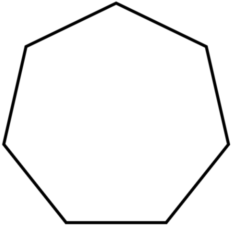
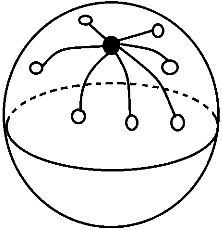
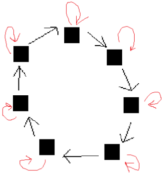
Introduction



$$\boxed{a^7 = 1}$$





geometry number theory surface topology combinatorics

What is a number ?

What is a number ?

Beware, quotes are truncated !
(and translated into english via DeepL, Google Translate and meself...)

Interesting books (you can also just read the Wikipedia page !)

Number Theory : An Approach Through History from Hammurapi to Legendre, André Weil (1984)

Histoire des nombres complexes : entre algèbre et géométrie, Dominique Flament (2003)

Éléments d'histoire des mathématiques, Nicolas Bourbaki (1960,1969,1974)

I. A little bit of history

« sometimes only imaginary, that is to say that we can always imagine as many as I said in each equation, but that there is sometimes no quantity which corresponds to that which we imagine », René Descartes (1637)

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⋆⋆⋆

« Itaque elegans et mirabile effugium reperit in illo analyseos miraculo, idealis mondi monstro, pene inter Ens et non Ens amphibium quo radicem imaginariam apellamus », Gottfried Wilhelm Leibniz (1702)

« Thus we find the elegant and admirable outcome in this miracle of analysis, monster of the world of ideas, almost amphibious between being and non-being, which we call imaginary root »

« masque géométrique appliqué sur des formes analytiques dont l'usage immédiat [est] plus simple et plus expéditif », Augustin-Louis Cauchy (1813)

« geometric mask applied to analytical terms whose immediate use [is] simpler and more expeditious »

II. Numbers as dessins d'enfants on a torus

Écoutons Grothendieck :

« The teaching duties for my university students (including so-called "advanced students") with a modest (and often less than modest) mathematical background, led me to drastically renew the themes of reflection, and gradually to myself. It seemed important to me to start from a common intuitive background, independent of any technical language that might express it, and even prior to such a language. The main emphasis is on the topological properties of surfaces, or on the combinatorial aspects that constitute their most down-to-earth technical expression, and not on the differential, even conformal, Riemannian, holomorphic and (hence) "complex algebraic curves" aspects. »

« Once this last step has been taken, however, algebraic geometry (my old lover!) suddenly bursts back into the scene, and through objects that can be considered the ultimate building blocks of all other algebraic varieties. Whereas in my pre-1970 research, my attention was systematically directed towards objects of maximal generality, in order to design an adequate set language for the world of algebraic geometry, and that I would dwell on algebraic curves only insofar as this proved indispensable (particularly in étale cohomology) in order to develop "all-purpose" techniques and statements valid in any dimension and in any place (I mean, on any basic scheme, or even any basic ringed topos...), so here I am, using objects so simple that a child can know them by playing, to the origins of algebraic geometry, familiar to Riemann and his disciples! »

« Such a supposition looked so crazy that I was almost embarassed to submit it to **compétences en la matière**. *Deligne found the supposition crazy indeed, but without a counter-example in his pocket. »*

« Such a supposition looked so crazy that I was almost embarassed to submit it to ***compétences en la matière****. Deligne found the supposition crazy indeed, but without a counter-example in his pocket. »*

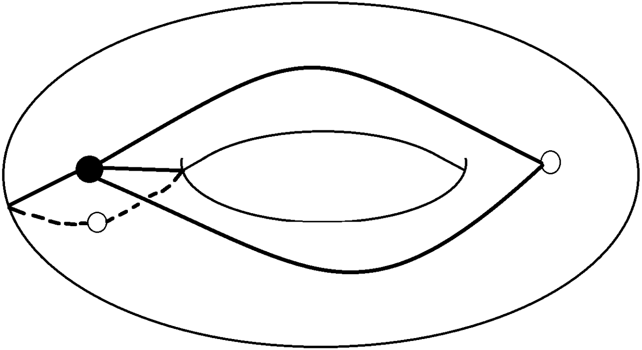
« We end up with this observation, which eight years later still seems as extraordinary as ever : ***every "finite" oriented map canonically realizes itself on a complex algebraic curve!*** *This discovery, which technically boils down to so little, made a very strong impression on me, and represented a decisive turning point in the course of my reflections, a displacement in particular of my center of interest in mathematics, which suddenly found itself strongly localized. I don't think a mathematical fact has ever struck me as much as this one, and had a comparable psychological impact. This is surely due to the very familiar, non-technical nature of the objects considered, of which any drawing of a child scribbling on a piece of paper provides a perfectly explicit example. »*

