# Groupe de travail : Espaces de modules

(IMJ-PRG, 24-25)

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La section *Joyeusetés* contient des précisions sur les arguments principaux avancés le long du texte principal.

The section *Joyeusetés* provides precisions to some of the arguments from the main text.

# 1 Introduction to moduli spaces. (Salim Alloun, 18th of November 24)

Before going right through the mathematics that this *groupe de travail* shall handle, namely algebraic spaces and stacks, we must first ask ourselves why there are such things as moduli spaces and why they are studied.

## 1.1 A philosophical point of view.

Mathematicians study abstract objects, namely those objects which are given by a precise definition enabling us to prove theorems analytically (which means manipulating only logical propositions without appealing to empirical data). Philosophers would say that mathematicians focus on the *essence*. We are rarely satisfied by a definition though. For instance for finite groups, the definition of a simple group has been understood as the best way to define what would be the fundamental bricks of them, similar to what prime numbers are for integers. However one wondered whether it could be possible to actually determine the exact list of finite simple groups, namely to answer in a different way to the question "What are the fundamental bricks of finite groups ?" What is important here, is that the list will be the one up to isomorphism. This shift from *essence* to *existence*, that is to say from the analytical world to a more down-to-earth explicit one, is well summarized by Hilary Putnam in *Realism with a human face* (p.$235-236$) :

*« one of the most important aspects of Darwin's new way of thinking about the world, the way based on the idea of natural selection, is what Mayr calls Darwin's "anti-essentialism." It is an interesting fact that the people who produced the theory of natural selection, Darwin and his co-dicoverer Wallace, and the people who early became converts to it were naturalists, not experimentalists ; they were people who had been to odd places and seen a lot of flora and fauna [...] what they were interested in was variation. The traditional view in biology [...] is that there is such a thing as the essence [of an animal] and this essence is what is of scientific importance and interest. [...] Now, this Dariwinian attitude [...] opens the way for the idea that species slide into one another. »*

In the case of finite groups, the list has been given (apparently). In this list some of them are given by taking points over a finite field in a common algebraic group, for instance $\mathrm{PSL}_n(\mathbb{F}_q)$ ( $q \geqslant 5, n \geqslant 2$). It teaches the following lesson : there can be a certain "space" whose "points" can be identified by the list of possible such objects. If this "space" has been constructed we can study it for itself. For the prime numbers it is always an open problem (Spec $\mathbb{Z}$ as a scheme being only a bit of the answer) but even taking the step to study the set of prime numbers inside the integers can be illuminating for now we can study asymptotic behaviors. Another striking point is that morphisms between different such "spaces" will amounts to the possible interaction between different class of objects ; and also we can loosen the conditions defining our objects, giving rise to singular objects, reaching the "boundary" of our "space". This space is what we call a **moduli space**.

**Remark 1.** *(Historical) The term **moduli** is the plural for **modulus** which roughly means parameter. So the moduli space is the space where live the parameters needed to describe entirely the object at study. The first one to use it was B.Riemann*[1]*. He meticulously describes the theory of what we call now Riemann surfaces and for such a surface of genus p he writes :*

1. *Theorie der Abel'schen Functionen*, *Journal fur die reine und angewändte Mathematik*, (1857)

*« und es hängt also eine Klasse von Systemen gleichvezweigter $2+p$ fach zusammenhängender Functionen und die zu ihr gehörende Klasse algebraischer Gleichungen von $3p-3$ stetig veränderlichen Grössen ab, welche die **Moduln** dieser Klasse gennant werden sollen. » (p.33)*

*« Diese Voraussetzung trifft nur zu, wenn $p>1$, und die Anzahl der **Moduln** ist nur dann $=3p-3$, für $p=1$ aber $=1$. » (p.34)*

*Further in the text he defines **Periodicitätsmoduln** which are the integrals along the $2p$ paths forming a basis for the first homology group.*

## 1.2 Fine and coarse moduli spaces.

The "Darwin way of thinking" consists in considering all the diversity of objects of a certain kind. In algebraic geometry it is translated in the language of families.

**Definition 2.** *A **family** of objects of certain kind is given by a morphism of schemes $F \to B$ such that for each close point $b \in B$, the fiber at $b$ (made of points lying over $b$ and noted $F_b$) is an object of this certain kind. $B$ is said **to parametrize** the family $(F_b)_b$.*

Now among all the possible family (all the diversity of objects) we would like to know if there is a universal family such that all the other ones are induced from it, up to our chosen equivalence (most of the times the equivalence is isomorphism equivalence). In a sense, the parameter space of this universal family would be the space containing "all" the information about the objects studied up to the chosen equivalence. We say that we have a **moduli problem** whose data is encoded by a contravariant functor

$$M : \mathsf{Sch} \longrightarrow \mathsf{Sets}$$

where $M(B)$ is the set of families parametrized by $B$ up to equivalence. The functor is contravariant because given a map $B' \to B$ each family parametrized by $B$ gives rise by fiber-product to a family parametrized by $B'$.

**Example 1.** *The moduli problem for smooth projective curves over $\mathbb{C}$ of genus $g$, for line bundles over a given curve $X$, etc.*

We can give our first formalization with the notion of fine moduli space :

**Definition 3.** *A **fine moduli space** for a moduli problem given by the functor $M$ is a scheme $X$ representing $M$, i.e. such that there is a natural equivalence*

$$\mathrm{Hom}_{\mathsf{Sch}}(-, X) \Longrightarrow M$$

*In particular for all $B$, $M(B)$ is in bijection with $\mathrm{Hom}_{\mathsf{Sch}}(B, X)$.*

**Remark 4.** *Now if $k$ is a field then the $k$–points of $X$ are given by $\mathrm{Hom}_{\mathsf{Sch}}(\mathrm{Spec}\ k, X)$ which is in bijection with $M(\mathrm{Spec}\ k)$, i.e. families parametrized by one $k$–point which is to say one object defined over $k$ up to equivalence over $k$.*

