What does it mean to formalise and why do it

Riccardo Brasca

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We will speak about Lean

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- it translates something written by human beings to something totally precise
- it can do certain simple computations automatically
- it checks the correctness of the proofs, starting from the axioms

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$\mathbb{R}$ is endowed with several sums as a field, ring, group...

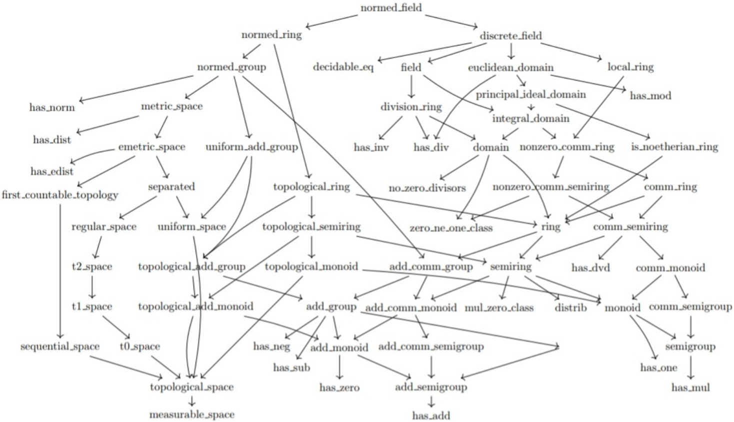

Image de Jeremy Avigad

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Bugs in the other parts of the proof assistant surely exist but this is less important

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Mathlib is Lean's official mathematical library
It has the level of an advanced undergraduate or first year graduate student in mathematics (around 1.2 millions lines of code)

Why formalize mathematics

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It invites us to rethink basic mathematical concepts from a different point of view

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Theorem (Clausen-Scholze)

Let $0 < p' < p \leq 1$ be real numbers, S a profinite set and V a p-Banach space.

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Theorem (Clausen-Scholze)

Let $0 < p' < p \leq 1$ be real numbers, S a profinite set and V a p-Banach space. Then

$$\mathrm{Ext}^1_{\mathrm{Cond(Ab)}}(\mathcal{M}_{p'}(S), V) = 0.$$

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- *... It is the kind of argument that needs to be closely inspected*
- *While I was very happy to see many study groups on condensed mathematics throughout the world, to my knowledge all of them have stopped short of this proof. (Yes, this proof is not much fun...)*

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In around six months we finished the most technical (and Scholze's main question) part of the theorem

Adam Topaz 7:58 PM

Yeah exactly. This is what I was worried about.

EDITED I *think* the order of the quantifiers can probably be reversed if one assumes completeness, because then for each epsilon you would get an element and eventually have to prove that those elements converge (I don't know if the details would actually work out). 8:01 PM

Riccardo Brasca 8:03 PM

I was thinking to it, but I am really sure this doesn't work for making the $\inf$ in the definition of the quotient norm a $\min$. Of course it can still work for our elements for some reasons but still, something has to be done

Peter Scholze 11:54 PM

Ah!

Sorry, indeed you caught something there. Let me think about this. 11:54 PM

One option might be to change the meaning of $\leq k$-exactness, to also include an inf 11:55 PM

Hmm. It should be possible to fix this by a small tweaking of some definition, but let me try to figure out a good global fix to this. I'll keep you posted. 11:59 PM

JAN 21, 2021 — JAN 22, 2021

Peter Scholze 12:50 AM

Probably this is just my mind making up a solution as I want to go to bed, but I think the following fix ought to work. Leave all the statements and definitions essentially unchanged, but replace all normed abelian groups with complete normed abelian groups. In particular, in 9.10, the quotient $N = M'/M$ is implicitly completed. Then I think 9.10 stays true as stated, except that one may have to replace $k^3 + k$ by something slightly different.

I'll try to update the file tomorrow 12:51 AM

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Mario Carneiro	Heather Macbeth
Patrick Massot	Bhavik Mehta
Scott Morrison	Filippo A.E. Nuccio
Joël Riou	Damiano Testa
Andrew Yang	many others

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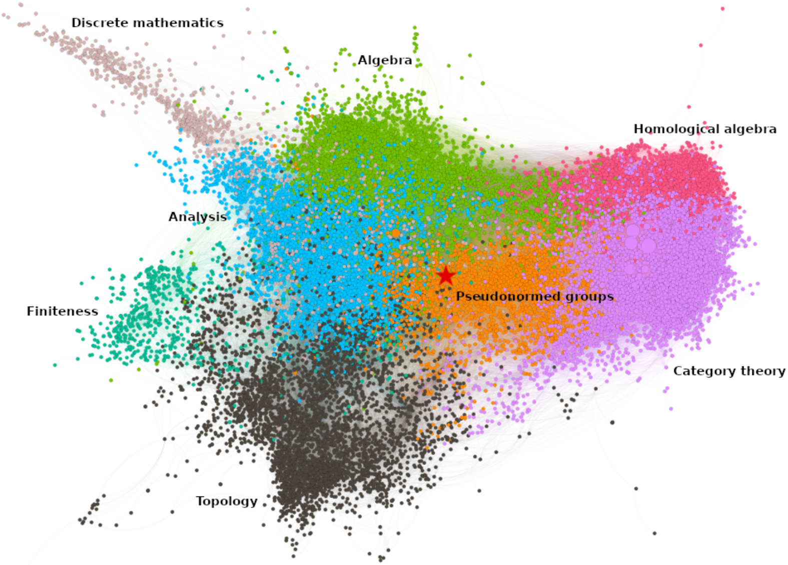
- *... When I wrote the blog post half a year ago, I did not understand why the argument worked*
- *The Lean Proof Assistant was really that: An assistant in navigating through the thick jungle that this proof is. Really, one key problem I had when I was trying to find this proof was that I was essentially unable to keep all the objects in my "RAM" ... So I think here we have witnessed an experiment where the proof assistant has actually assisted in understanding the proof*


Discrete mathematics
Algebra
Homological algebra
Analysis
Finiteness
Pseudonormed groups
Category theory
Topology

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Consider the following easy lemma

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Let (u_n) *and* (v_n) *be sequences of real numbers and let* $\ell \in \mathbb{R}$. *If* $\lim u_n = \ell^+$ *and* $\lim v_n = -\infty$ *then*

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Breen-Deligne resolution in LTE

Theorem

let $f : X \to Y$ be a continuous function, where X and Y are metric spaces with X compact. Then f is uniformly continuous.

Proof.

Let $\varepsilon > 0$ and let

$$K = \{(A, B) \in X \times X \mid \varepsilon \leq d(f(A), f(B))\}.$$

We have that K is closed and hence compact. Let $(P_1, P_2) \in K$ be a minimum of the distance function and let $\delta = d(P_1, P_2)$. If $A, B \in X$ are such that $d(A, B) < \delta$ but $d(f(A), f(B)) \geq \varepsilon$ then $(A, B) \in K$, so $\delta = d(P_1, P_2) \leq d(A, B)$, that is absurd. □