What is a — ?

These notes are written after the fact, but only once. So there must be a lot of errors. That is fine anyway because, here I am partially quoting Serre, a mathematician's duty is to

"remplacer les théorèmes faux par d'autres."

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1 10$^{\text{th}}$ April 2024

I will first begin with some anecdote. I once discussed with William Sarem about the book *A Mathematician's Lament* (Paul Lockhart). In a nutshell, the author was arguing that mathematics can be considered as a form of art. William disagreed and proposed me to have a discussion. I was very excited, as I always am to bring my long pre-cooked arguments and to expose them in order to share some understanding that I think have gathered (in this case, mainly rephrasing what Lockhart had written). However William found out to be in my point of view an unusual strong-minded discussion partner. I told him "It may be that our disagreement only stems from different definitions of *art* ." His answer was basically that common language[1] does not work that way. No definition can entirely grasp a word, and dictionaries are always auto-referential. I was struck by that approach, also because he would not take a little step out of it. That marked me forever, as I understood that I should be a careful listener to surprising new frames of thought, and in fact especially when it is not embodied by a strong-minded person.

My mind was too much entrenched in the mathematical world of axioms and strict definitions. In this world the question *Is mathematics a form of art ?* is indeed considered meaningful only if you have already defined *mathematics*, *art* and the relation *A is a form of B*. All of that is obviously out of anybody's capacity[2], and for that matter an intellectually honest person, which I manifestly was not, could not have born that contradiction.[3]

Ontology of mathematics

In the mathematical language, a question like *What is a — ?* seems meaningless. Or rather the question for instance *What is a scheme ?* will have this general answer : *Look in [some reference], there you will have the definition you are looking for.* And if sometimes definitions are contradictory — like the terms pre-schemes and schemes from EGA edition 1960 to EGA edition 1971 — then some halting algorithm will give you a new redaction where every word has a proper definition[4]. Now, is there anything to be saved from the sort of ontological question like *What is a curve ?* I will argue that yes, and that the study of moduli spaces is a mathematical well-defined problem that sheds light on ontological questions.

Moduli spaces

[5] Solving a **moduli problem** amounts to find a classification of a certain kind of objects, and more precisely it is to find a **moduli space** whose points will roughly correspond to isomorphism classes of those objects. For now take this as a common-language[6] definition. The following examples will motivate the need to compute moduli spaces.

[1]Here french

[2]One might be able to have crystal-clear internal definitions, but as far as I know it seems out of hand to be able to express it clearly and rigourously in any kind of human language.

[3]Pour le lecteur francophone, je conseille ces deux vidéos de MonsieurPhi **[2]**.

[4]At least mathematicians would generally consider it to be the case, but linguists may strongly disagree.

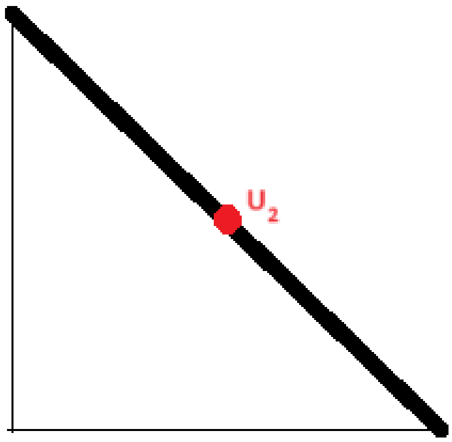
[5]I have been inspired by **[1]**. I recommend to read the beginning, but for me after page 6 it has become completely unreadable.

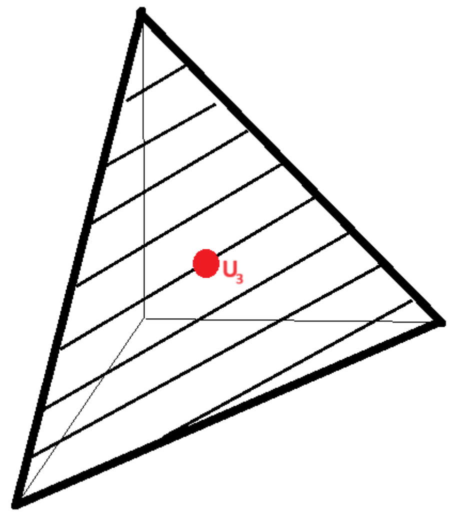
[6]Is it really though ?