### 1.2.1 There is no fine moduli space for elliptic curves over $\mathbb{C}$.

Recall that an elliptic curve over $\mathbb{C}$ is a smooth projective curve of genus 1 with a marked closed point. We can always chose a $\lambda \neq 0, 1$ such that the curve is inside $\mathbb{P}^2$ given by

$$Y^2Z = X(X - Z)(X - \lambda Z)$$

with the marked point being $(0 : 1 : 0)$. However $\lambda$ is not unique, we can make $\mathfrak{S}_3$ acts on $\mathbb{A}^1\backslash\{0, 1\}$ via the group

$$\langle\ \lambda \mapsto 1 - \lambda\ ,\ \lambda \mapsto \lambda^{-1}\ \rangle$$

and a straight-forward computation gives that $\mathfrak{S}_3\backslash\mathbb{A}^1_\lambda\backslash\{0, 1\}$ is isomorphic to $\mathbb{A}^1_j$ () where

$$j(\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$$

Up to the $\mathfrak{S}_3$–action we recover the isomorphism class of the elliptic curve. So $j$ gives a bijection between $\mathbb{C}$ and $M(\text{Spec } \mathbb{C})$ where $M$ is associated to the moduli problem of smooth projective curves of genus 1 with one marked point. But is $\mathbb{A}^1_j$ a fine moduli space for $M$? The answer is no for the following reason. There are particular objects with more automorphisms than the generic ones. First consider $E \to A^1_t\backslash\{0\}$ the family given by the equation

$$E_t : Y^2Z = X^3 - tZ^3$$

then for all closed point you have that $j(E_t) = 0$. Elliptic curves have $\mathbb{Z}/2\mathbb{Z}$ as automorphism group generated by $Y \mapsto -Y$ **except** two of them, namely $\mathbb{Z}/6\mathbb{Z}$ for $j = 0$ by adding the automorphism $X \mapsto \exp(2i\pi/3)X$ and $j = 1728$ given by the equation

$$Y^2Z = X(X^2 - Z^2)$$

by adding the automorphism $X \mapsto -X$. So what happens for example with the family $(E_t)_{t\neq 0}$ is that the parameter $t$ enables us to produce a non-trivial monodromy (remember that the topological fundamental group of $\mathbb{C}\backslash\{0\}$ is $\mathbb{Z}$) which will give an identification of $E_1$ with itself by something different from the identity. So now if the $j$–line were a fine moduli space then the representability would induce that the map $E \to A^1_t\backslash\{0\}$ is isomorphic to the trivial family $E_1 \times A^1_t\backslash\{0\} \to A^1_t\backslash\{0\}$, but for the trivial family the monodromy is trivial, i.e. the identification $E_1 \to E_1$ is the identity.

However, the $j$–line is what we call a coarse moduli space over $\mathbb{C}$.

**Definition 5.** *A **coarse moduli space** for a moduli problem over $\mathbb{C}$ given by the functor $M$ is a scheme $X$ over $\mathbb{C}$ maximal in terms of representability of $M$, i.e. there is a natural transformation*

$$M \Longrightarrow \text{Hom}_{\mathsf{Sch}_{/\mathbb{C}}}(-, X)$$

*such that (1) for all scheme $X'$ together with a natural transformation $\text{Hom}_{\mathsf{Sch}_{/\mathbb{C}}}(-, X') \Longrightarrow M$ there exists a map $X' \to X$ such that the following diagram commute*

$$\begin{array}{ccc} \text{Hom}_{\mathsf{Sch}_{/\mathbb{C}}}(-, X') & \Longrightarrow & \text{Hom}_{\mathsf{Sch}_{/\mathbb{C}}}(-, X) \\ & \searrow\!\!\searrow & \Downarrow \\ & & M \end{array}$$

*and (2) the induced map $X(\mathbb{C}) \to M(\text{Spec } \mathbb{C})$ is a bijection.*

In the definition above we can still put $\mathsf{Sch}$ and replace the bijection with the $\mathbb{C}$–points by the condition on any algebraically closed field.

A refinement of the moduli problem is given by looking at a contravariant functor

$$\mathscr{M} : \mathsf{Sch} \longrightarrow \mathsf{Grpd}$$

such that $\mathscr{M}(B)$ is the groupoid whose objects are families parametrized by $B$ and whose maps are equivalences between families. It is clear that $\mathscr{M}$ lies over $M$ for we have defined $M$ by forgetting the equivalences and only focusing on the set of equivalence classes of families. This will the whole point of looking at stacks.

## 1.3 The case of compact Riemann surfaces.

The category of compact Riemann surfaces is equivalent to the category of smooth projective curves over $\mathbb{C}$. These examples are easy to handle compared with the case where $\mathbb{C}$ is replaced by any field, but still we will see all the possible phenomena that can happen on moduli spaces. For the following study, moduli spaces have to be understood first as a set defined to be the class of isomorphic structures on a given object. Indeed, since the Riemann surfaces considered here are topologically classified by their genus, or by the additional data of the finite number of closed disks removed, then we can fix a topological surface, for instance $S_g$ the compact orientable surface with $g$ holes, and look at all the possible structures. We rigorously end up with a set on which we could put a topology, a metric, an orbifold structure, etc. Of course we would like this space to be "similar" to the scheme representing the functor for the given moduli problem, and in the following we identify moduli problem and moduli space.

### 1.3.1 Moduli space for marked tori

A torus is a compact surface of genus 1, and each of them together with a marked point and a Riemann surface structure corresponds to an elliptic curve over $\mathbb{C}$, let's call $M_{1,1}$ the associated moduli problem. Their universal cover is the complex plane and we can get all of them by looking at $\mathbb{C}/\Lambda$ where $\Lambda$ is a lattice in $\mathbb{C}$, 0 being the marked Each $\Lambda$ can be written as $\mathbb{Z} + \tau\mathbb{Z}$ where $\tau$ lies in the upper half-plane $\mathbb{H}$. Given $\tau, \tau'$, there exists an isomorphism between the Riemann surfaces if and only if their associated lattices are equal which is the same as saying $\tau$ and $\tau'$ are in the same orbit for the action of $\mathrm{PSL}_2(\mathbb{Z})$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}$$

We would want to say that the moduli space for $\mathscr{M}_{1,1}$ is $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$. Well it is the case but in which sense ? As a set it is the case by definition of the quotient space. The action of $\mathrm{PSL}_2(\mathbb{Z})$ is properly discontinuous but however free only if two points are removed ! They are exactly the points associated to $j = 0$, $j = 1728$, namely $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ is isomorphic to the complexe plane as topological space and if you pullback the complexe structure with the condition that the class of $i = \sqrt{-1}$ is sent to 1728 and $\exp(2i\pi/3)$ to 0, then we recover the $j$–invariant. In that context we say that we have a map between orbifolds $\mathbb{H} \longrightarrow \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ where 0 and 1728 are the two orbifold points.