« *For me, its essential message was that* ***there is a profound identity between the combinatorics of finite maps, on the one hand, and the geometry of algebraic curves over number fields, on the other***. »

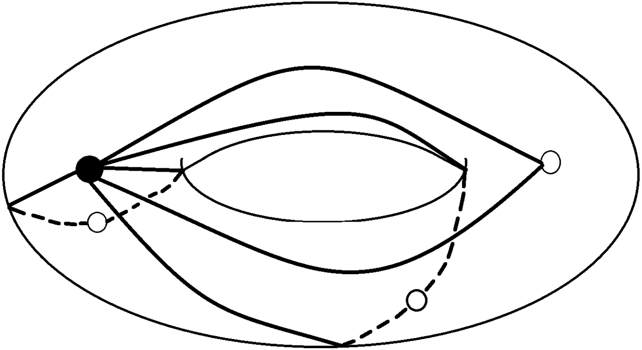
Consider a dessin on a torus. Then it defines a map $\beta : E \longrightarrow \mathbb{P}^1$, where E is an elliptic curve. Since we know what an elliptic curve looks like, people tend to look for equations defined over an algebraic closure $\overline{\mathbb{Q}}$ (they see as already constructed) i.e. $\lambda \in \overline{\mathbb{Q}} \backslash \{0, 1\}$ such that in $\mathbb{P}^2$ the curve is given by the Legendre equation

$$v^2 w = u(u - w)(u - \lambda w)$$

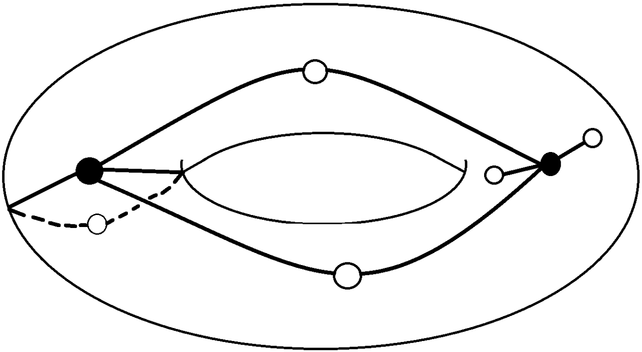
and β a homogenous fraction in $K(u, v, w)$, where K is a sufficiently large number field. Problem is that those equations are not unique. However some numbers are better because do not depend on the equations, for instance the j–invariant.



$j = 1728$

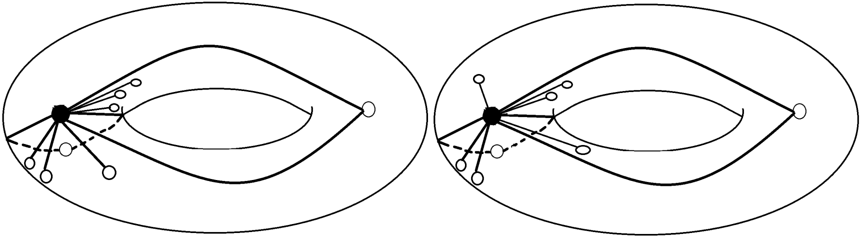


$j = 0$



$\boxed{j = 8000}$

horrible computation : $$\frac{4^4\left(1-\frac{1+\sqrt{2}}{2}+\left(\frac{1+\sqrt{2}}{2}\right)^2\right)^3}{\left(\frac{1+\sqrt{2}}{2}\right)^2\left(1-\frac{1+\sqrt{2}}{2}\right)^2} = 8000$$