First examples

Probability measure on finite sets.

A probability measure on the set $\{1, \ldots, n\}$ is exactly the data of n non-negative real numbers $p_1, \ldots, p_n$ such that $\sum p_i = 1$. For $n = 2$ and $n = 3$ the moduli spaces are the following





On this geometrical space we can instantaneously determine symmetries and specific points like the uniform distribution $\mathcal{U}_n$. We are also able to understand that the space is complete and cannot easily give rise to degeneracy, as we will see later.

Derivations.

A useful tool to build something similar to a moduli space, is to represent a functor.

Definition 1. *A functor $F : \mathsf{C} \longrightarrow \mathsf{Sets}$ is **representable** in a supcategory D if it is equivalent to the right–$\mathrm{Hom}_{\mathsf{D}}$–functor on C of some object in D, i.e. if there exists an object Y in D and for all object X in C, a bijection*

$$F(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(Y, X)$$

such that the functorial diagrams commute. We can define the same for a contravariant functor by considering instead the left–$\mathrm{Hom}_{\mathscr{D}}$–functor.

Let A be a commutative ring and B a commutative A–algebra. An A–linear **derivation** D on B is an A–linear map $D : B \longrightarrow M$, where M is a B–module, satisfying the Leibniz rule, i.e. for all $b, b' \in B$,

$$D(bb') = bD(b') + b'D(b)$$

In this case there exists a universal module such that every derivation comes from it.

Theorem 1. *Let $F : B-\mathsf{Mod} \longrightarrow \mathsf{Sets}$[7] be the functor defined by*

$$F(M) = \{A-\textit{linear derivations of } B \textit{ on } M\}$$

[7] We could in fact have replaced Sets by $B-\mathsf{Mod}$.

Then F is representable by a B–module, called the module of Kähler differentials, denoted by $\Omega_{B|A}$. The bijection is given by a universal derivation $d : B \longrightarrow \Omega_{B|A}$ such that for all $D \in F(M)$, there exists a unique map making the diagram commute

$$\begin{array}{ccc} B & \xrightarrow{d} & \Omega_{B|A} \\ & \searrow^{D} & \downarrow \\ & & M \end{array}$$

Here we can see how properties of B are encoded in $\Omega_{B|A}$. For instance, if $A = \mathbb{C}$ and $B = \mathbb{C}[x_1, \ldots, x_n]$ then we can compute $\dim_{\mathbb{C}}(\Omega_{B|A}) = n$ which is exactly the dimension of $\operatorname{Spec} B$.

Precise definitions

Let us get into the purely geometrical cases. We are considering a moduli problem which is translated into a contravariant functor F : Schemes $\longrightarrow$ Sets such that $F(B)$ is the set of families, up to isomorphism, parametrized by B of some geometrical kind of object. For instance, it can be " smooth projective curves", "finite étale cover above a scheme S", etc.

Definition 2. *A **fine moduli space** M is a scheme representing the functor F in the category* Schemes *(also noted* Sch*).*

Example 1. *The kind of object we are looking at is a line in the plane that goes through 0. By definition the projective line $\mathbb{P}^1$ is a fine moduli space for this moduli problem. In this case $\mathbb{P}^1$ is complete, meaning that we cannot have degenerate lines in the plane. I first imagined that changing 0 in the moduli problem would give degeneracy. However the moduli space for "lines in the plane that goes through M" is also $\mathbb{P}^1$, and the moduli space for "lines in the plane that goes through M, for some point M" is therefore $\mathbb{P}^1 \times \mathbb{A}^2$. Meaning that there is a smooth way to go from one moduli problem to another one.*

For some moduli problems, a moduli space does not exist, and for that reason we consider coarse moduli spaces. Those will be schemes that approximate the best non-existing moduli space.