Another observation is that $\mathscr{M}_{1,1}$ is not compact, we must add $j = \infty$. What does that mean for this particular moduli space ? It means that one must add the missing values of $\lambda$, i.e. $\lambda \in \{0, 1\}$, which gives two isomorphic singular nodal cubics

$$Y^2Z = X^2(X - Z) \qquad Y^2Z = X(X - Z)^2$$

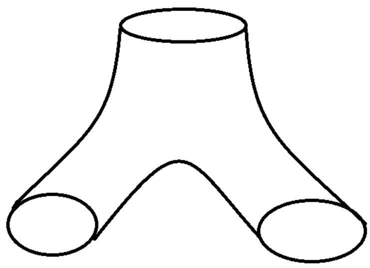
The remaining $\lambda = \infty$ gives the cuspidal cubic

$$Y^2Z = X^3$$

and does not appear on the compactification of $\mathscr{M}_{1,1}$. We will later see (if one day) that for $g \geqslant 2$, on the boundary of $\mathscr{M}_g$ lie the reducible and nodal curves.

### 1.3.2 Moduli space of compact Riemann Surfaces of genus $g \geqslant 2$

As mentioned in the philosophical *entrée*, if we can observe interactions between the objects from different moduli problems then their associated moduli spaces should also "interact". One case is when objects for a moduli problem can be broken down into fundamental objects for an easy moduli problem. In the case of Riemann surfaces of genus $g \geqslant 2$, we can make everything boil down to the following surface $P$, called a **pair of pants** :

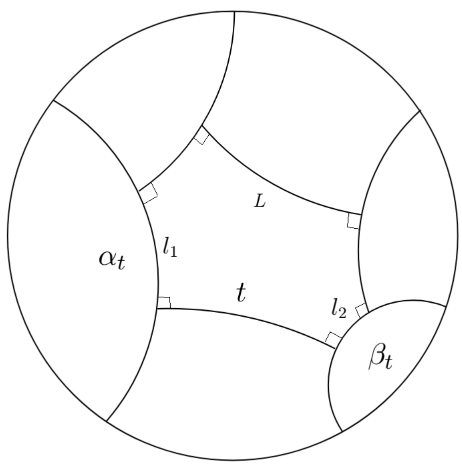


The surface $P$ is without boundary. Called $\gamma_1, \gamma_2, \gamma_3$ the three homotopy classes of loops going around the three circles on the picture (the three components of the compactification of $P$ minus $P$). When there is a hyperbolic structure on a surface then there is only one closed simple geodesic in each homotopy class of loops. The surface $P$ has Euler characteristic $-1$ therefore a Riemann surface structure on it is necessarily hyperbolic. The length $l(\gamma_i)$ is then defined to be the length of the associated geodesic.

**Proposition 1.** *Given three positive numbers $l_1, l_2, l_3 > 0$, there exists one and only one Riemann surface structure (up to biholomorphism) on $P$ such that*

$$l(\gamma_i) = 2l_i$$

*Proof.* (The proof and the picture, yet slightly edited, are taken from [2].) Assume that the $l_i$ are given. In the Poincaré disk with its hyperbolic metric construct a right-angled hexagon in the following manner.



Take two continuous families of lines $\alpha_t, \beta_t$ such that there mutual distance is of length $t > 0$. The segment realizing this distance is perpendicular to both of them. Move along $\alpha_t$ (resp. $\beta_t$) by the length $l_1$ (resp. $l_2$) such as in the picture. Now draw the two perpendicular lines to the segments of length $l_1$ and $l_2$. For a certain $t_0 > 0$ those two lines intersect on the boundary of the disk. For $t > t_0$, we can form the segment realizing the distance, call its lentgh $L$ as in the picture. Finally $L$ varies continuously with respect to $t$, and

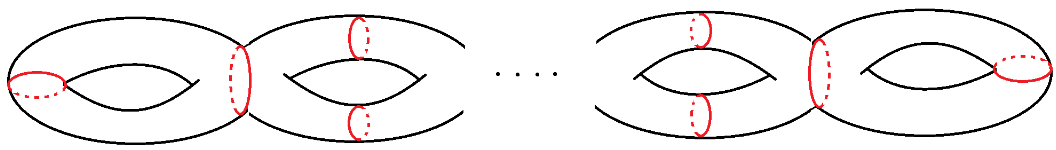
$$L \underset{t\to+\infty}{\longrightarrow} +\infty \qquad L \underset{t\to t_0}{\longrightarrow} 0$$

Therefore there exists a $t > 0$ such that $L = l_3$, the three segments of unkown lengths are called the seams. By gluing two copies of this hexagon along the seams, we end up with a pair of pants plus the boundary. We therefore have to extend locally the surface such to obtain a Riemann surface structure on $P$, and by definition we obtain three closed simple geodesics of lengths $2l_1, 2l_2, 2l_3$ corresponding to the homotopy classes $\gamma_1, \gamma_2, \gamma_3$. □

This has to be understood as a statement on a moduli space.

**Corollary 2.** *The moduli space of Riemann surface structure on $P$ is in bijection with $\mathbb{R}^3_{>0}$.*

Let $g \geqslant 2$. The decomposition of $S_g$ into pair of pants is summarized in the following picture.



We have chosen $3(g-1)$ homotopy classes $\gamma_1, \dots, \gamma_{3(g-1)}$ so that we get $S_g$ to be the union of $2 + 2(g-2) = 2(g-1)$ pairs of pants up to isotopy.

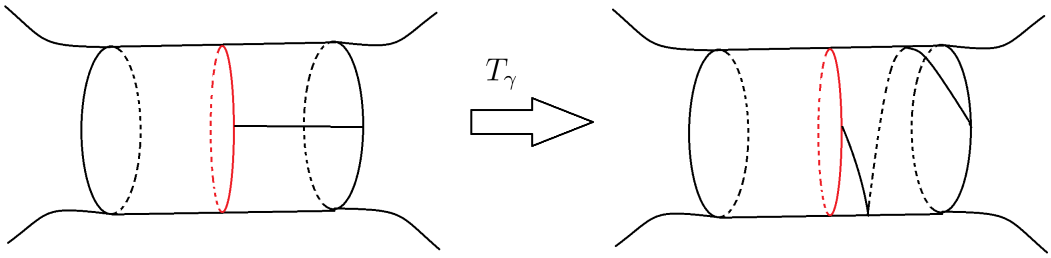
**Theorem 1.** *The moduli space $\mathscr{M}_g$ of compact Riemann surface structures on $S_g$ (up to biholomorphism) is of real-dimension $6(g-1)$.*