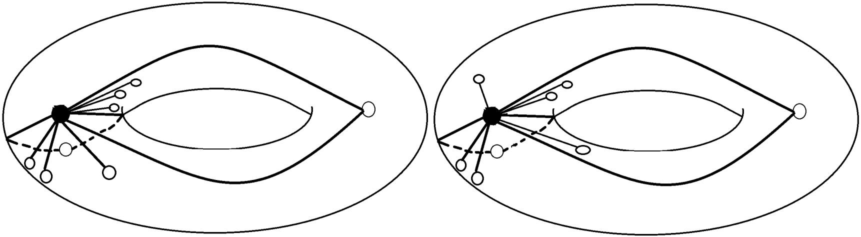


If I were old-school, I would write

$$j = -4^3 \frac{131547 \pm 24887\sqrt{5}}{9 \cdot 7}$$

and that the Galois group is given by the group

$$\mathrm{Gal}(\mathbb{Q}(\sqrt{5})|\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \simeq (\mathbb{Z}/5\mathbb{Z})^\times/\{\pm 1\}$$



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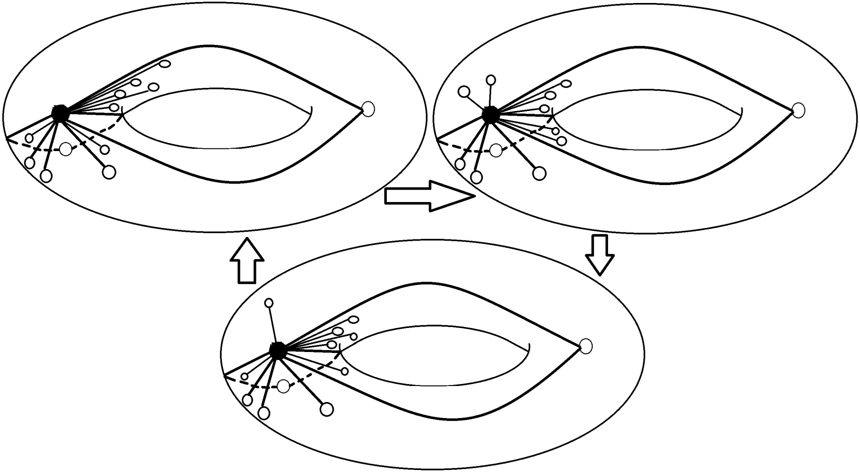
$$\boxed{(9 \cdot 7)^2 j^2 + 1060795008j + 70779645095659 = 0}$$

$$\boxed{\mathrm{Gal} \simeq \mathbb{Z}/2\mathbb{Z}}$$



$$j^3 + 2880j^2 + 1531904j + 216793088 = 0$$

$$\mathrm{Gal} \simeq \mathbb{Z}/3\mathbb{Z}$$

« ***Thus the Galois group*** $\mathrm{G}_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ ***is realized as a group of automorphisms of a most concrete profinite group****, respecting certain essential structures of this group. The most fascinating tasks here is precisely to learn the necessary and sufficient condition for an automorphism to come from a Galois element - which would provide a "purely algebraic" description of* $\mathrm{G}_{\mathbb{Q}}$*, in terms of profinite groups and without reference to the Galois theory of number fields.* » - A.Grothendieck, Esquisse d'un programme (Sketch of a programme)

III. Felix Klein's 1879 article

Felix Klein actually discovered dessins in 1879. He called them **Linienzuges**, roughly meaning sequences of lines (lignes brisées)

Felix Klein actually discovered dessins in 1879. He called them **Linienzuges**, roughly meaning sequences of lines (lignes brisées)

$\mathrm{PSL}_2(\mathbb{Z}) \curvearrowright \mathbb{H}$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau := \frac{a\tau + b}{c\tau + d}$$

inducing a map

$$\begin{array}{c} \mathbb{H} \\ \Big\downarrow \\ \mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H} \ \left(\underset{\simeq}{\overset{j}{\to}} \mathbb{C} \right) \end{array}$$

Now if the class of i and the class of $\exp(2i\pi/3)$ are removed and j replaced by $j/1728$ then it is a unramified cover with base equal to $\mathbb{P}^1(\mathbb{C}) \backslash \{0, 1, \infty\}$.