Definition 3. *A **coarse moduli space** M is a scheme along with a natural transformation*

$$\Psi : F \Longrightarrow \operatorname{Hom}_{\mathsf{Sch}}(-, M)$$

such that

1) *If K is algebraically closed then $F(\operatorname{Spec} K) \longrightarrow \operatorname{Hom}(\operatorname{Spec} K, M) = M(K)$ is a bijection.*
2) *If M' is a scheme along with a natural transformation $\Psi' : F \Longrightarrow \operatorname{Hom}_{\mathsf{Sch}}(-, M')$ then there is a unique $M \longrightarrow M'$ such that the diagram commutes*

$$\begin{array}{ccc} & & \operatorname{Hom}(-, M) \\ & \overset{\Psi}{\nearrow} & \Downarrow \\ F & & \\ & \underset{\Psi'}{\searrow} & \\ & & \operatorname{Hom}(-, M') \end{array}$$

Condition 2) is of a universal kind, therefore we can check that if a coarse moduli space exists then 1) $\implies$ 2).

Elliptic curves

This the first non-trivial case of moduli space to compute. The functor is

$$F(B) = \{\text{family of elliptic curves parametrized by } B\}/\simeq$$

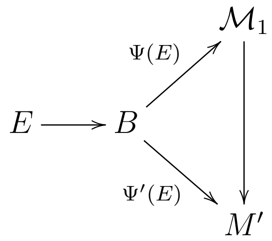
An element in $F(B)$ is an isomorphism class of schemes $E \longrightarrow B$ such that each fiber $E(b)$ over a closed point $b = \mathrm{Spec}\, K$ in B is an elliptic curve over K. Furthermore, the letter j will denote the j–invariant of an elliptic curve.

Theorem 2. $\mathbb{A}^1$ *along with* $\Psi_B : F(B) \longrightarrow \mathrm{Hom}_{\mathsf{Sch}}(B, \mathbb{A}^1)$ *defined on closed points by*

$$\Psi_B(E)(b) = j(E(b))$$

is a coarse moduli space for F. This moduli space is denoted by $\mathcal{M}_1$.

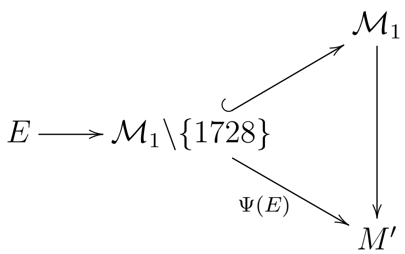
Proof. Condition 1) is for instance proved in **[4]**, where j is a rational function of the coefficients of a Weierstrass equation and thus Ψ indeed takes values in $\mathrm{Hom}_{\mathsf{Sch}}$. Let us turn now to condition 2) and let M' be a scheme along with Ψ'[8]. We want to have a map $\mathcal{M}_1 \longrightarrow M'$ such that the following diagram commutes



where $E \longrightarrow B$ is any family of elliptic curves. Now take the family $E \longrightarrow \mathcal{M}_1 \backslash \{1\}$ of elliptic curves $E(t)$ defined by the equation

$$y^2 = 4x^3 - \frac{27t}{t - 1728}(x - 1)$$

It has been chosen because $j(E(t)) = t$ and so enables you to define $\mathcal{M}_1 \backslash \{1728\} \longrightarrow M'$ via



Well now the trivial family $E \longrightarrow \mathrm{Spec}\, \mathbb{Q}$ defined by the equation $y^2 = x^3 - x$ has j–invariant 1728. The point $\mathrm{Spec}\, \mathbb{Q} \longrightarrow M'$ completes the map $\mathcal{M}_1 \longrightarrow M'$. It is straightforward to check that the map is unique and verifies the universal property, however one also has to check that we have a map between schemes... □

We will prove in the next lecture why $\mathcal{M}_1$ is **not** a fine moduli space. In fact the obstructions are exactly curves with non-trivial automorphisms. In this case these are the curves with $j = 1728$ and $j = 0$.