*Proof.* A Riemann surface structure on $S_g$ is completely determined by the structure the $2(g-1)$ pairs of pants. We make their definition more precise, by choosing to cut along the simple closed geodesics associated to the $\gamma_i$. Using **Proposition 1** the Riemann surface structure seems to be determined by 3 times $2(g-1)$ positive real numbers. However the lengths on common geodesics must coincide so we have to divide by 2, i.e. we need to specify the lengths of the $3(g-1)$ geodesics. And secondly the gluing can be more subtle that we would think. We can do a continuous twist (explained below) for each of the $3(g-1)$ gluings, so that we have

$$\mathscr{M}_g \simeq \mathbb{R}^{3(g-1)}_{>0} \times \mathbb{R}^{3(g-1)}$$

□

Dehn Twists. If $S$ is a Riemann surface and $\gamma$ a closed simple geodesic in $S$ then we define the **Dehn twist** around $\gamma$ to be a transformation $T_\gamma$ identity on $S$ minus a tubular neighborhood $A$ of $\gamma$ in the following manner.




The transformation is identity on the left part of $A$. The right part is taken to be the annulus $\mathbb{R}/2\pi\mathbb{Z} \times [0,1[$ and

$$T_\gamma(\theta, t) = (\theta + 2\pi t, t)$$

To not be depended on $A$ and only on the homotopy class of $\gamma$ then $T_\gamma$ is well-defined up to isotopy. We will see later where those Dehn Twists appear, but the point here is to say that to be defined the Dehn Twist have to be continuous, hence the $2\pi$. In the proof of **Theorem 1** we can do a continuous transformation with $\theta$ any real number and then glue the pairs of pants.

### 1.3.3 Teichmüller space

# 2 Fibered categories in groupoids. (Tangi Pasquer, 25th of Nov. 24 and 2nd of Dec. 24)

## 2.1 Introduction

Let $\mathsf{C}$ be a small category[2]. We have seen in the first talk that a moduli problem is encoded by a so-called moduli functor. Such a functor $F$ is a contravariant functor with values in $\mathsf{Sets}$, also called a **presheaf** in sets. It has to be understood as

$$F : S \longmapsto \{\text{isomorphism classes of "objects" over } S.\}$$

However, this functor erases all information concerning the automorphisms of the classified objects, leading to bad geometric behaviour as in the example of $M_{1,1}$ mentioned before. Therefore, the new paradigm is to replace $F$ by something similar to a presheaf in groupoids

$$\mathscr{F} : \mathsf{C} \longrightarrow \mathsf{Grpd},$$

where $\mathscr{F}(S)$ is the category made of "objects" over $S$, and morphisms taken are only isomorphisms over $S$. Recall that a **groupoid** is nothing but a category where morphisms are all isomorphisms, with morphisms between groupoids being just usual functors.

**Example 2.** *Let $G$ be a group. Then we can form the groupoid $\mathrm{B}G$ with one object $\star_G$ and*

$$\mathrm{Hom}_{\mathrm{B}G}(\star_G, \star_G) \simeq G$$

*with the composition coming from the group law of $G$. As an exercice prove that a groupoid is always equivalent (as a category) to a disjoint union of $\mathrm{B}G_i$.*

**Example 3.** *Let $G$ be a group acting on a set $X$. Then we define a groupoid by considering the category whose objects are elements of $X$ and whose morphisms $x \to y$ are labeled by elements $g$ such that $g \cdot x = y$. Prove that $\mathrm{B}G$ is the groupoid associated to a trivial action.*

**Example 4.** *Let $X$ be a topological space. The* ***fundamental groupoid*** *$\Pi_1(X)$ is the groupoid whose objects are points of $X$ and morphisms are paths $x \to y$ up to homotopy (the composition being the concatenation up to homotopy).*

Given a groupoid we can look a the set of isomorphism classes of its objects, we end up with a functor

$$\mathrm{Isocl} : \mathsf{Grpd} \longrightarrow \mathsf{Sets}$$

and $F = \mathrm{Isocl} \circ \mathscr{F}$.

Along with this nice presentation should however come the following warning : in general, it will be hard if not impossible to produce an actual functor $\mathcal{F} \colon \mathsf{C} \to \mathsf{Grpd}$ in the context of moduli spaces. The issue is due to the fact that the formation of pullbacks of objects over $S$ along morphisms $S' \to S$ will not respect a strict composition rule.

To make this formalism work, one should use higher category theory. Unlike $\mathsf{Set}$, there is a notion of 2-morphisms between morphisms of groupoids : these are the natural transformations of functors. This actually defines a "2-category" $\mathsf{Grpd}$ of groupoids, and one should instead let $\mathcal{F}$ satisfy a weaker property than being a functor, namely, a quasi-functor between 2-categories.

We won't discuss this point of view here, and will instead focus on an equivalent presentation using categories fibered in groupoids to avoid any such ill-defined functors.

2. in the following all generic categories are small, or with a small skeleton, except the obvious ones such as the category of small categories, etc.

## 2.2 Fibered categories

**Definition 6.** $\mathsf{F}$ *is said to be a* ***category over*** $\mathsf{C}$ *if it comes along with a functor* $p : \mathsf{F} \longrightarrow \mathsf{C}$*. We call* $p$ *the* ***projection*** *and we say that* $U$ *is* ***over*** $X$ *when* $p(U) = X$*. In this case, let* $\varphi : V \to U$ *be a map in* $\mathsf{F}$*. We said that* $\varphi$ *is* ***cartesian*** *if for any map* $\varphi' : V' \to U$ *and a factorisation*

$$\begin{array}{ccc} p(V') & \xrightarrow{g} & p(V) \\ & \underset{p(\varphi')}{\searrow} & \downarrow{\scriptstyle p(\varphi)} \\ & & p(U) \end{array}$$

*there exists a unique* $\psi : V' \to V$ *such that*

$$p(\psi) = g,$$

*or in simpler terms if* $p$ *lifts factorisations uniquely. If* $\varphi$ *is cartesian we say that* $V$ *is a* ***pullback*** *of* $U$ *along* $p(\varphi)$*.*

**Proposition 3.** *Let* $\mathsf{F}$ *be a* ***category over*** $\mathsf{C}$*,* $f : Y \to X$ *a map in* $\mathsf{C}$ *and* $V$ *is an object over* $Y$*. There is a unique, up to unique isomorphism, cartesian map* $\varphi : V \to U$ *such that* $V$ *is the pullback of* $U$ *along* $f$*. When such a* $V$ *is chosen we usually denote it by* $f^*U$*.*