Will man nun wissen, wie viele verschiedene elfblättrige Flächen es giebt, welche die von uns verlangte Lage und Multiplicität der Verzweigungspunkte besitzen, so ist die Frage augenscheinlich die (Annalen XIV, pag. 424): *Auf wie viele Weisen ist es möglich, die 22 Halbkanten der in Figur a vorhandenen inneren Begränzung derart zu einem aus 11 Stücken bestehenden, doppelt überdeckten Linienzuge zusammenzubiegen, dass von den 11 Punkten* $J = 0$ *dreimal drei und von den 11 Punkten* $J = 1$ *viermal zwei zusammenkommen?* — Die Figur b (die wohl ohne besondere Erläuterung verständlich ist) soll an einem Beispiele erläutern, wie dieses Zusammenbiegen gemeint ist.

four white vertices of order 2

three black vertices of order 3

He doesn't mention the vertices of order 1, so in total :

$3 + 2$ black vertices and $4 + 3$ white vertices.

von der soeben die Rede war, ist, wie ich l. c. zeigte, so in Bezug auf J verzweigt, dass bei $J = \infty$ sämmtliche elf Blätter im Cyklus zusammenhängen, bei $J = 0$ dreimal drei, bei $J = 1$ viermal zwei. Nun behauptete ich ebendort, *dass es nicht weniger als zehn wesentlich verschiedene Riemann'sche Flächen giebt, welche dieselbe Eigenschaft besitzen* (von denen aber nur zwei bei der Transformationstheorie in Betracht kommen). Es ist heute meine nächste Aufgabe, diese Behauptung auf dem damals bereits angedeuteten, rein geometrischen Wege zu beweisen.

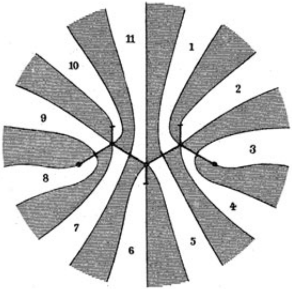


Fig. 3.

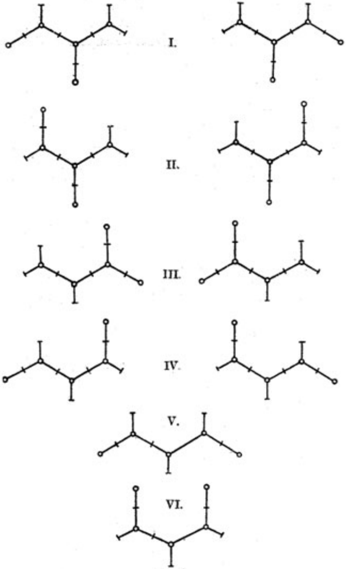


Fig. 2.

in Figur b dargestellten Fall; die Schemata I sind, meinen früheren Erläuterungen zufolge, die einzigen, welche auf die aus der Transformationstheorie hervorgehenden Gleichungen elften Grades passen. —

Es giebt, diesen Schematen nach, in der That *zehn* Möglichkeiten der Zusammenbiegung. Dass es auch nicht mehr giebt, ist ebenso evident; denn offenbar gelingt es nicht, noch andere elfgliedrige Linienzüge der von uns gewünschten Art herzustellen. — Somit ist der zu Eingang des Paragraphen ausgesprochene Satz bewiesen.

Im Verfolg meiner Untersuchungen über die Transformation der elliptischen Functionen behandele ich im Nachstehenden die Transformation *elfter* Ordnung. Es ist dabei mein besonderes Ziel gewesen, die Gleichung *elften* Grades, welche in diesem Falle auftritt, in einfachster Form explicite herzustellen. Im XIV[ten] Bande dieser Annalen, p. 423—424, habe ich bereits gezeigt, dass man dieser Gleichung folgende Gestalt geben kann:

$$J = F(z),$$

eine Curve, auszuscheiden, die bei den 660 Collineationen in sich übergeht, und das specielle auf sie bezügliche Problem der y durchzuführen.