[8] $\Psi_B(E)$ will be denoted by $\Psi(E)$, consider B as part of the data of the family E

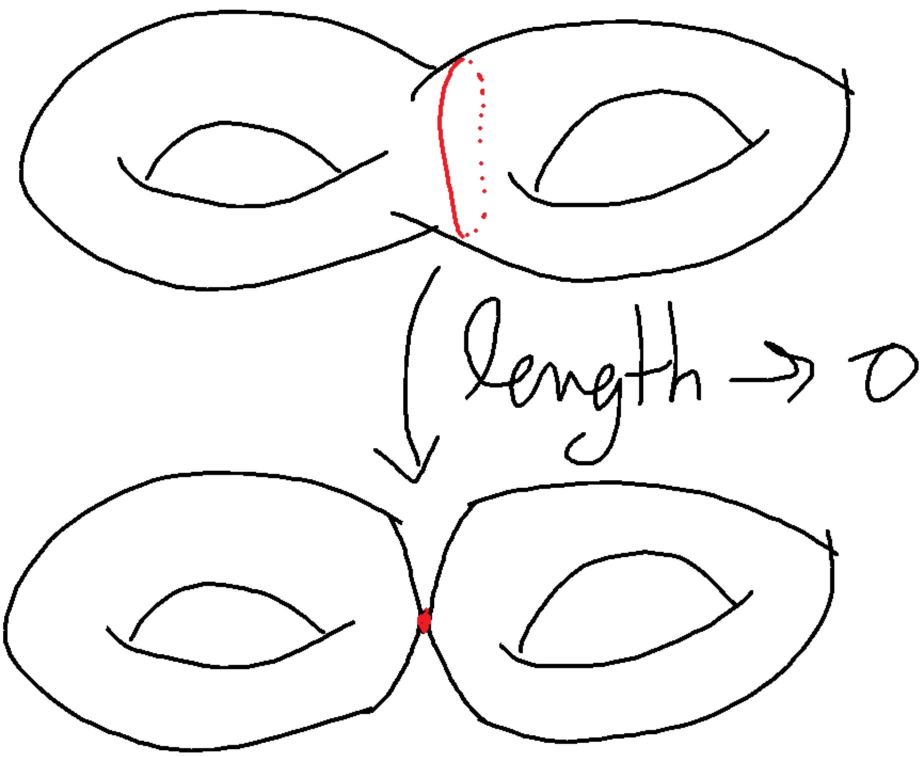
This example of moduli spaces will enable me to stress an important fact. From the definition of the kind of objects I want to classify then by its, let's say, linguistic properties I can imagine what would be a potential "degeneracy." If I am looking at smooth curves then it is natural that singular curves is at least one way of considering "degenerate" smooth curves. What is wonderful, is that the singularities can be found geometrically directly on the moduli space, or rather **on the boundary** of the moduli space.

Let us consider the moduli problem of classifying genus g smooth curves over $\mathbb{C}$. Then we have the following (see **[5]**)

Theorem 3. *Taking $\mathbb{C}$–points gives you an equivalence of categories*

$$\{\textit{connected smooth projective curves defined over } \mathbb{C}\} \xrightarrow{\sim} \{\textit{connected compact Riemann surfaces}\}$$

Fix $g \geqslant 2$ an integer and S_g denote the "NATIONAL BUREAU OF STANDARD's *compact oriented surface of genus g*" [9]. Then an isomorphism class of smooth curves of genus g is the same as an isomorphism class of Riemann surface structure on S_g which is the same as an isomorphism class of hyperbolic metric on S_g. Those metrics are fixed by the lengths of closed paths on S_g, and $6g - 6$ well-chosen paths suffice (as explained by Debattam last week). Look at the case $g = 2$, where I have chosen a path whose length I am making it go to zero :



