

A FAMILY OF QUATERNIONIC MONODROMY GROUPS OF THE KONTSEVICH–ZORICH COCYCLE

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ABSTRACT. For all d belonging to a density-1/8 subset of the natural numbers, we give an example of a square-tiled surface conjecturally realizing the group $SO^*(2d)$ in its standard representation as the Zariski-closure of a factor of its monodromy. We prove that this conjecture holds for the first elements of this subset, showing that the group $SO^*(2d)$ is realizable for every $11 \leq d \leq 299$ such that $d \equiv 3 \pmod{8}$, except possibly for $d = 35$ and $d = 203$.

1. INTRODUCTION

A *translation surface* (X, ω) is a compact Riemann surface equipped with a non-zero Abelian differential. Away from its zeroes, ω induces an atlas on X all whose changes of coordinates are translations, called a *translation atlas*. Translation surfaces can be packed into a moduli space endowed with a natural $SL(2, \mathbb{R})$ -action, given by post-composing with the coordinate charts of the translation atlas. The geometric and dynamical properties of this action have been extensively studied. We refer the reader to the surveys by Forni–Matheus [FM14], Wright [Wri15] and Zorich [Zor06] for excellent introductions to the subject.

Integrating over ω provides coordinate charts for the moduli space of translation surfaces, called *period coordinates*. An *affine invariant manifold* \mathcal{M} is an immersed connected suborbifold of the moduli space of translation surfaces which is locally defined by linear equations having real coefficients and zero constant terms in period coordinates. By the landmark work of Eskin–Mirzakhani [EM18] and Eskin–Mirzakhani–Mohammadi [EMM15], affine invariant manifolds coincide with orbit closures of the $GL^+(2, \mathbb{R})$ -action.

The *Hodge bundle* is the vector bundle over \mathcal{M} whose fibre over $X \in \mathcal{M}$ is the complex vector space of Abelian differentials $H^{1,0}(X)$. If M is the underlying topological surface of the elements of \mathcal{M} , such vector spaces are isomorphic to $H_1(M, \mathbb{R})$ as real vector spaces and by abuse of terminology we will also refer to the resulting bundle as the Hodge bundle. The Gauss–Manin connection provides a natural way to compare fibres of the Hodge bundle. The *Kontsevich–Zorich cocycle* over \mathcal{M} is the dynamical cocycle over the Hodge bundle induced by the $SL(2, \mathbb{R})$ -action. This cocycle is flat for the Gauss–Manin connection.

The *monodromy group* of \mathcal{M} is the group arising from the action of the (orbifold) fundamental group of \mathcal{M} on the Hodge bundle. These groups can be also defined by the action of the Kontsevich–Zorich cocycle on an $SL(2, \mathbb{R})$ -invariant subbundle of the Hodge bundle. Moreover, the Kontsevich–Zorich cocycle is semisimple and its decomposition respects the Hodge structure [Fil16]. Using these facts, Filip [Fil17] showed

that the possible Zariski-closures of the monodromy groups arising from $\mathrm{SL}(2, \mathbb{R})$ -(strongly-)irreducible subbundles, at the level of real Lie algebra representations and up to compact factors, belong to the following list:

- (i) $\mathfrak{sp}(2g, \mathbb{R})$ in the standard representation;
- (ii) $\mathfrak{su}(p, q)$ in the standard representation;
- (iii) $\mathfrak{su}(p, 1)$ in an exterior power representation;
- (iv) $\mathfrak{so}^*(2d)$ in the standard representation; or
- (v) $\mathfrak{so}_{\mathbb{R}}(n, 2)$ in a spin representation.

Nevertheless, it is not known whether every Lie algebra representation in this list is realizable as a monodromy group [Fil17, Question 1.5]. Indeed, it is well-known that every group in the first item is realizable. The groups in the second item were shown to be realizable by Avila–Matheus–Yoccoz [AMY17]. Moreover, the group $\mathrm{SO}^*(6)$ in its standard representation (which coincides with $\mathrm{SU}(3, 1)$ in its second exterior power representation) is also realizable by the work of Filip–Forni–Matheus [FFM18].

The main theorem of this article is the following:

Theorem 1.1. *For each d belonging to a density-1/8 subset of the natural numbers, there exists a square-tiled surface conjecturally realizing the group $\mathrm{SO}^*(2d)$ as the monodromy group of an $\mathrm{SL}(2, \mathbb{R})$ -(strongly-)irreducible piece of its Kontsevich–Zorich cocycle. This conjecture depends on certain linear-algebraic conditions, which can be computationally shown to be true for small values of d . In this way, we show that $\mathrm{SO}^*(2d)$ is realizable for every $11 \leq d \leq 299$ in the congruence class $d = 3 \pmod{8}$, except possibly for $d = 35$ and $d = 203$.*

Indeed, as was done by Filip–Forni–Matheus [FFM18], we will show that these groups seem to arise in quaternionic covers of simple square-tiled surfaces.

This article is organized as follows. In Section 2, we cover the required background on monodromy groups of square-tiled surfaces. Section 3 shows the construction of the explicit family of square-tiled surfaces arising as quaternionic covers. Finally, we compute the desired monodromy groups in Section 4.

2. PRELIMINARIES

2.1. Monodromy groups. Monodromy groups are, in general, a way to encode how a space relates to its universal cover. In the case of the Kontsevich–Zorich cocycle, they are defined as follows: given an affine invariant manifold (or orbit closure) \mathcal{M} , we define its monodromy group as the image of the natural map $\pi_1(\mathcal{M}, X) \rightarrow \mathrm{Sp}(H_1(M; \mathbb{R}))$, where M is an underlying topological surface and $\pi_1(\mathcal{M}, X)$ is the orbifold fundamental group based at $X \in \mathcal{M}$. This means that they encode the homological action of the mapping classes identifying different points of the Teichmüller space to the same point X of the moduli space. Since \mathcal{M} is connected, any choice of basepoint yields an isomorphic group.

By the Hodge bundle we mean the vector bundle over \mathcal{M} having $H_1(M; \mathbb{R})$ as the fibre over every point. The Kontsevich–Zorich cocycle is the dynamical cocycle defined over the Hodge bundle by the $\mathrm{SL}(2, \mathbb{R})$ -action. An $\mathrm{SL}(2, \mathbb{R})$ -invariant subbundle E is a subbundle for which $g \cdot E_X = E_{g \cdot X}$ for every $X \in \mathcal{M}$ and $g \in \mathrm{SL}(2, \mathbb{R})$. A flat subbundle E is a subbundle which is flat for the Gauss–Manin connection. Observe that a flat subbundle is necessarily $\mathrm{SL}(2, \mathbb{R})$ -invariant, since if the curvature vanishes then the parallel transport is done along $\mathrm{SL}(2, \mathbb{R})$ -orbits in the “obvious” way. The converse is

not true in general: the flatness condition requires no curvature in every possible direction, including those which are not reachable by the $\mathrm{SL}(2, \mathbb{R})$ -action. The classification of the $\mathrm{SL}(2, \mathbb{R})$ -invariant subbundles which are not flat is known [EFW18].