Von dieser Curve wissen wir, *dass sie das Bild der Galois'schen Resolvente der Transformationsgleichung sein muss.* Nun ist, wie ich früher ausführte (Ann. XIV, p. 151) die Galois'sche Resolvente vorgestellt durch eine Riemann'sche Fläche, die 660-blättrig über der Ebene J ausgebreitet ist und deren Blätter bei $J = 0$ zu je 3, bei $J = 1$ zu je 2, bei $J = \infty$ zu je 11, sonst aber nirgends zusammenhängen, deren Geschlecht also $= 26$ ist. Auf unserer Curve muss es dementsprechend eine rationale Function J geben, welche jeden

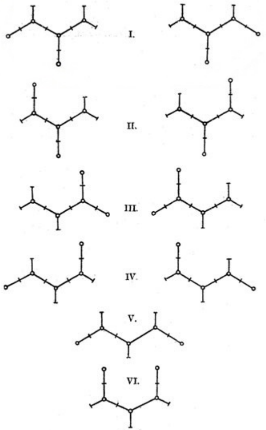
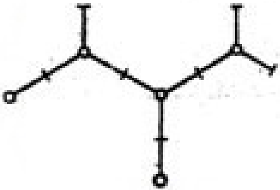
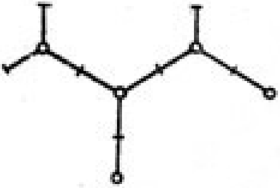


Fig. 2.



I.



bination der folgenden zwei Operationen $\left(\varrho = e^{\frac{2i\pi}{11}}\right)$:

$$
(2)\left\{
\begin{array}{ll}
S) & y_1' = \varrho y_1, \quad y_4' = \varrho^4 y_4, \quad y_5' = \varrho^5 y_5, \quad y_9' = \varrho^9 y_9, \quad y_3' = \varrho^3 y_3, \\
T) & \left\{
\begin{aligned}
\sqrt{-11}\cdot y_1' &= (\varrho^9 - \varrho^2) y_1 + (\varrho^4 - \varrho^7) y_4 + (\varrho^3 - \varrho^8) y_5 \\
&\qquad + (\varrho^5 - \varrho^6) y_9 + (\varrho^1 - \varrho^{10}) y_3, \\
\sqrt{-11}\cdot y_4' &= (\varrho^4 - \varrho^7) y_1 + (\varrho^3 - \varrho^8) y_4 + (\varrho^5 - \varrho^6) y_5 \\
&\qquad + (\varrho^1 - \varrho^{10}) y_9 + (\varrho^9 - \varrho^2) y_3, \\
\sqrt{-11}\cdot y_5' &= (\varrho^3 - \varrho^8) y_1 + (\varrho^5 - \varrho^6) y_4 + (\varrho^1 - \varrho^{10}) y_5 \\
&\qquad + (\varrho^9 - \varrho^2) y_9 + (\varrho^4 - \varrho^7) y_3, \\
\sqrt{-11}\cdot y_9' &= (\varrho^5 - \varrho^6) y_1 + (\varrho^1 - \varrho^{10}) y_4 + (\varrho^9 - \varrho^2) y_5 \\
&\qquad + (\varrho^4 - \varrho^7) y_9 + (\varrho^3 - \varrho^8) y_3, \\
\sqrt{-11}\cdot y_3' &= (\varrho^1 - \varrho^{10}) y_1 + (\varrho^9 - \varrho^2) y_4 + (\varrho^4 - \varrho^7) y_5 \\
&\qquad + (\varrho^3 - \varrho^8) y_9 + (\varrho^5 - \varrho^6) y_3,
\end{aligned}
\right.
\end{array}
\right.
$$

denen sich als einfachste Combination die cyklische Vertauschung zugesellt:

2) *Die Function dritten Grades:*

$$
\begin{aligned}
(10)\qquad f_0 = {} & (y_1^3 + y_4^3 + y_5^3 + y_9^3 + y_3^3) \\
& + 3(y_1^2 y_3 + y_4^2 y_1 + y_5^2 y_4 + y_9^2 y_5 + y_3^2 y_9) \\
& - 3(y_1 y_4 y_9 + y_4 y_5 y_3 + y_5 y_9 y_1 + y_9 y_3 y_4 + y_3 y_1 y_5) \\
& + \frac{1+\sqrt{-11}}{2} (y_1^2 y_5 + y_4^2 y_9 + y_5^2 y_3 + y_9^2 y_1 + y_3^2 y_4) \\
& - \frac{1+\sqrt{-11}}{2} (y_1 y_4 y_5 + y_4 y_5 y_9 + y_5 y_9 y_3 + y_9 y_3 y_1 + y_3 y_1 y_4) \\
& - (1+\sqrt{-11})(y_1^2 y_4 + y_4^2 y_5 + y_5^2 y_9 + y_9^2 y_3 + y_3^2 y_1).
\end{aligned}
$$