*Proof.* If $\varphi' : V' \to U$ is another cartesian morphism over $f$, then $g = \mathrm{id}_X$ is a factorisation that should lift to a unique morphism $V' \to V$, and similarly we get a unique map $V \to V'$ making the following diagram commute

$$\begin{array}{ccccc} V' & \longrightarrow & V & \longrightarrow & V' \\ & \searrow & \downarrow & \swarrow & \\ & & U & & \end{array}$$

and since $\mathrm{id}_V$ (resp. $\mathrm{id}_{V'}$) and $V \to V' \to V$ (resp. $V' \to V \to V$) are lifts of $\mathrm{id}_X$ (resp. $\mathrm{id}_Y$) then both maps are equal, hence the result. □

**Definition 7.** *A* ***fibered category over*** $\mathsf{C}$ *is a category* $\mathsf{F}$ *over* $\mathsf{C}$ *such that all maps in* $\mathsf{C}$ *are projections of cartesian maps. In this case, the* ***fiber over*** $X$ *is defined to be the category, denoted* $\mathsf{F}(X)$*, made of objects* $U$ *over* $X$ *and with morphisms the ones projecting to* $\mathrm{id}_X$*.*

One should compare this point of view to the one of a $k$-vector bundle $E$ over a base $B$. Instead of viewing it as a locally free sheaf $E\colon \mathsf{Op}(B) \to \mathsf{Vect}_k$, one can think about it as a space $E$ with a map $p\colon E \to B$. We are doing something similar here : instead of viewing a stack as a sheaf $\mathsf{C} \to \mathsf{Grpd}$, we define it as some big space (a category actually) $\mathsf{F}$ with a functor $p\colon \mathsf{F} \to \mathsf{C}$.

**Example 5.** *If* $F$ *is the presheaf of sets in the first section then the category* $\mathsf{F}$ *constructed as the disjoint union of* $F(X)$ *for* $X$ *in* $\mathscr{C}$ *(and the projection being the obvious one, i.e.* $U \in F(X) \longmapsto X$*) is a fibered category.*

We want to construct $\mathsf{Fib}_{\mathsf{C}}$ the 2–category of fibered categories over $\mathsf{C}$. This means that we will define morphisms between fibered categories and also morphisms between morphisms. This would amount, for instance, in the case of the fundamental groupoid, to have morphisms between paths, i.e. to remember all homotopies.

**Definition 8.** *A (1–)morphism of fibered categories over* $\mathsf{C}$ *is a functor* $\Phi : \mathsf{F} \longrightarrow \mathsf{G}$ *commuting with the projections*

$$\begin{array}{ccc} \mathsf{F} & \xrightarrow{\Phi} & \mathsf{G} \\ & {\scriptstyle p_{\mathsf{F}}} \searrow & \downarrow {\scriptstyle p_{\mathsf{G}}} \\ & & \mathsf{C} \end{array}$$

*and sending cartesian morphisms in* $\mathsf{F}$ *to cartesian morphisms in* $\mathsf{G}$*.*

*Let* $\Phi, \Phi'$ *be morphisms* $\mathsf{F} \longrightarrow \mathsf{G}$ *of fibered categories over* $\mathsf{C}$*. A 2–morphism* $\alpha : \Phi \Longrightarrow \Phi'$ *is a natural transformation* ***preserving bases****, which means that if* $U$ *is over* $X$ *then* $p_{\mathsf{G}}(\alpha_U) = \mathrm{id}_X$*.*

$\mathrm{Hom}_{\mathsf{Fib}_{\mathsf{C}}}(\mathsf{F}, \mathsf{G})$ *is therefore a category whose objects are morphisms of fibered categories over* $\mathsf{C}$ *from* $\mathsf{F}$ *to* $\mathsf{G}$*, and whose morphisms are 2–morphisms between them.*

**Example 6.** *Let* $\rho : G \longrightarrow H$ *be a group morphism. It induces* $\mathrm{B}\rho : \mathrm{B}G \longrightarrow \mathrm{B}H$ *a structure of category over* $\mathrm{B}H$*. Then* $\mathrm{B}G$ *is a fibered category over* $\mathrm{B}H$ *if and only if* $\rho$ *is a surjection. Futhermore, if* $K$ *is the kernel of* $\rho$ *then we can canonically identify* $\mathrm{B}K$ *with the sub-groupoid* $\mathrm{B}G(\star_H)$*.*

**Example 7.** *Let* $X$ *be an object in* $\mathsf{C}$*. We define the category relative to* $X$*, noted* $\mathsf{C}_{/X}$*, whose objects are morphisms* $Y \longrightarrow X$*, and whose morphisms are maps* $Y \longrightarrow Y'$ *such that the following commute*

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ & \searrow & \downarrow \\ & & X \end{array}$$

*The forgetful functor* $\mathsf{C}_{/X} \longrightarrow \mathsf{C}$ *sending* $(Y \longrightarrow X)$ *to* $Y$ *induces a structure of fibered category over* $\mathsf{C}$*. The associated presheaf of sets is nothing but* $h_X : Y \longmapsto \mathrm{Hom}_{\mathsf{C}}(Y, X)$*.*

The following is a lemma that should remind you the setting of sheaves.

**Lemma 4.** *Let* $\Phi : \mathsf{F} \longrightarrow \mathsf{G}$ *be a morphism of fibered categories. It induces an obvious functor*

$$\Phi(X) : \mathsf{F}(X) \longrightarrow \mathsf{G}(X)$$

*for any object* $X$ *of* $\mathsf{C}$*. Then* $\Phi$ *is full and faithful (resp. an equivalence) if and only if for all* $X$ *in* $\mathsf{C}$*,* $\Phi(X)$ *is full and faithful (resp. an equivalence).*

*Proof.* □

The Yoneda lemma states that the functor $X \longmapsto h_X$ from $\mathsf{C}$ to $\mathrm{Presh}(\mathsf{C})$ category of presheaves of sets on $\mathsf{C}$ (morphisms are natural transformations) is a faithful functor. It comes from the more general statement below.

**Lemma 5.** *(Yoneda) Let* $F$ *be a presheaf of sets on* $\mathsf{C}$ *and* $X$ *in* $\mathsf{C}$*. Then* $\gamma \mapsto \gamma_X(\mathrm{id}_X)$ *is a bijection*

$$\mathrm{Hom}_{\mathrm{Presh}(\mathsf{C})}(h_X, F) \xrightarrow{\sim} F(X).$$

In our context we are interested in a higher Yoneda lemma whose corollary is that $X \mapsto \mathsf{C}_{/X}$ is a faitfhul functor $\mathsf{C} \longrightarrow \mathsf{Fib}_{\mathsf{C}}$.