The Hodge bundle can be decomposed into irreducible pieces and monodromy groups can be defined for such pieces. One then has the following [Fil17, Theorem 1.1]:

Theorem 2.1. *Let E be a strongly irreducible flat subbundle of the Hodge bundle over some affine invariant manifold \mathcal{M} . Then, the presence of zero Lyapunov exponents implies that the Zariski-closure of the monodromy group has at most one non-compact factor, which, up to finite-index, is equal at the level of Lie group representations to:*

- $\mathrm{SU}(p, q)$ in the standard representation;
- $\mathrm{SU}(p, 1)$ in any exterior power representation; or
- $\mathrm{SO}^*(2d)$ in the standard representation for some odd d .

Observe that this is a “refined” version of the constraints in the previous section, under stronger hypotheses.

2.2. Square-tiled surfaces. A square-tiled surface is a particular kind of translation surface defined as a finite cover of the unit square torus branched over a single point. That is, we say that a translation surface (X, ω) is *square-tiled* if there exists a covering map $\pi: X \rightarrow \mathbb{R}^2/\mathbb{Z}^2$, which is unramified away from $0 \in \mathbb{R}^2/\mathbb{Z}^2$, and $\omega = \pi^*(dz)$, where dz is the Abelian differential on $\mathbb{R}^2/\mathbb{Z}^2$ induced by the natural identification $\mathbb{R}^2 \simeq \mathbb{C}$. We will often write X to refer to (X, ω) for simplicity.

Combinatorially, a square-tiled surface can be defined as a pair of *horizontal* and *vertical* permutations $h, v \in \mathrm{Sym}(\mathrm{Sq}(X))$, where $\mathrm{Sq}(X)$ is some finite set that we interpret as the *squares* of X . These two permutations can be obtained from our original definition as the deck transformations induced respectively by the curves $t \mapsto (t, 0)$ and $t \mapsto (0, t)$, with $t \in [0, 1]$, and the set of squares $\mathrm{Sq}(X)$ can be defined to be the set of connected components of $\pi^{-1}((0, 1)^2)$. Conversely, we can glue squares horizontally using h and vertically using v and define ω to be the pullback of dz in each square to obtain a square-tiled surface as in the original definition.

2.2.1. $\mathrm{SL}(2, \mathbb{R})$ -action and monodromy groups. Every square-tiled surface X is a Veech surface, that is, its $\mathrm{SL}(2, \mathbb{R})$ -orbit is closed. In particular, this implies that any $\mathrm{SL}(2, \mathbb{R})$ -invariant subbundle of the Hodge bundle over the orbit $\mathrm{SL}(2, \mathbb{R}) \cdot X$ is actually flat. Therefore, Theorem 2.1 can be applied for any $\mathrm{SL}(2, \mathbb{R})$ -(strongly-)irreducible subbundle.

We say that square-tiled surface X is *reduced* if the covering map π cannot be factored through another non-trivial covering of the torus. In this case, the elements $g \in \mathrm{SL}(2, \mathbb{R})$ such that $g \cdot X$ is a square-tiled surface are exactly $\mathrm{SL}(2, \mathbb{Z})$. It is often the case that we study the action of $\mathrm{SL}(2, \mathbb{Z})$ on X instead of the entire $\mathrm{SL}(2, \mathbb{R})$ -action, since square-tiled surfaces can be represented in purely combinatorial terms. The *Veech group* of X , usually denoted $\mathrm{SL}(X)$, is the subgroup of $\mathrm{SL}(2, \mathbb{Z})$ stabilizing X . It is always a finite-index subgroup of $\mathrm{SL}(2, \mathbb{Z})$ and its index coincides with the cardinality of $\mathrm{SL}(2, \mathbb{Z}) \cdot X$. Every square-tiled surface that we will consider is reduced.

A square-tiled surface may also have non-trivial automorphisms. In this case, the $\mathrm{SL}(2, \mathbb{Z})$ -action does not immediately induce a homological action on the Hodge bundle. Indeed, automorphisms are precisely the reason why orbit closures are, in general, orbifolds and not manifolds. More precisely, we define an *affine homeomorphism* as an orientation preserving homeomorphism of X whose local expressions (with respect to

the translation atlas) are affine maps of \mathbb{R}^2 . We denote the group of affine homeomorphisms by $\text{Aff}(X)$. We may extract the linear part of an affine homeomorphism to get a surjective homomorphism $\text{Aff}(X) \rightarrow \text{SL}(X)$. The kernel of this homomorphism is the group $\text{Aut}(X)$ of automorphisms of X . This can be encoded in the form of a short exact sequence:

$$1 \rightarrow \text{Aut}(X) \rightarrow \text{Aff}(X) \rightarrow \text{SL}(X) \rightarrow 1.$$

In other words, if M is the underlying topological surface of X , then $\text{Aut}(X)$ is precisely the subgroup of $\text{Mod}(M)$ stabilizing a lift of X to the Teichmüller space of translation surfaces. In this sense, it measures to which extent the $\text{Mod}(M)$ -action fails to be free at X . Automorphisms can also be defined combinatorially: they are the elements of $\text{Sym}(\text{Sq}(X))$ that commute with both h and v . It is well-known that if X has only one singularity, then it has no non-trivial automorphisms.

The homology group $H_1(M; \mathbb{R})$ admits a splitting $H_1^{\text{st}}(M) \oplus H_1^{(0)}(M)$ into symplectic and mutually symplectically orthogonal subspaces. The subspace $H_1^{\text{st}}(M)$ is two-dimensional and is usually called the *tautological plane*. It is spanned by the following two cycles: the sum of all bottom horizontal sides of the squares of X oriented rightwards, and the sum of all left vertical sides of the squares of X oriented upwards. The subspace $H_1^{(0)}(M)$ consists of the *zero-holonomy cycles*, that is, the cycles c such that $\int_c \omega = 0$.

Let $\rho: \text{Aff}(X) \rightarrow \text{Sp}(H_1(M; \mathbb{R}))$ be the representation induced by the homological action of $\text{Aff}(X)$. By restricting this representation to an invariant subspace, we obtain a monodromy representation of a subbundle of the Hodge bundle. We define the *monodromy group* of this subbundle to be the image of this representation.

The group $\rho(\text{Aff}(X))$ preserves the splitting $H_1(M; \mathbb{R}) = H_1^{\text{st}}(M) \oplus H_1^{(0)}(M)$. Moreover, the space $H_1^{\text{st}}(M)$ is also irreducible and its monodromy group is a finite-index subgroup of $\text{SL}(2, \mathbb{Z}) = \text{Sp}(2, \mathbb{Z})$ which can be identified with $\text{SL}(X)$. The subspace $H_1^{(0)}(M)$ is in general reducible. Therefore, understanding monodromy groups means understanding the irreducible pieces of $H_1^{(0)}(M)$ and the way $\rho(\text{Aff}(X))$ acts on them.

2.2.2. Constraints for monodromy groups. Let $G = \text{Aut}(X)$. The vector space $H_1(M; \mathbb{R})$ has a structure of a G -module induced by the representation $G \rightarrow \text{Sp}(H_1(M; \mathbb{R}))$. Since G is a finite group, it possesses finitely many irreducible representations over \mathbb{R} which we denote $\text{Irr}_{\mathbb{R}}(G)$. The G -module $H_1(M; \mathbb{R})$ can be decomposed as a direct sum of irreducible representations. That is:

$$H_1(M; \mathbb{R}) = \bigoplus_{\alpha \in \text{Irr}_{\mathbb{R}}(G)} V_{\alpha}^{\oplus n_{\alpha}},$$

where each V_{α} is an irreducible subspace of $H_1(M; \mathbb{R})$ on which G acts as the representation α .