Die Function φ_0 stimmt mit $\frac{-1+\sqrt{-11}}{12}\,\Sigma p^2$ überein; die Function f_0 ist von $\frac{-\sqrt{-11}}{6}\,\Sigma p^3$ nur um ein Glied verschieden, das ein numerisches Multiplum von ∇ (3) ist. — Die elf Werthe, welche φ_0 oder f_0 bei den 660 Collineationen annimmt, und die ich φ_ν, bez. f_ν nennen will, erwachsen aus φ_0 und f_0, wenn man der Collineation S^ν entsprechend statt y_{κ^2} einträgt $\rho^{\kappa^2\nu}\cdot y_{\kappa^2}$. — Aendert man in diesen Formeln

§ 10.

Zusammenstellung der bisherigen Resultate.

Fassen wir zusammen, so sind wir für *die Transformation elfter Ordnung der elliptischen Functionen* nunmehr zu folgenden Resultaten gekommen:

1) *Die Galois'sche Resolvente* 660^{ten} *Grades lässt sich folgendermassen anschreiben:* Man unterwerfe die fünf Verhältnissgrössen

$$y_1 : y_4 : y_5 : y_9 : y_3$$

den 15 Relationen $H_{ik} = 0$ (vergl. (12) resp. (13)) und setze:*)

$$\frac{-C^3}{1728\,\nabla^{11}} = J,$$

wo ∇ die Function dritten Grades (2), C die Function elften Grades (4) bezeichnet. — Hat man *ein* Lösungssystem dieser Gleichungen gefunden, so ergeben sich alle anderen durch die Collineationen des § 3.

2) *Es giebt zwei einfachste Formen der Resolvente elften Grades.* Die eine, von uns zu Anfang allein betrachtete, lautet (20):

$$J:J-1:1=(z^2-3z+(5-\sqrt{-11}))\cdot$$
$$\cdot\left(z^3+z^2-3\cdot\frac{1+\sqrt{-11}}{2}\cdot z+\frac{7-\sqrt{-11}}{2}\right)^3$$
$$:\left(z^3+4z^2+\frac{7-5\sqrt{-11}}{2}\cdot z+(4-6\sqrt{-11})\right)\cdot$$
$$\cdot\left(z^4-2z^3+3\cdot\frac{1-\sqrt{-11}}{2}\cdot z^2+(5+\sqrt{-11})z-3\cdot\frac{5+\sqrt{-11}}{2}\right)^2$$
$$:-1728;$$

ihre 11 Wurzeln sind durch die Formel gegeben:

$$z_\nu=\frac{f_\nu}{\nabla},$$

wo f_ν durch Gleichung (10) definirt ist.

Die zweite Form wird durch (25) vorgestellt:

$$0=\xi^{11}-22\cdot\xi^8+11(9-2\sqrt{-11})\cdot\xi^5-11\cdot\frac{12\,g_2}{\sqrt[3]{\Delta}}\cdot\xi^4+88\cdot\sqrt{-11}\cdot\xi^2$$

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Die zweite Form wird durch (25) vorgestellt:

$$0 = \xi^{11} - 22 \cdot \xi^{8} + 11(9 - 2\sqrt{-11}) \cdot \xi^5 - 11 \cdot \frac{12\, g_2}{\sqrt[3]{\Delta}} \cdot \xi^4 + 88 \cdot \sqrt{-11} \cdot \xi^2$$
$$- 11 \cdot \frac{-3 + \sqrt{-11}}{2} \cdot \frac{12\, g_2}{\sqrt[3]{\Delta}} \cdot \xi - \frac{144\, g_2^2}{\sqrt[3]{\Delta^2}};$$

und ihre Wurzeln sind:

$$\xi_\nu = \frac{\varphi_\nu}{\nabla^{\frac{2}{3}}},$$

unter φ_ν die Functionen (9) verstanden.