**Lemma 6.** *(2–Yoneda) The functor* $\Phi \mapsto \Phi(\mathrm{id}_X)$ *on objects and* $(\Phi \overset{\alpha}{\Longrightarrow} \Phi') \mapsto \alpha_{\mathrm{id}_X}$ *on morphisms is an equivalence of categories*

$$\mathrm{Hom}_{\mathsf{Fib}_{\mathsf{C}}}(\mathsf{C}_{/X}, \mathsf{F}) \xrightarrow{\sim} \mathsf{F}(X).$$

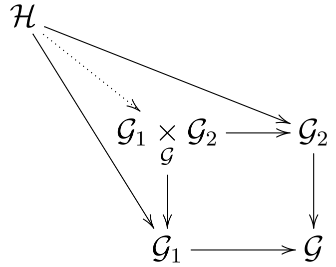
## 2.3 Over groupoids

We are now about to consider the special case of fibered categories whose fibers are groupoids. As we will soon need to perform some operations on those fibered categories, we start by defining them "fiber by fiber" and work with groupoids.

**Definition 9.** *Let $\mathcal{G}_1, \mathcal{G}_2$ be groupoids over a groupoid $\mathcal{G}$. Define the* ***fiber product*** *of $\mathcal{G}_1$ and $\mathcal{G}_2$ over $\mathcal{G}$, noted $\mathcal{G}_1 \underset{\mathcal{G}}{\times} \mathcal{G}_2$, the groupoid whose objects are $(g_1, g_2, \sigma)$ where $g_i$ is an object in $\mathcal{G}_i$ and $p_1(g_1) \xrightarrow{\sigma} p_2(g_2)$ an isomorphism, and whose morphisms are $g_i \longrightarrow g_i'$ inducing a commutative diagram*

$$\begin{array}{ccc} p_1(g_1) & \longrightarrow & p_1(g_1') \\ \sigma \downarrow & & \downarrow \sigma' \\ p_2(g_2) & \longrightarrow & p_2(g_2') \end{array}$$

**Proposition 7.** *The fiber product is a groupoid satisfying the universal property of diagrams commuting up to natural equivalence*



*The map $\mathcal{H} \dashrightarrow \mathcal{G}_1 \underset{\mathcal{G}}{\times} \mathcal{G}_2$ is unique up to a unique natural equivalence.*

**Example 8.** *Let $G_1$ and $G_2$ be two groups with morphisms $G_i \to G$ to a third group. We can consider the fiber product of the associated groupoids $\mathrm{B}G_1$ and $\mathrm{B}G_2$ over $\mathrm{B}G$. The result is left for the reader to compute as an exercise.*

**Definition 10.** *A fibered category $\mathsf{F}$ is said to be* ***in groupoids*** *if the categories $\mathsf{F}(X)$ for any object $X$ are groupoids. We note by $\mathsf{FibGrpd_C}$ the fullsubcategory of $\mathsf{Fib_C}$ they form.*

**Proposition 8.** *Let $\mathsf{G}_1, \mathsf{G}_2$ be categories fibered in groupoids over $\mathsf{C}$. Then $\mathrm{Hom}_{\mathsf{Fib_C}}(\mathsf{G}_1, \mathsf{G}_2)$ is a groupoid.*

Let $\mathsf{F}$ be a category fibered in groupoids over $\mathsf{C}$. For any morphism $f : Y \to X$ in $\mathsf{C}$ and any object $U$ over $X$, we choose an arbitrary pullback $f^*U$ over $Y$ in $\mathsf{F}(Y)$, with its associated cartesian morphism $f^*U \to U$.

**Definition 11.** *Let $X$ be an object of $\mathsf{C}$ and $U, U'$ be over $X$. We define a pseudo-contravariant-functor on the category $\mathsf{C}_{/X}$ as follows :*

- *for a morphism $Y \xrightarrow{f} X$ we associate the set $\mathrm{Hom}_{\mathsf{F}(Y)}(f^*U, f^*U')$ ;*
- *for a commutative diagram (i.e. a map $f \circ g \longrightarrow f$ )*

$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ & \searrow & \downarrow f \\ & & X \end{array}$$

*we associate the map*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{F}(Y)}(f^*U, f^*U') & \longrightarrow & \mathrm{Hom}_{\mathsf{F}(Y')}\left((f\circ g)^*U, (f\circ g)^*U'\right) \\ \left(f^*U \xrightarrow{u} f^*U'\right) & \longmapsto & \left((f\circ g)^*U \xrightarrow{v} (f\circ g)^*U'\right) \end{array}$$

*where $v$ is the unique arrow fitting in the diagram*

$$\begin{array}{ccc} (f\circ g)^*U & \xrightarrow{v} & (f\circ g)^*U' \\ \downarrow & & \downarrow \\ g^*(f^*U) & \xrightarrow[g^*u]{} & g^*(f^*U') \end{array}$$

*with vertical arrows being the unique isomorphisms given by the comparison between the two cartesian maps $g^*(f^*U) \to U$ and $(g\circ f)^*U \to U$ (resp. $g^*(f^*U') \to U'$ and $(g\circ f)^*U' \to U'$) over $g\circ f$.*

The choices of pullbacks having been made once and for all, the above defines a presheaf without any additional choices being made. In short, pulling back *objects* is not canonical, but when choices of those pullbacks are made, pulling back *maps* is. Therefore, this is a strong composition rule $g^*(f^*(u)) = (f\circ g)^*u$ when pulling back maps, whereas there is is general none for objects.

Despite those choices of pullbacks being made, we still have the following result.

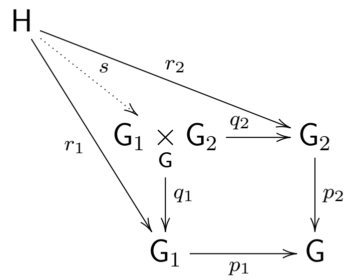
**Proposition 9.** *This matching defines a presheaf in sets on the category $\mathsf{C}_{/X}$. Given solely $U$ and $U'$ above $X$ and no prefered set of choices of pullbacks, any two sets of choices of pullbacks of $U$, $U'$ are related by a unique natural equivalence.*

This presheaf (or any of its equivalent copies) is denoted $\underline{\mathbf{Isom}}_{\mathsf{C}}(U,U')$. It is called the *isomorphism* presheaf associated to $U$ and $U'$. When $U = U'$, we write $\underline{\mathbf{Aut}}_{\mathsf{C}}(U)$ the *automorphism* presheaf associated to $U$.