We can collect the same G -irreducible representations into the so-called *isotypical components*. That is, let $W_{\alpha} = V_{\alpha}^{\oplus n_{\alpha}}$ and then:

$$H_1(M; \mathbb{R}) = \bigoplus_{\alpha \in \text{Irr}_{\mathbb{R}}(G)} W_{\alpha}.$$

The group $\rho(\text{Aff}(X))$ does not, a priori, respect this decomposition because a general affine homeomorphism may not commute with every automorphism. However, since G is a finite group, there exists a finite-index subgroup of $\text{Aff}_*(X) \leq \text{Aff}(X)$ whose every element commutes with every element of G . Replacing $\text{Aff}(X)$ by some finite-index

subgroup preserves the connected component of the identity of the Zariski-closure of the resulting monodromy group.

Given an irreducible representation α of G , we may define an associative division algebra D_α : the centralizer of $\alpha(G)$ inside $\text{End}_{\mathbb{R}}(V_\alpha)$. Up to isomorphism, there are three associative real division algebras:

- $D_\alpha \simeq \mathbb{R}$, and α is said to be real;
- $D_\alpha \simeq \mathbb{C}$, and α is said to be complex; or
- $D_\alpha \simeq \mathbb{H}$, and α is said to be quaternionic.

The following theorem [MYZ14, Section 3.7; MYZ16] relates these cases to constraints for monodromy groups:

Theorem 2.2. *The Zariski-closure of the group $\rho(\text{Aff}_*(X))|_{W_\alpha}$ is contained in:*

- $\text{Sp}(2g_\alpha, \mathbb{R})$ if α is real;
- $\text{SU}(p_\alpha, q_\alpha)$ if α is complex; or
- $\text{SO}^*(2d_\alpha)$ if α is quaternionic.

We will exploit these constraints to find the desired groups.

3. CONSTRUCTION OF THE FAMILY OF SQUARE-TILED SURFACES

In this section, we will construct the quaternionic covers that realize the desired groups as the Zariski-closure of the monodromy of a specific flat irreducible subbundle of the Hodge bundle.

Let $d \geq 3$ be an odd integer. We consider a “staircase” $X^{(d)}$ with d squares: the square-tiled surface induced by the horizontal permutation $(2, 1)(4, 3) \dots (d-1, d-2)(d)$ and the vertical permutation $(1)(2, 3)(4, 5) \dots (d-1, d)$. It belongs to the connected component $\mathcal{H}_{(d+1)/2}(d-1)^{\text{hyp}}$. Its automorphism group is trivial, since it belongs to a minimal stratum.

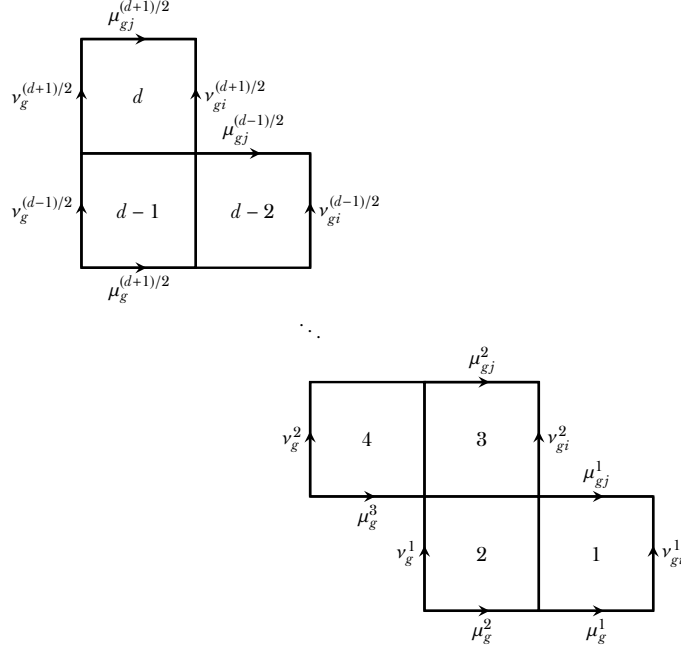
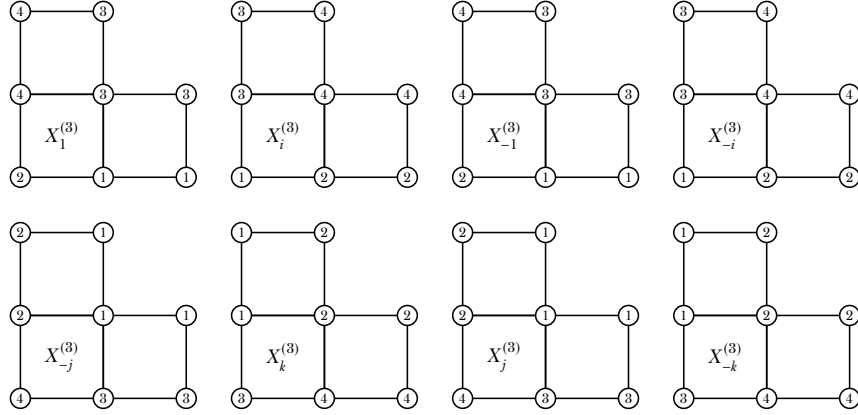
We construct a cover $\tilde{X}^{(d)}$ of $X^{(d)}$ as follows: for each element g of the quaternion group $Q = \{1, -1, i, -i, j, -j, k, -k\}$, we take a copy $X_g^{(d)}$ of $X^{(d)}$. We glue the r -th right vertical side of $X_g^{(d)}$ to the r -th left vertical side of $X_{gi}^{(d)}$. Similarly, we glue the r -th top horizontal side of $X_g^{(d)}$ to the r -th bottom horizontal side of $X_{gj}^{(d)}$. See Figure 1. This construction coincides, up to relabelling, with that of Filip–Forni–Matheus for $d = 3$ [FFM18, Section 5.1].

For each $g \in Q$, we can define an automorphism φ_g of $\tilde{X}^{(d)}$ by mapping $X_h^{(d)}$ to $X_{gh}^{(d)}$ in the natural way, that is, preserving the covering map $\tilde{X}^{(d)} \rightarrow X^{(d)}$ for each $h \in Q$. Indeed, the gluings are defined by multiplication on the right, which commutes with multiplication on the left. These are the only automorphisms of $\tilde{X}^{(d)}$: an automorphism ψ of $\tilde{X}^{(d)}$ induces an automorphism of $X^{(d)}$ by “forgetting the labels”. Since the only automorphism of $X^{(d)}$ is the identity, $X_1^{(d)}$ is mapped to some $X_g^{(d)}$ for $g \in Q$ in a way that preserves the covering map $\tilde{X}^{(d)} \rightarrow X^{(d)}$. Thus, $\psi = \varphi_g$ and $\text{Aut}(\tilde{X}^{(d)}) \simeq Q$. We will denote $\text{Aut}(\tilde{X}^{(d)})$ by G .

