :

V.I. Arnold.– Méthodes mathématiques de la mécanique classique, Editions Mir.
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Examen cours de Systèmes Dynamiques 1988-1989

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EXAMEN du 28/02/89

Cours de 3e cycle : Introduction aux systèmes dynamiques

Dans la suite, X est un espace métrique compact. On désigne par $\mathcal{M}(X)$ l'ensemble des mesures de probabilité sur X ; pour une application continue $f : X \to X$, on note $\mathcal{M}(f)$ l'ensemble des mesures de probabilité invariantes par f, et par $\mathcal{M}_e(f) \subset \mathcal{M}(f)$ le sous-ensemble des mesures ergodiques. Sauf mention contraire, les exercices qui suivent sont indépendants.

1. Soient $f : X \to X$ et $\varphi : X \to \mathbb{R}^k$ deux applications continues. Définissons

$$C = \{ \alpha \in \mathbb{R}^k \mid \alpha = \lim_{i \to +\infty} \frac{1}{n_i} \sum_{j=0}^{n_i - 1} \varphi(f^j(x_i)),$$

où $(x_i)_{i \geq 0}$ est une suite dans X et $(n_i)_{i \geq 0}$ une

Salamanca, July 6th 1991