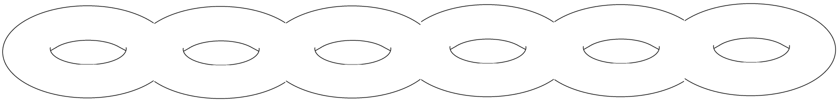
```
gap> x_1:=(1,2);
(1,2)
gap> y_1:=();
()
gap> x_2:=(1,2,3);
(1,2,3)
gap> y_2:=();
()
gap> genreduproduitfibre(x_1,y_1,x_2,y_2);
 0
```

```
gap> x_1:=(1,2,3)(4,5,6)(7,8,9);
(1,2,3)(4,5,6)(7,8,9)
gap> y_1:= (2,6)(4,10)(5,7)(8,11);
(2,6)(4,10)(5,7)(8,11)
gap> x_2:=(1,2,3)(4,5,6)(7,8,9)(10,11,12);
(1,2,3)(4,5,6)(7,8,9)(10,11,12)
gap> y_2:=(2,10)(3,4)(5,7)(6,8)(9,1)(12,11);
(1,9)(2,10)(3,4)(5,7)(6,8)(11,12)
gap> genreduproduitfibre(x_1,y_1,x_2,y_2);
6
```

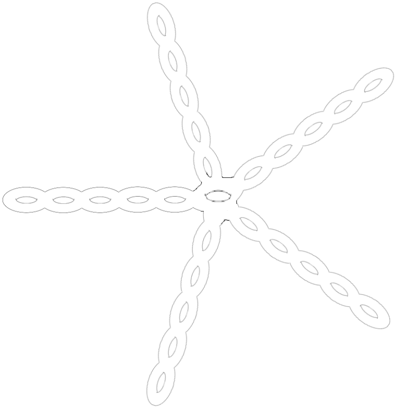
```
x = (1,13,25)(2,14,23)(3,12,24)(4,16,28)
(5,17,26)(6,15,27)(7,19,31)(8,20,29)
(9,18,30)(10,21,32)(11,22,33)(34,46,58)
(35,47,56)(36,45,57)(37,49,61)(38,50,59)
(39,48,60)(40,52,64)(41,53,62)(42,51,63)
(43,54,65)(44,55,66)(67,79,91)(68,80,89)
(69,78,90)(70,82,94)(71,83,92)(72,81,93)
(73,85,97)(74,86,95)(75,84,96)(76,87,98)
(77,88,99)(100,112,124)(101,113,122)(102,111,123)
(103,115,127)(104,116,125)(105,114,126)(106,118,130)
(107,119,128)(108,117,129)(109,120,131)(110,121,132)

y = (1,89)(2,94)(3,91)(4,98)(5,95)(6,90)(7,93)(8,99)(9,97)(10,92)(11,96)(12,100)(13,105)
(14,102)(15,109)(16,106)(17,101)(18,104)(19,110)(20,108)(21,103)(22,107)(23,34)(24,39)(25,36)
(26,43)(27,40)(28,35)(29,38)(30,44)(31,42)(32,37)(33,41)(45,67)(46,72)(47,69)(48,76)(49,73)
(50,68)(51,71)(52,77)(53,75)(54,70)(55,74)(56,78)(57,83)(58,80)(59,87)(60,84)(61,79)(62,82)
(63,88)(64,86)(65,81)(66,85)(111,122)(112,127)(113,124)(114,131)(115,128)(116,123)(117,126)
(118,132)(119,130)(120,125)(121,129)
```

With enough time a human being can draw the dessin on this surface :



Finally the genus 26 surface is a 5–sheet cover of the genus 6 one. Therefore it should canonically look like this :



$$g = (6 - 1) \cdot 5 + 1 = 26$$

III.bis. The Grothendieck-Teichmüller group $\mathcal{GT}$

$\mathcal{GT}$ is defined as a subset of $\mathrm{Aut}(\widehat{F_2})$ (group automorphisms of $\widehat{F_2}$ which are continuous). Let $\gamma \in \mathrm{Aut}(\widehat{F_2})$, and z is such that $xyz = 1$.

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Definition. γ is an element of $\mathcal{GT}$ if and only if there exists $\lambda \in (\widehat{\mathbb{Z}})^{\times}$ and $f \in \widehat{F_2}$ such that :

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REMARKS. (I& II) $\iff$ γ commute in $\mathrm{Out}(\widehat{F_2})$ with the subgroup $\mathfrak{S}_3$ which "permutes" $\{0, 1, \infty\}$. $\mathcal{GT}$ is a subgroup. λ is unique, and f is unique if chosen in the derived subgroup $\widehat{F_2}'$.