From now on, we will restrict to the case $d \equiv 3 \pmod{8}$. The surface $\tilde{X}^{(d)}$ has four singularities, each of order $2d-1$. Therefore, $\tilde{X}^{(d)}$ belongs to the (connected) stratum $\mathcal{H}_{4d-1}((2d-1)^4)$.

Since the automorphism $\varphi_{-1} \in G$ is an involution, it induces a splitting

$$H_1(\tilde{X}^{(d)}; \mathbb{R}) = H_1^+(\tilde{X}^{(d)}) \oplus H_1^-(\tilde{X}^{(d)}),$$

FIGURE 1. An illustration of $X_g^{(d)}$.FIGURE 2. An illustration of $\tilde{X}^{(3)}$ showing its four singularities. Horizontally, the each copy of $X^{(3)}$ is cyclically glued to the copy on its right or left, but this does not hold for the vertical gluings (as the top sides of $X_k^{(3)}$, for example, are glued to the bottom sides of $X_{-i}^{(3)}$).

where $H_1^\pm(\tilde{X}^{(d)})$ is the subspace of $H_1(\tilde{X}^{(d)})$ where φ_{-1} acts as $\pm \text{Id}$. These subspaces are symplectic and symplectically orthogonal. The subspace $H_1^+(\tilde{X}^{(d)})$ contains $H_1^{\text{st}}(\tilde{X}^{(d)})$ and is naturally isomorphic to $H_1(X_\pm^{(d)}; \mathbb{R})$, where $X_\pm^{(d)} = \tilde{X}^{(d)}/\varphi_{-1}$. This latter surface is an intermediate cover of $X^{(d)}$ over the group $Q/\{1, -1\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since

every singularity of $\tilde{X}^{(d)}$ is fixed by φ_{-1} , $X_{\pm}^{(d)}$ belongs to the stratum $\mathcal{H}_{2d-1}((d-1)^4)$. Therefore, $H_1^+(\tilde{X}^{(d)})$ is a $(4d-2)$ -dimensional subspace of the $(8d-2)$ -dimensional space $H_1(\tilde{X}^{(d)}; \mathbb{R})$ and we obtain that the dimension of $H_1^-(\tilde{X}^{(d)})$ is $4d$. The irreducible representations (over \mathbb{C}) of the group Q can be summarized in the following character table:

	Dimension	1	-1	$\pm i$	$\pm j$	$\pm k$
χ_1	1	1	1	1	1	1
χ_i	1	1	1	1	-1	-1
χ_j	1	1	1	-1	1	-1
χ_k	1	1	1	-1	-1	1
$\text{tr } \chi_2$	2	2	-2	0	0	0

TABLE 1. Character table of Q .

As detailed in Section 2.2.2, $H_1(\tilde{X}^{(d)}; \mathbb{R})$ can be split into isotypical components associated with such representations. From the character table, we obtain that $H_1^-(\tilde{X}^{(d)})$ corresponds to $2d$ copies of a G -irreducible representation whose character is the quaternionic character χ_2 , that is, $H_1^-(\tilde{X}^{(d)}) = W_{\chi_2}$. Indeed, φ_{-1} acts as the identity for any other representation in the table. We obtain the following:

Lemma 3.1. *The Zariski-closure of the monodromy group of the flat subbundle induced by $H_1^-(\tilde{X}^{(d)})$ is a subgroup of $\text{SO}^*(2d)$. Moreover, Kontsevich–Zorich cocycle over this subbundle has at least four zero Lyapunov exponents.*

Proof. The first statement is a direct consequence of Theorem 2.2. The second statement is a consequence of the first since d is odd [Fil17, Corollary 5.5, Section 5.3.4]. \square

We will prove that, for certain d with $d \bmod 8 = 3$, such Zariski-closure is actually $\text{SO}^*(2d)$.

4. COMPUTATION OF THE MONODROMY GROUPS

4.1. Dimensional constraints. In the presence of zero Lyapunov exponents, Theorem 2.1 states that the only possible Lie algebra representations of the Zariski-closure of the monodromy group of a flat subbundle are $\mathfrak{so}^*(2d)$ in the standard representation $\rho_{2d}: \mathfrak{so}^*(2d) \rightarrow \mathbb{R}^{4d}$, $\mathfrak{su}(p, q)$ in the standard representation $\sigma_{p,q}: \mathfrak{su}^*(p, q) \rightarrow \mathbb{R}^{2(p+q)}$, and $\mathfrak{su}(p, 1)$ in some exterior power representation $\tau_{r,p}: \mathfrak{su}(p, 1) \rightarrow \mathbb{R}^{2\binom{p+1}{r}}$. The representations $\tau_{r,p}$ are irreducible and faithful: by complexifying, one obtains $\mathfrak{sl}(p+1, \mathbb{C})$ whose exterior power representations are known to satisfy these properties.

Let

$$\mathcal{D} = \left\{ d \in \mathbb{N} \mid 2d \neq \binom{p+1}{r} \text{ for every } p \text{ and } 1 < r < p \right\}.$$

We have the following dimensional constraints:

Lemma 4.1. *Let $d \in \mathcal{D}$. If $2d = p + q$, then $\dim_{\mathbb{R}} \mathfrak{so}^*(2d) < \dim_{\mathbb{R}} \mathfrak{su}(p, q)$. Moreover, if $2d = \binom{p+1}{r}$, then $\dim_{\mathbb{R}} \mathfrak{so}^*(2d) < \dim_{\mathbb{R}} \mathfrak{su}(p, 1)$.*

Proof. We have that

$$\dim_{\mathbb{R}} \mathfrak{so}^*(2d) = d(2d - 1)$$

$$\begin{aligned}\dim_{\mathbb{R}} \mathfrak{su}(p, q) &= (p + q)^2 - 1 \\ \dim_{\mathbb{R}} \mathfrak{su}(p, 1) &= p(p + 2).\end{aligned}$$

If $p + q = 2d$, then $\dim_{\mathbb{R}} \mathfrak{su}(p, q) = 4d^2 - 1$. Thus, $\dim_{\mathbb{R}} \mathfrak{so}^*(2d) < \dim_{\mathbb{R}} \mathfrak{su}(p, q)$ for every d . If $2d = \binom{p+1}{r}$ for $d \in \mathcal{D}$, then we have that $r \in \{1, p\}$. Therefore, $2d = p + 1$ and we conclude as in the previous case. \square

Remark 4.2. To obtain the strict inequality in the previous proof, it is necessary to assume that $d \in \mathcal{D}$. Indeed, if $d \notin \mathcal{D}$ then $2d = \binom{p+1}{r}$ with $1 < r < p$. This implies that $\dim_{\mathbb{R}} \mathfrak{su}(p, 1) \leq \dim_{\mathbb{R}} \mathfrak{so}^*(2d)$. Indeed, since $2d = \binom{p+1}{r} \geq \binom{p+1}{2}$ we have that $p(p + 1) \leq 4d$ and it is easy to check that this results in $p(p + 2) \leq d(2d - 1)$ if $d \geq 3$.