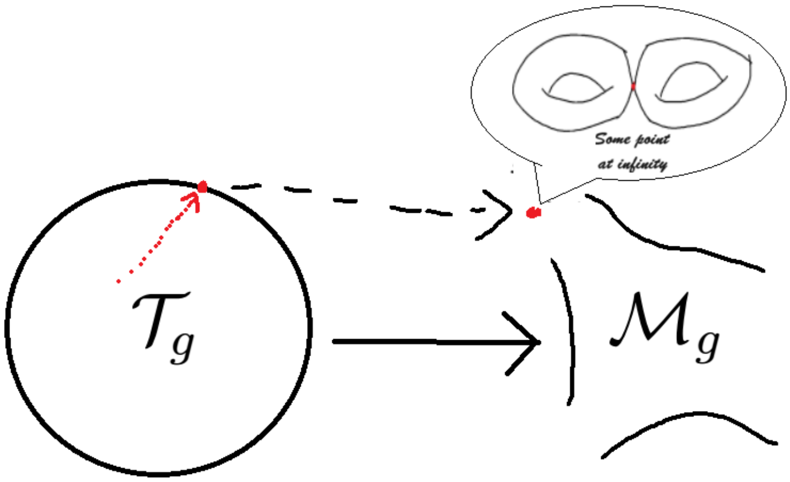
So now we can clearly see that each path produces a singular Riemann surface, and thus should produce some singular curves. Debattam has defined last week the Teichmüller space which is the

[9] Okay, it's in page 43 of **[1]** ...

universal cover of the moduli space

$$\begin{array}{c} \mathcal{T}_g \\ \big\downarrow \\ \mathcal{M}_g \end{array}$$

The Teichmüller space $\mathcal{T}_g$ is in fact homeomorphic to the unit ball of dimension $6g-6$ and therefore singular curves can be obtained from $S^{(6g-6)}$, the sphere in dimension $6g-6$, made of the natural compact points we want to add :



2 17^{th} April 2024

Things were so badly explained (and also, only three attended...) that I should understand how important it is to prepare talks beforehand ! Therefore there will not be a written rendering of what has been said. However, two things :

- Benjamin Fleuriault pointed out that $\mathbb{P}^1 \times \mathbb{A}^1$ is not the fine moduli space of "lines in the plane that goes through M, for some point M" but rather that the moduli space of "couples of a point M and a line in the plane going through M" is isomorphic to the trivial bundle $\mathbb{P}^1 \times \mathbb{A}^1$.
- A nice exercise is to see that the moduli space of "lines that go through a line" is the wedge product of copies of $\mathbb{P}^1$ along that line.

... and a anecdote. When I first heard about the classification of finite simple groups, I was very disappointed to the point of making that meme

It felt like sporadic groups should not be mixed with the alternating groups and so on. Also the fact that the first ones were not simple made that list too particular, as if this list should have not corresponded to a *nice family* of groups. At the same time I knew the definition of a finite simple group, which is very easy and very natural when you want to decompose finite groups. Again this was (intellectually) difficult for me to bear that paradox. I told that to Frédéric Cuvellier, my maths teacher in *sup*. He told me that on the contrary this was a nice result because now we know exactly what are the finite simple groups. I could not understand that, because in my point of view I already knew the definition of a finite simple group. So at the end of the day remember which meaning is hidden behind the question :

What is a — ?

Bibliography.

[1] : J.Harris & I.Morrison, *Moduli of Curves*, Springer (1998)

[2] : https://www.youtube.com/watch?v=ayyrtMdPyi4
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[3] : F.Benson & D.Margalit, *A Primer on Mapping Class Groups*, Princeton Mathematical Series (2011)

[4] : J.H.Silverman, *The Arithmetic of Elliptic Curves*, Springer New York (1986)

[5] : R. Miranda, *Algebraic curves and Riemann surfaces*, American Mathematical Society (1995)