We end this paragraph by using what we defined for groupoids to build and understand the universal property of the fiber product of categories fibered in groupoids.

**Proposition 10.** *Let $p_1\colon \mathsf{G}_1 \to \mathsf{G}$ and $p_2\colon \mathsf{G}_2 \to \mathsf{G}$ be two morphisms of categories fibered in groupoids over $\mathsf{C}$. Recall that natural transformations in groupoids, are all.*

1. *There exists a fibered category $\mathsf{G}_1 \underset{\mathsf{G}}{\times} \mathsf{G}_2$ with projections $q_1\colon \mathsf{G}_1 \underset{\mathsf{G}}{\times} \mathsf{G}_2 \to \mathsf{G}_1$ and $q_2\colon \mathsf{G}_1 \underset{\mathsf{G}}{\times} \mathsf{G}_2 \to \mathsf{G}_2$, and a 2–morphism $\alpha\colon p_1\circ q_1 \Longrightarrow p_2\circ q_2$ over $\mathsf{C}$ satisfying the universal property summarized by the diagram below :*




*In words, for any quadruple $(\mathsf{H}, r_1, r_2, \beta)$ with $\mathsf{H}$ a category fibered in groupoids over $\mathsf{C}$, morphisms of fibered categories $r_i\colon \mathsf{H} \to \mathsf{G}_i$ and a 2–morphism $\beta\colon p_1\circ r_1 \Longrightarrow p_2\circ r_2$, there is a triple $(s, \gamma_1, \gamma_2)$ with $\mathsf{H} \overset{s}{longrightarrow} \mathsf{G}_1 \underset{\mathsf{G}}{\times} \mathsf{G}_2$ a morphism of fibered categories and two 2–morphisms $r_i \overset{\gamma_i}{\Longrightarrow} q_i\circ s$.*

2. *Given a quadruple* $(\mathsf{H}, r_1, r_2, \beta)$ *as above, the resulting data* $(s, \gamma_1, \gamma_2)$ *is unique up to a unique isomorphism. This means that given another such triple* $(s', \gamma_1', \gamma_2')$*, there is a unique 2–morphism* $\lambda\colon s \Longrightarrow s'$ *such that* $\gamma_i'$ *are induced by* $\lambda$*.*
3. *Using fiber products of groupoids, these two properties sum up as follows : the functor*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{Fib}_\mathsf{C}}(\mathsf{H}, \mathsf{G}_1 \underset{\mathsf{G}}{\times} \mathsf{G}_2) & \longrightarrow & \mathrm{Hom}_{\mathsf{Fib}_\mathsf{C}}(\mathsf{H}, \mathsf{G}_1) \underset{\mathrm{Hom}_{\mathsf{Fib}_\mathsf{C}}(\mathsf{H},\mathsf{G})}{\times} \mathrm{Hom}_{\mathsf{Fib}_\mathsf{C}}(\mathsf{H}, \mathsf{G}_2) \\ \left(s\colon \mathsf{H} \to \mathsf{G}_1 \underset{\mathsf{G}}{\times} \mathsf{G}_2\right) & \longmapsto & (\ q_1 \circ s\ ,\ q_2 \circ s\ ,\ p_1 \circ q_1 \circ q \overset{\alpha \circ s}{\Longrightarrow} p_2 \circ q_2 \circ q\ ) \\ (\lambda\colon s \Longrightarrow s') & \longmapsto & (q_i(\lambda)\colon q_i \circ s \Longrightarrow q_i \circ s') \end{array}$$

*between groupoids is an equivalence of categories.*

To put it short, the fiber product of categories fibered in groupoids in unique up to a 1–isomorphism that is unique up to a unique 2–morphism.

## 2.4 Splittings of fibered categories

If $\mathsf{F}$ is a fibered category over $\mathsf{C}$, we described a pseudo-functor $X \mapsto \mathsf{F}(X)$ sending maps $Y \xrightarrow{f} X$ to the functor $\mathsf{F}(Y) \Longrightarrow \mathsf{F}(X)$ which send $V$ to $f^*V$ a choice of pullback. Remember it is only a pseudo-functor because it does not satisfy the morphism property for composition. This section is dedicated to defining a property of choices of pullbacks that make this pseudo-functor an actual functor.

This property is not essential for the following arguments and can be seen as a technical commodity. However, it seems relevant to the authors as it helps see what goes wrong if one would like to select pullbacks in a coherent way to get a functor $X \mapsto \mathsf{F}(X)$.

**Definition 12.** *Let* $\mathsf{F} \xrightarrow{p} \mathsf{C}$ *be a fibered category. A* ***splitting*** *of* $p$ *is a subcategory* $\mathsf{K}$ *of* $\mathsf{C}$ *such that the following properties hold :*

— *for any* $X$ *in* $\mathsf{C}$ *and* $U$ *above it,* $U$ *and* $\mathrm{id}_U$ *are in* $\mathsf{K}$ *;*
— *for any morphism* $Y \xrightarrow{f} X$ *and* $U$ *above* $X$*, there is a unique cartesian morphism* $f^*U \to U$ *above* $f$ *;*
— *all morphisms in* $\mathsf{K}$ *are cartesian.*

Given a splitting $\mathsf{K}$ of $\mathsf{F}$, there is a way to define a functor $F\colon \mathsf{C} \to \mathsf{Grpd}$ using the pullbacks in $K$. Conversely, a coherent choice of pullbacks which makes $F$ an actual functor defines a splitting of $\mathsf{F} \to \mathsf{C}$. The reader in encouraged to check this on his own.

**Example 9.** *Let* $\rho : G \longrightarrow H$ *be a group morphism. Then the fibered category* $\mathrm{B}G$ *over* $\mathrm{B}H$ *splits if and only if* $\rho$ *has a section.*

The example above shows that not all fibered categories split. Nevertheless, the next theorem ensures a weaker condition that may suit most applications.