The previous lemma shows that, assuming (strong) irreducibility, $\mathfrak{so}^*(2d)$ is the only possible Lie algebra of the Zariski-closure of the flat subbundle induced by $H_1(\tilde{X}^{(d)})$. Indeed, we already know by Lemma 3.1 that its Lie algebra \mathfrak{g} is contained in $\mathfrak{so}^*(2d)$, so $\dim_{\mathbb{R}} \mathfrak{g} \leq \dim_{\mathbb{R}} \mathfrak{so}^*(2d)$. Moreover, the corresponding Lie algebra representation is either ρ_{2d} , $\sigma_{p,q}$ or $\tau_{r,p}$. These representations act irreducibly on real vector spaces of dimensions $4d$, $2(p + q)$ and $2\binom{p+1}{r}$, respectively. Hence, if the sought representation is $\sigma_{p,q}$ or $\tau_{r,p}$, the previous lemma implies that $\dim_{\mathbb{R}} \mathfrak{so}^*(2d) < \dim_{\mathbb{R}} \mathfrak{g}$, a contradiction. We finish this section by showing that the set \mathcal{D} is large inside \mathbb{N} :

Lemma 4.3. *The set \mathcal{D} has full density in \mathbb{N} .*

Proof. Let

$$B_{p+1} = \left\{ \binom{p+1}{r} \mid 1 < r < p \right\} \text{ and } B = \bigcup_{p \geq 3} B_{p+1}.$$

We will show that $|B \cap \{1, \dots, n\}|/n \rightarrow 0$. Observe that

$$|B \cap \{1, \dots, n\}| \leq \sum_{p \geq 3} |B_{p+1} \cap \{1, \dots, n\}|$$

Now, observe that:

- If $p \geq 3$ and $\binom{p+1}{2} > n$, then $|B_{p+1} \cap \{1, \dots, n\}| = 0$;
- If $p \geq 5$ and $\binom{p+1}{4} > n$, then $|B_{p+1} \cap \{1, \dots, n\}| \leq 2$.

We will split the sum in this way to obtain a bound for $|B \cap \{1, \dots, n\}|$. Let p_2 be the smallest $p \geq 3$ such that $\binom{p+1}{2} > n$ and let p_4 be the smallest $p \geq 5$ such that $\binom{p+1}{4} > n$. We have that

$$\begin{aligned}|B \cap \{1, \dots, n\}| &\leq \sum_{p=3}^{p_4-1} (p+1) + \sum_{p=p_4}^{p_2-1} 2 \leq p_4(p_4 - 1) + 2(p_2 - 1) \\ &= O(n^{1/4})O(n^{1/4}) + O(n^{1/2}) = O(n^{1/2}) = o(n).\end{aligned}$$

\square

4.2. Dehn multi twists. We will use Dehn multi twists along specific rational directions to prove irreducibility. Assume that there exist rational directions (p_r, q_r) for $0 \leq r < d$ such that:

- (1) the cylinder decomposition along (p_r, q_r) consists of eight cylinders with waist curves c_g^r , for $g \in Q$, of the same length. Thus, the Dehn multi twist along (p_r, q_r) can be written as $T_r v = v + n_r \sum_{g \in G} \langle v, c_g^r \rangle c_g^r$; and

- (2) the action of G on the labels is “well-behaved”, that is, $(\varphi_h)_*c_g^r = c_{hg}^r$ for every $0 \leq r < d$, and $g, h \in Q$.

Let $Q^+ = \{1, i, j, k\}$ and $\tilde{c}_g^r = c_g^r - c_{-g}^r$ for each $g \in Q^+$. If $v \in H_1^-(\tilde{X}^{(d)})$ we have that

$$\begin{aligned} \langle v, \tilde{c}_g^r \tilde{c}_g^r \rangle &= \langle v, c_g^r - c_{-g}^r \rangle (c_g^r - c_{-g}^r) \\ &= \langle v, c_g^r \rangle c_g^r - \langle v, c_{-g}^r \rangle c_{-g}^r - \langle v, c_g^r \rangle c_{-g}^r + \langle v, c_{-g}^r \rangle c_g^r \\ &= \langle v, c_g^r \rangle c_g^r + \langle v, c_{-g}^r \rangle c_{-g}^r + \langle v, c_{-g}^r \rangle c_g^r + \langle v, c_g^r \rangle c_{-g}^r \\ &= 2(\langle v, c_g^r \rangle c_g^r + \langle v, c_{-g}^r \rangle c_{-g}^r), \end{aligned}$$

where we used that $(\varphi_{-1})_*v = -v$ and that $(\varphi_{-1})_*$ is a symplectic automorphism. Therefore, $T_r v = v + \frac{n_r}{2} \sum_{g \in Q^+} \langle v, \tilde{c}_g^r \rangle \tilde{c}_g^r$. Let $C_r = \langle \tilde{c}_g^r \rangle_{g \in Q^+}$, the span of the \tilde{c}_g^r for $g \in Q^+$. We will also assume the following:

- (3) $\{\tilde{c}_g^r\}_{g,r}$ is a basis of $H_1^-(\tilde{X}^{(d)})$;
(4) for each $0 \leq r \neq s < d$ and $v \in C_r \setminus \{0\}$, $T_s v \neq v$; and
(5) for any $v \in C_0 \setminus \{0\}$, $C_0 = \langle \{v\} \cup \{(T_0 - \text{Id})(T_r - \text{Id})(T_1 - \text{Id})v\}_{r=2}^4 \rangle$.

The last three conditions can be stated in terms of intersection numbers:

- (3) the matrix of intersection numbers of the \tilde{c}_g^r is non-singular;
(4) for each $0 \leq r \neq s < d$, there exists $g \in Q^+$ such that $\langle \tilde{c}_g^r, \tilde{c}_g^s \rangle \neq 0$; and
(5) let $v = \sum_{g \in Q^+} \mu_g \tilde{c}_g^0 \in C_0 \setminus \{0\}$ and $v_r = (T_0 - \text{Id})(T_r - \text{Id})(T_1 - \text{Id})v$ for $2 \leq r \leq 4$. Write $v_r = \sum_{g \in Q^+} \mu_{g,r} \tilde{c}_g^0$ and put $\mu_{g,1} = \mu_g$. Then, the matrix of coefficients $(\mu_{g,r})_{g \in Q^+, 1 \leq r \leq 4}$ is non-singular.

These conditions are enough to prove that $H_1^-(\tilde{X}^{(d)})$ is strongly irreducible for the action of $\text{Aff}_*(\tilde{X}^{(d)})$:

Lemma 4.4. *Assume that (1)–(5) hold. Then, $\text{Aff}_{**}(\tilde{X}^{(d)})$ acts irreducibly on $H_1^-(\tilde{X}^{(d)})$, where $\text{Aff}_{**}(\tilde{X}^{(d)})$ is any finite-index subgroup of $\text{Aff}_*(\tilde{X}^{(d)})$.*

Proof. Let $V \neq \{0\}$ be a subspace of $H_1^-(\tilde{X}^{(d)})$ on which $\text{Aff}_{**}(\tilde{X}^{(d)})$ acts irreducibly. By (3), it is enough to prove that $C_r \subseteq V$ for each $0 \leq r < d$.