« *Whatever form is algebraically equivalent to an another when expressed in general symbols, must continue to be equivalent, whatever those symbols denote* », Reverend George Peacock (1791-1858)

Usually we start with :

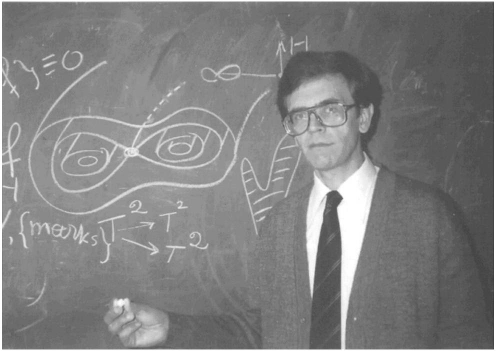
"Let $\overline{\mathbb{Q}}$ be **an** algebraic closure of $\mathbb{Q}$."

Or worse :

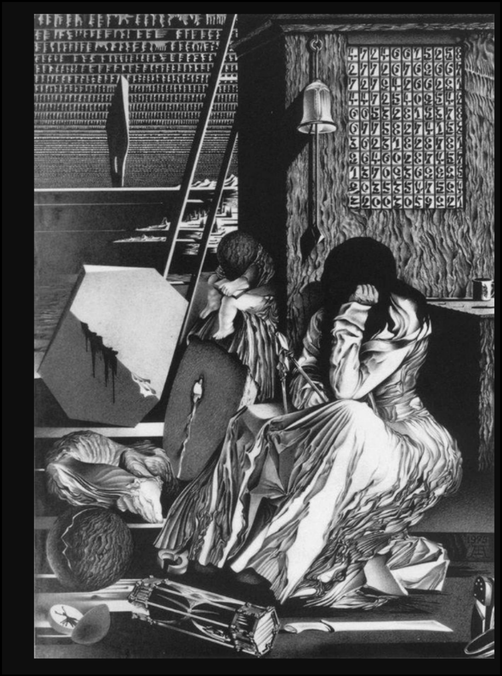
"Let $\overline{\mathbb{Q}}$ be **the** algebraic closure of $\mathbb{Q}$."

This, for me, is a **nightmare**.

Anatoli Timofeïevitch Fomenko (1945-????) :



Beware of his plot theory about history...

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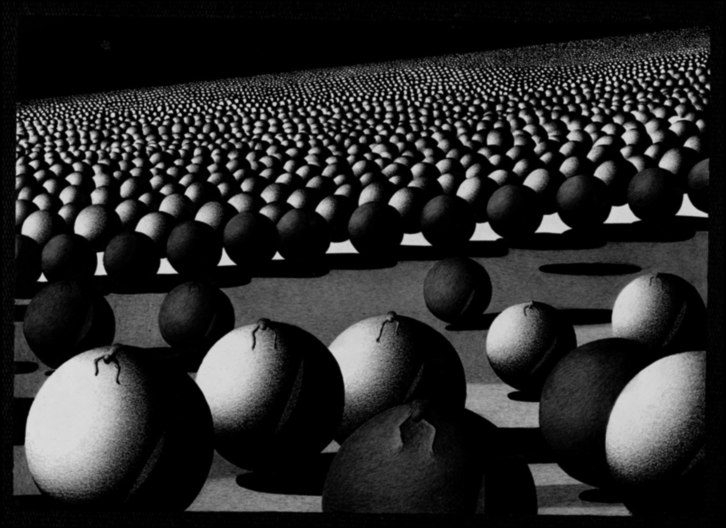

Let's construct an algebraic closure of $\mathbb{Q}$, appropriate to the situation. I call it Ω to make the difference. The goal is to construct a canonical map

$$\mathcal{GT} \longrightarrow \mathrm{Gal}(\Omega|\mathbb{Q})$$

Only after that, we could set $\overline{\mathbb{Q}} := \Omega$ and the étale exact sequence will give the map

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The idea is more general but I will restrict myself to say

$$\Omega := \text{generated by all the } j-\text{invariants}$$

The nightmare ends and now we're happy!