Wandering intervals in affine interval exchange maps: nonreal eigenvalues

Rodolfo Gutiérrez (rodolfo.gutierrez@imj-prg.fr)
Université Paris-Diderot 7, France

Wandering intervals and interval exchange maps

Let $T : X \to X$ be a map, where $X$ is either an interval or a circle. A wandering interval for $T$ is an interval $I \subseteq X$ satisfying:
1. The iterates of $I$ by $T$ are pairwise disjoint;
2. The set of $I$ is finite.
Let $I = [0, 1)$. A bijective map $T : I \to I$ is an interval exchange map (i.e.m.) if it is a piecewise translation. A bijective map $f : I \to I$ is an affine i.e.m. if it is a piecewise affine map with positive slopes.

Invariantsubspacesof $M$

An i.e.m., showing $I$ before and after the transformation.

Camelier and Gutiérrez [CG97] showed that a particular i.e.m. admits an extension which is an affine i.e.m. with wandering intervals.

Question: Given an i.e.m. $T$, do there exist affine i.e.m. with wandering intervals which are extensions of $T$?

Affine extensions with wandering intervals

Let $T$ be a fixed self-similar i.e.m., that is, there exists $x < 1$ such that the first-return map on $[0, x)$ is, up to rescaling, equal to $T$. Let $(I_0)_{x<1}$ be the finite partition of $I$ on the definition of an i.e.m.

A self-similar i.e.m. In this example, the interval $[0, x)$ is equal to $I_1$.

There is a natural way to construct a primitive substitution $\sigma$ on $\mathbb{Z}$ from a self-similar i.e.m. Let $M$ be the matrix associated with $\sigma$. Camelier and Gutiérrez [CG97] showed that any affine extension of $M$ must have a slope vector $\beta$ such that its real part lies in the invariant subspace corresponding to the eigenvalues different from the Perron one.

Previous results

Log $\ell \in$ Invariant subspaces of $M$

Stable space

Neutral space

Unstable space

Log $\ell \in$ Invariant subspaces of $M$

Eigenspace of $1$

Eigenspace of $\beta$ which generates the fractals

Main result (Cobo–G.–Maass)

Let $\beta$ be a nonreal unstable eigenvalue of $M$ such that $|\beta|/|1|$ is not a root of unity. Let $\Gamma$ be an eigenvector for $\beta$. If $T$ satisfies the unique representation property for $\beta$ and $\Gamma$, then, for almost every $\gamma$ in the complex subspace generated by $\Gamma$, the vector $\exp(-\gamma T)$ is the slope vector of an affine extension of $T$ with wandering intervals.

Main result (technical version)

Assume that the conditions of the theorem hold. If $y$ is a good eigenvector, one has that there exists $\omega \in \Omega$ and $\rho > 0$ such that
\[
\liminf_{n \to \infty} \frac{\text{Re}(y_n(\omega)))}{n^\rho} > 0, \quad \liminf_{n \to \infty} \frac{\text{Re}(y_n(\omega)))}{n^\rho} > 0,
\]
with $y_n(\omega) = \sum_{m=1}^{n} Y_{\omega m}$ and $y_n(\omega) = \sum_{m=1}^{n} Y_{\omega m}$. The set of full measure in the statement theorem, which we call the set of good eigenvectors, is related to Diophantine properties of $\beta$ and the fractals.

References


Fixed an eigenvector $\Gamma$ for $\beta$ and a letter $a \in \mathcal{A}$. We define the fractal $\bar{\mathcal{A}}_a$ arising from $T$, $\beta$, $\Gamma$ as the set of certain complex series over the coordinates of $\Gamma$, renormalised by powers of $\beta^{-1}$. A representation of $\omega \in \bar{\mathcal{A}}_a$ essentially a possible way to sum $\omega$. Points with multiple representations always exist. An extreme point of $\bar{\mathcal{A}}_a$ is a point with minimal real part for a rotated version of $\bar{\mathcal{A}}_a$.

Unique representation property

We say that $T$ has the unique representation property for $\beta$ and $\Gamma$ if every extreme point of each fractal has a unique representation.

Proof strategy

Let $\Omega$ be the substitutive subshift arising from $\sigma$. Let $y$ be an eigenvector for $\beta$. An affine extension with slope vector $\exp(-\gamma T)$ exists if there exists $\omega \in \Omega$ and $\rho > 0$ such that
\[
\liminf_{n \to \infty} \frac{\text{Re}(y_n(\omega)))}{n^\rho} > 0, \quad \liminf_{n \to \infty} \frac{\text{Re}(y_n(\omega)))}{n^\rho} > 0,
\]
with $y_n(\omega) = \sum_{m=1}^{n} Y_{\omega m}$ and $y_n(\omega) = \sum_{m=1}^{n} Y_{\omega m}$. The set of full measure in the statement theorem, which we call the set of good eigenvectors, is related to Diophantine properties of $\beta$ and the fractals.

The strategy of the proof is as follows:
1. Construct $\omega \in \Omega$ with $\text{Re}(\gamma(\omega))) \geq 0$.
2. Choose a suitable $\rho > 0$ considering the exponential growth of the words by the substitution.
3. Find a representation $x$ of an extreme point by manipulating $\omega$.
4. Assume by contradiction that $\eta_\omega^{\rho} \text{Re}(\gamma(\omega))) \to 0$.
5. Find another representation $y$ of an extreme point by manipulating shifts $\gamma(\omega) \omega)$. One has $x \neq y$, but that they are extreme points for the same rotation of $\bar{\mathcal{A}}_a$.
6. By the contradiction hypothesis, the real parts of truncated and rotated sums on $x$ and $y$ converge exponentially fast the same value.
7. By the unique representation property: $x$ and $y$ represent different extreme points. Together with (+), this contradicts that $y$ is a good eigenvector.

The cubic Arnoux-Yoccoz map

Let $\alpha$ be the real number such that $\alpha + 2^\alpha + 3^\alpha = 1$.

The cubic Arnoux-Yoccoz map $T$ is $[AY81]$.

This map is not self-similar, but a primitive substitution can still be associated with it. The matrix $M$ has an simple eigenvalue $\beta$ satisfying our hypotheses.

The Arnoux-Yoccoz map satisfies the unique representation property for $\beta$. Thus, for almost every eigenvector $y$ for $\beta$, $\exp(-\gamma T)$ is the slope vector of an affine extension of $T$ with wandering intervals.

Fractals for the cubic Arnoux-Yoccoz map

References


Some extreme points.