Since the index of $\text{Aff}_{**}(\tilde{X}^{(d)})$ is finite, some power of T_r belongs to $\text{Aff}_{**}(\tilde{X}^{(d)})$ for every $0 \leq r < d$. Without loss of generality, we can assume $T_r \in \text{Aff}_{**}(\tilde{X}^{(d)})$, since the number $n_r \neq 0$ in the formula for T_r is irrelevant for the proof.

We will first show that $C_0 \subseteq V$. Let $u \in V \setminus \{0\}$. Since $H_1^-(\tilde{X}^{(d)})$ is symplectic, by (3) there exists $0 \leq r < d$ such that $T_r u \neq u$. Clearly, $w = (T_r - \text{Id})u \in C_r \setminus \{0\}$. Now, by (4), $v = (T_0 - \text{Id})w \in C_0 \setminus \{0\}$. Finally, by (5) we have that $C_0 \subseteq V$.

Now, it is enough to show that $(T_r - \text{Id})C_0 = C_r$. Indeed, this implies that $C_r \subseteq V$ for each $0 \leq r < d$ as $C_0 \subseteq V$ and V is closed under addition and is irreducible for the action of $\text{Aff}_{**}(\tilde{X}^{(d)})$. Let $v = (T_r - \text{Id})\tilde{c}_1^0 \in C_r \setminus \{0\}$. Observe that (2) implies that C_0 is G -invariant. Since $\text{Aff}_{**}(\tilde{X}^{(d)})$ commutes with G , we have that $(\varphi_g)_*v \in V$ for each $g \in Q^+$. Write $v = \sum_{g \in Q^+} \mu_g \tilde{c}_g^r$. By (2), we have that

$$\begin{aligned} (\varphi_i)_*v &= -\mu_i \tilde{c}_1^r + \mu_1 \tilde{c}_i^r - \mu_k \tilde{c}_j^r + \mu_j \tilde{c}_k^r \\ (\varphi_j)_*v &= -\mu_j \tilde{c}_1^r + \mu_k \tilde{c}_i^r + \mu_1 \tilde{c}_j^r - \mu_i \tilde{c}_k^r \\ (\varphi_k)_*v &= -\mu_k \tilde{c}_1^r - \mu_j \tilde{c}_i^r + \mu_i \tilde{c}_j^r + \mu_1 \tilde{c}_k^r. \end{aligned}$$

Therefore, the matrix of coefficients of $(\varphi_g)_*v$ for $g \in Q^+$ is

$$\begin{pmatrix} \mu_1 & \mu_i & \mu_j & \mu_k \\ -\mu_i & \mu_1 & -\mu_k & \mu_j \\ -\mu_j & \mu_k & \mu_1 & -\mu_i \\ -\mu_k & -\mu_j & \mu_i & \mu_1 \end{pmatrix}$$

whose determinant is $\left(\sum_{g \in Q^+} \mu_g^2\right)^2 \neq 0$. We obtain that $\langle (\varphi_g)_*v \rangle_{g \in Q^+} = C_r \subseteq V$, which completes the proof. \square

Remark 4.5. The action of the automorphism group $G \simeq Q$ on $H_1^-(\tilde{X}^{(d)})$ induces a structure of an \mathbb{H} -module on $H_1^-(\tilde{X}^{(d)})$. Indeed, the map $(\varphi_g)_*$ can be interpreted as the multiplication by g . The blocks C_r satisfy $C_r = \langle (\varphi_g)_*v \rangle_{g \in Q^+}$, as shown in the proof of the previous lemma, so they can be interpreted as the span of single vectors by the coefficients in \mathbb{H} . In this context, conditions (1)–(4) are a “moral analogue” of Deligne’s criterion to compute Zariski-closures [Del80; PS03]: the group contains appropriate transvections along a basis of the vector space and this basis is transformed by the transvections in a sufficiently rich manner. These conditions are indeed enough to prove the strong irreducibility of $H_1^-(\tilde{X}^{(d)})$ in the quaternionic setting, that is, by the action of the group generated by $\text{Aff}_{**}(\tilde{X}^{(d)})$ and G . Nevertheless, since we are interested in the irreducibility over the *reals* (as opposed to the irreducibility over the *quaternions*) of the monodromy group, we also need condition (5) to ensure that any invariant subspace contains an entire quaternionic block C_0 . Of course, the block C_0 could be replaced by any other block, but we stated condition (5) in this way for simplicity.

We can now show that this conditions are enough for the monodromy group to be $\text{SO}^*(2d)$:

Proposition 4.6. *Assume that $d = 3 \pmod 8$, that $d \in \mathcal{D}$ and that (1)–(5) hold. Then, the Zariski-closure of the group $\rho(\text{Aff}_*(X))|_{H_1^-(\tilde{X}^{(d)})}$ is $\text{SO}^*(2d)$.*

Proof. By Lemma 3.1, exactly four Lyapunov exponents of the Kontsevich–Zorich cocycle are zero, so the hypotheses of Theorem 2.1 are satisfied. To conclude by Lemma 4.1, it is enough for $\text{Aff}_*(X)$ to act strongly irreducibly on $H_1^-(\tilde{X}^{(d)})$, which follows from the previous lemma. \square

The next section is then devoted to finding the desired Dehn multi twists.

4.3. Suitable rational directions. In this section, we will find the desired rational directions (p_r, q_r) and prove (1)–(5) for the specific values of d mentioned in the statement of the main theorem to conclude the proof. Assume that $d = 3 \pmod 8$ for the rest of the section.

The matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

generate $\text{SL}(2, \mathbb{Z})$ and, thus, can be used to understand the $\text{SL}(2, \mathbb{Z})$ -orbit of a square-tiled surface. The orbit of the “staircase” $X^{(d)}$ consists of three elements, which we call $Z^{(d)}$, $X^{(d)}$ and $Y^{(d)}$. See Figure 3. Its Veech group has index three and is the so-called theta subgroup of $\text{SL}(2, \mathbb{Z})$ consisting on the elements whose modulo-two reductions are either Id or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. See Figure 4.

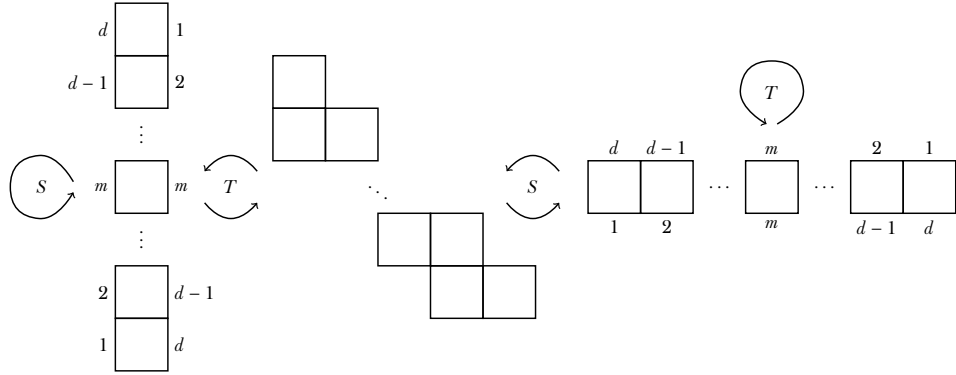


FIGURE 3. The $SL(2, \mathbb{Z})$ -orbit of $X^{(d)}$ using T and S as generators. It consists of three distinct square-tiled surfaces, which we call $Z^{(d)}$, $X^{(d)}$ and $Y^{(d)}$ from left to right. The labels in the $Y^{(d)}$ and $Z^{(d)}$ show the identification of the sides. Unlabelled horizontal sides are identified with the only horizontal having the same horizontal coordinates, and similarly for unlabelled vertical sides.

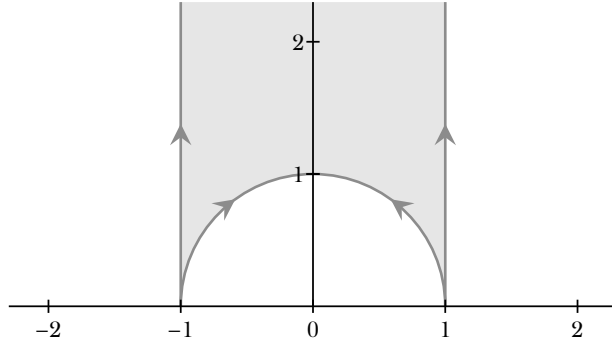


FIGURE 4. The “canonical” fundamental domain of the action of the theta subgroup on the upper half-plane. The resulting Teichmüller curve has genus zero and two cusps.

d	Index	Genus	Cusps
3	12	0	3
11	16896	225	960
19	1867776	30721	94208

TABLE 2. The index of $SL(\tilde{X}^{(d)})$ and the genus and number of cusps of the resulting Teichmüller curve for small values of d .

Since $\tilde{X}^{(d)}$ is a cover of $X^{(d)}$, the graph induced by the action on T and S on $\tilde{X}^{(d)}$ is a cover of the graph in Figure 3. In other words, if $g \in SL(2, \mathbb{Z})$ then $g \cdot \tilde{X}^{(d)}$ is a degree-eight cover of $g \cdot X^{(d)}$. Moreover, since the graph in Figure 3 has only three vertices, writing g in terms of T and S and following the arrows of the graph us to compute

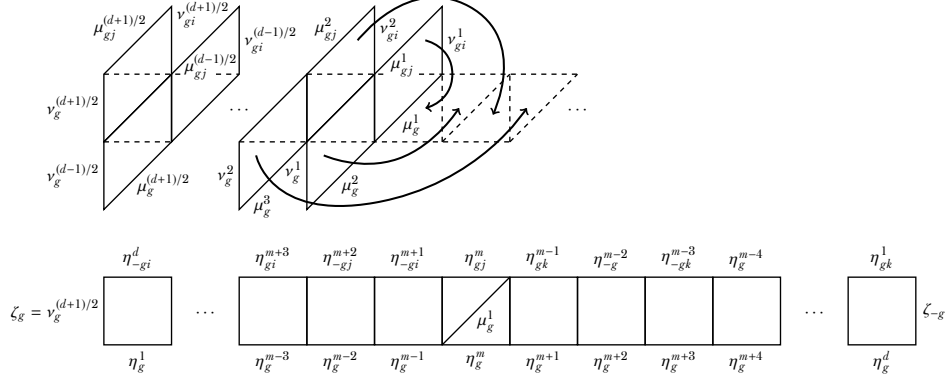


FIGURE 5. An illustration of $Y_g^{(d)}$ and of the cut-and-paste operations used to obtain this description.

$g \cdot X^{(d)}$, which is useful to understand $g \cdot \tilde{X}^{(d)}$. The index of $\mathrm{SL}(\tilde{X}^{(d)})$ and the genus and number of cusps of the resulting Teichmüller curve grow rapidly with d , as shown by Table 2

We will use the following rational directions: $(p_r, q_r) = (-(4r+1), 4r+3)$ for $0 \leq r < d$.

Observe that the matrix $\begin{pmatrix} 2r+1 & 2r \\ 4r+3 & 4r+1 \end{pmatrix}$ maps the direction (p_r, q_r) to $(-1, 0)$. Moreover, this matrix can be written as $S^2 T^{2r} S$. By Figure 3, $S^2 T^{2r} S \cdot X^{(d)} = Y^{(d)}$, so this surface has only one horizontal cylinder.

The matrix S maps (p_r, q_r) to $(-(4r+1), 2)$. The surface $S \cdot \tilde{X}^{(d)}$, which we call $\tilde{Y}^{(d)}$, is a degree-eight cover of $S \cdot X^{(d)} = Y^{(d)}$, which we will now describe explicitly.

For each $g \in Q$, consider a copy $Y_g^{(d)}$ of $Y^{(d)}$. Each of these copies consists of d squares.

We label the r -th bottom side of each square of $Y_g^{(d)}$ with η_g^r and the left side of the leftmost square with ζ_g .

Let $m = (d+1)/2$, which satisfies $m \equiv 2 \pmod{4}$ since $d \equiv 3 \pmod{8}$. There are $m-1$ squares to the left and to the right of m in $Y_g^{(d)}$. The labels of the top sides of the squares to the right of m are:

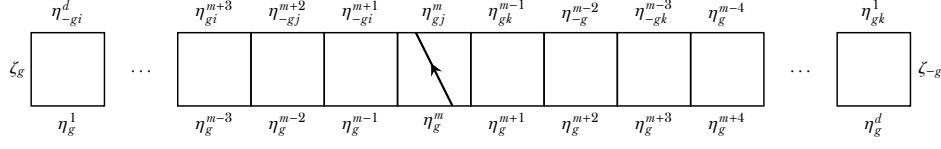
$$\eta_{gk}^{m-1}, \eta_{-g}^{m-2}, \eta_{-gk}^{m-3}, \eta_g^{m-4}, \dots, \eta_{gk}^5, \eta_{-g}^4, \eta_{-gk}^3, \eta_g^2, \eta_{gk}^1.$$

The labels of the top sides of the squares to the left of m are:

$$\eta_{-gi}^d, \eta_{-gj}^{d-1}, \eta_{-gi}^{d-2}, \eta_{-gj}^{d-3}, \eta_{-gi}^{d-4}, \dots, \eta_{gi}^{m+3}, \eta_{-gj}^{m+2}, \eta_{-gi}^{m+1}, \eta_{gj}^m.$$

In the two previous lists, the group elements in Q follow a 4-periodic pattern. Finally, we label the rightmost square of $Y_g^{(d)}$ with ζ_{-g} . See Figure 5 for an illustration.

By a slight abuse of notation, from now on we will use the names η_g^r and ζ_g to refer to the elements of $H_1(\tilde{Y}^{(d)}, \Sigma; \mathbb{R})$ induced by the horizontal or vertical curves joining the two vertices of the side labelled η_g^r or ζ_g , oriented either rightwards or upwards. We have that $\mathrm{Aut}(\tilde{Y}^{(d)}) \simeq Q$, which can be proved in the exact same way as for $\tilde{X}^{(d)}$. That is, we define an automorphism φ_g by mapping $Y_h^{(d)}$ to $Y_{gh}^{(d)}$ and these are the only automorphisms of $\tilde{Y}^{(d)}$ since $\mathrm{Aut}(Y^{(d)})$ is trivial. The automorphism φ_{-1} induces a splitting $H_1(\tilde{Y}^{(d)}; \mathbb{R}) = H^+(\tilde{Y}^{(d)}) \oplus H^-(\tilde{Y}^{(d)})$. The space $H^-(\tilde{Y}^{(d)})$ is $4d$ -dimensional and it is


 FIGURE 6. Direction $(-1, 2)$ on $Y_g^{(d)}$.

exactly the image of $H^-(\tilde{X}^{(d)})$ by S . Let $\hat{\eta}_g^r = \eta_g^r - \eta_{-g}^r$ for $g \in Q^+ = \{1, i, j, k\}$ and $1 \leq r \leq d$. We have that each $\hat{\eta}_g^r$ is an absolute cycle since φ_{-1} fixes every singularity. Therefore, $\hat{\eta}_g^r \in H_1^-(\tilde{Y}^{(d)})$ and we obtain that $\{\hat{\eta}_g^r\}_{g \in Q^+, 1 \leq r \leq d}$ is a basis of $H_1^-(\tilde{Y}^{(d)})$.

Observe that $\tilde{Y}^{(d)}$ has four horizontal cylinders. The matrix T^{2r} maps the direction $S(p_r, q_r) = (-4r + 1, 2)$ to $(-1, 2)$. Therefore, understanding the direction (p_r, q_r) on $\tilde{X}^{(d)}$ is equivalent to understanding the direction $(-1, 2)$ on $T^{2r} \cdot \tilde{Y}^{(d)}$.

We will start the analysis for $r = 0$. For $g \in Q$, consider the trajectory induced by the direction $(-1, 2)$ on $Y_g^{(d)}$ as in Figure 6. The resulting cylinder decomposition consists on eight cylinders. Indeed, observe that each cycle η_g^r is intersected twice by such trajectories. Therefore, the total number of intersections of all the η_g^r by all trajectories is $16d$. To obtain that there are exactly eight cylinders in this decomposition, it is therefore enough to show that each trajectory intersects exactly $2d$ cycles η_g^r .

The trajectory in Figure 6 intersects the following cycles:

$$\begin{aligned} & \eta_{g_1}^m, \eta_{g_2}^{m+1}, \eta_{g_3}^{m-1}, \eta_{g_4}^{m+2}, \eta_{g_5}^{m-2}, \dots, \eta_{g_{d-1}}^d, \eta_{g_d}^1, \zeta_{g_d}, \\ & \eta_{g_{d+2}}^1, \eta_{g_{d+3}}^d, \dots, \eta_{g_{2d-3}}^{m-2}, \eta_{g_{2d-2}}^{m+2}, \eta_{g_{2d-1}}^{m-1}, \eta_{g_{2d}}^{m+1}, \eta_{g_{2d+1}}^m \end{aligned}$$

where the sequence g_1, \dots, g_{2d+1} is obtained by (right-)multiplying g successively by

$$(*) \quad j, -i, k, -j, -1, i, -k, \dots, j, 1, -i, k, \boxed{-1}, k, -i, 1, j, \dots, -k, i, -1, -j, k, -i, j.$$

The boxed -1 comes from the intersection with the vertical side labelled as ζ_{g_d} .

This sequence indeed describes a closed trajectory as $g_{2d+1} = g$. Indeed, the product can be computed from “inside out”, using that -1 is in the centre of Q . We obtain that $g_{2d+1} = -g \cdot k^2 \cdot (-i)^2 \cdot 1^2 \cdot j^2 \dots (-i)^2 j^2$. Moreover, the number of times 1^2 and $(-1)^2$ occur in this product is $(m-2)/2$, which is an even number as $m \bmod 4 = 2$, and the total number of terms is $d+1$, which is also an even number. Thus, $g_{2d+1} = g$ and we conclude that the cylinder decomposition induced by $(-1, 2)$ has exactly eight cylinders. Moreover, we obtain that the action of $\text{Aut}(\tilde{Y}^{(d)})$ on these waist curves is “well-behaved” in the sense of (2): naming the trajectory starting on $Y_g^{(d)}$ as c_g^0 , we get that $(\varphi_h)_* c_g^0 = c_{hg}^0$.

Now, if $r = 1$ then $Y_g^{(d)}$ is sheared horizontally in such a way that the labels η_g^{m+1} and η_{-gi}^{m+1} end up on the same square. We will consider this square to be the “middle” square and reglue the surface accordingly. The surface $T^2 \cdot \tilde{Y}^{(d)}$ is the union of sheared and reglued versions of $Y_g^{(d)}$, for $g \in Q$, that we call $T^2 \cdot Y_g^{(d)}$. See Figure 7 for an illustration.

In general, $T^{2r} \cdot \tilde{Y}^{(d)}$, for $0 \leq r < d$, is the surface obtained from $\tilde{Y}^{(d)}$ by cyclically shifting the labels on the top sides r times to the right, the ones on the bottom sides r times to the left, and changing the signs of the elements of Q of every label of the form η_g^s for $1 \leq s \leq r$. We conclude that the cylinder decomposition of $T^{2r} \cdot \tilde{Y}^{(d)}$ induced by

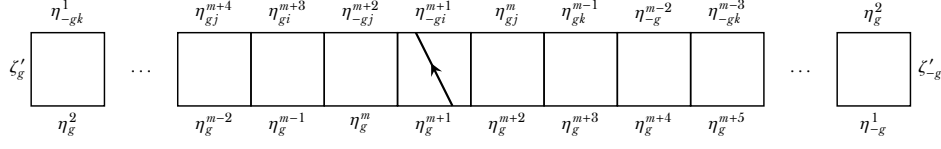


FIGURE 7. Direction $(-1, 2)$ on $T^2 \cdot Y_g^{(d)}$. The gluings are cyclically shifted and the signs of elements of Q on the labels η_\bullet^1 are changed.

the direction $(-1, 2)$ consists of exactly eight cylinders in the same way as for the case $r = 0$ and denote their waist curves by c'_g . The action of G is then well-behaved in the sense of (2). By construction, (1) also holds.

Let $\hat{c}'_g = c'_g - c'_{-g}$ for $g \in Q^+$. It remains to prove (3), (4) and (5) to conclude the proof. We conjecture that these two conditions hold for every d belonging to the congruence class $d = 3 \pmod 8$.

Nevertheless, the previous discussion allows us to compute the intersection numbers explicitly using a computer. To this end, we use the versions of these conditions in terms of intersection numbers. To obtain (3) we can compute the numbers $\langle \hat{c}'_g, \hat{\eta}_h^s \rangle$ for each $0 \leq r, s \leq d$. Then, we can compute the determinant of the resulting matrix to show that it is not singular. This matrix also allows us to compute $\langle \hat{c}'_g, \hat{c}'_h \rangle$ by expressing each \hat{c}'_g in terms of the basis $\{\hat{\eta}_g^r\}_{g \in Q^+, 1 \leq r \leq d}$ to show (4) and (5). The computations were done for $11 \leq d \leq 299$. Observe that $d = 35$ and $d = 203$ are the only elements of $\{11, \dots, 299\}$ satisfying $d = 3 \pmod 8$ and not belonging to \mathcal{D} , as $2 \times 35 = \binom{8}{4}$ and $2 \times 203 = \binom{29}{2}$. In this way, Theorem 1.1 is proved.

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