**Theorem 2.** *Given* $\mathsf{F}$ *a fibered category, we can always find an isomorphism of fibered categories* $\mathsf{F} \to \mathsf{F}'$ *and a splitting* $\mathsf{K}'$ *of* $\mathsf{F}'$*.*

# 3 Reminder on topos.

**Definition 13.** *A **site** is a category in which all finite fiber products exist together with the additional data of which families $\{U_i \to U\}$ are called **coverings of** $U$ such that*

- *if $U \to V$ is a morphism and $\{U_i \to U\}$ a covering of $U$ then $\{U_i \underset{U}{\times} V \to V\}$ is a covering of $V$.*
- *if $\{U_i \to U\}$ is a covering of $U$ and $\{V_{j,i} \to U_i\}$ a covering of $U_i$ then $\{V_{j,i} \to U_i \to U\}$ is a covering of $U$.*
- *isomorphisms are coverings.*

**Definition 14.** *A **sheaf** on a site $\mathsf{C}$ is a presheaf $\mathscr{F}$ such that for all coverings $\{U_i \to U\}$, the following diagram is an equalizer*

$$\mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \underset{U}{\times} U_j)$$

We have therefore the category of sheaves defined to be the full subcategory of the category of presheaves. The inclusion map has an adjoint, called the sheafification functor which we will not present here.

**Definition 15.** *A **topos** is a category equivalent to the category of sheaves of sets on a site.*

# 4 Algebraic spaces. (Matteo Verni, 9th of Dec. 24)

## 4.1 Beyond schemes

The Yoneda lemma asserts that any category is faithfully embedded in the associated category of presheaves via the functor $X \mapsto h_X := \mathrm{Hom}_S(-, X)$. If $S$ is a scheme then there is a more precise statement. Recall that fppf means faithfully flat and of finite presentation.

**Proposition 11.** *The functor $X \mapsto h_X$ is faithful from the category $\mathsf{Sch}_S$ of schemes over $S$ to the category of fppf sheaves on $\mathsf{Sch}_S$, or as a consequence to the category of étale sheaves.*

In this manner we can extend the notion of space by looking at some sheaves. As seen in the previous chapter we want those sheaves to take their values in groupoids, therefore it requires a different definition.

**Definition 16.** *Let $P$ be a property of scheme morphisms. One says that $P$ is **local on domain** (resp. **local on codomain**) if for all morphism $X \to Y$ the following are equivalent*

$$X \to Y \text{ has } P \iff \forall \text{ coverings } \{U_i \to X\}, U_i \to X \to Y \text{ has } P)$$

$$(\text{resp. } X \to Y \text{ has } P \iff \forall \text{ coverings } \{V_i \to Y\}, V_j \underset{Y}{\times} X \to V_i \text{ has } P \;)$$

Being proper, of finite type, affine and surjective are properties local on domain. Being smooth, flat, locally of finite type and locally faithfully flat are properties local both on domain and codomain. For all of those one covering is sufficient for the morphism to have that property.

In the following $S$ is a fixed scheme, all sheaves are sheaves of set and one should identify $X$ and $h_X$ (therefore a morphism of sheaves $X \to G$ is rather a morphism $h_X \to G$, etc.).

**Definition 17.** *Let $F, G$ be sheaves on $\mathsf{Sch}_{S,et}$, and $f : F \to G$ a morphism. We say that $f$ is **representable** in $\mathsf{Sch}_{S,et}$ when for all morphism $T \to G$, the morphism $f_T : F \underset{G}{\times} T \to T$ is representable in $\mathsf{Sch}_{S,et}$. In that case, we say that $f$ **has a property** when $f_T$ has the property for all morphism $T \to G$.*

**Example 10.** *Let $F$ be a sheaf such that $F \to F \underset{S}{\times} F$ is representable. Then all morphism $T \to F$ are representable.*

**Definition 18.** *An **algebraic space** (over $S$) is a sheaf on $\mathsf{Sch}_{S,et}$ such that $F \to F \underset{S}{\times} F$ is representable and there exists an étale surjective morphism $U \to F$.*

**Remark 19.** *The second condition is local, so there exists an étale surjective morphism if and only if there exists a covering by étale morphisms. One can check that if the Zariski topology replaces the étale topology then we get only schemes.*

**Definition 20.** *Let $X$ a scheme over $S$ and a monomorphism $R \to X \underset{S}{\times} X$. We say that $R$ defines a **étale equivalence relation on** $X$ if for all schemes $T$ over $S$, the inclusion $R(T) \subset X(T) \underset{S}{\times} X(T)$ defines a equivalence relation and if the two morphisms*

$$R \to X \underset{S}{\times} X \underset{\mathrm{pr}_i}{\to} X$$

*are étale. In that case, $X/R$ denotes the sheafification of the presheaf $T \mapsto X(T)/R(T)$.*

**Proposition 12.** *Let $R$ be an equivalence relation on $U$. Then $U/R$ is an algebraic space. Conversely, let $F$ be an algebraic space with $U \to F$ an étale surjective morphism. Then the canonical morphism $U \underset{F}{\times} U \to U \underset{S}{\times} U$ defines an equivalence relation and*

$$F \simeq U \underset{S}{\times} U/(U \underset{F}{\times} U)$$

**Example 11.** *Let $G$ be a discrete group acting freely on a scheme $X$. Then the canonical morphism $G \times X \to X \times X$ induces an equivalence relation. We denote $X/G$ for the associated algebraic space.*

## 5 Algebraic stacks. (Enrico Lampetti, 13th of Jan. 25)

## 6 Joyeusetés

**About the $j-line$.** What we precisely have is the following. The family $E$ is locally trivial and with a parameter scheme connected, therefore we have a representation of the étale fundamental group

$$\widehat{\mathbb{Z}} \simeq \pi_{1,\text{ét}}(\mathbb{A}^1_t \backslash \{0\}) \longrightarrow \mathrm{Aut}(E_1) \simeq \mathbb{Z}/6\mathbb{Z}$$

which is non-trivial, whereas the representation associated to the trivial family $\mathbb{A}^1\backslash\{0\} \times E_1$ gives of course the trivial representaton. If we wanted to have an isomorphism one would needed to take a 6–th root of $t$, call it $t^{1/6}$ and $t^{1/3}$ its square and $t^{1/2}$ its cube and write

$$(Y/t^{1/3})^2 Z = (X/t^{1/2})^3 - Z^3$$

How can we see that in the representations ? Well, if you want to have the same representatiosn then restricts them to the intersection of their kernel subgroups which is in that case $6\widehat{\mathbb{Z}} \subset \widehat{\mathbb{Z}}$ and this inclusion is nothing but the induced map from $\mathbb{A}^1_{t^{1/6}}\backslash\{0\} \longrightarrow \mathbb{A}^1_t\backslash\{0\}$

$$\pi_{1,\text{ét}}(\mathbb{A}^1_{t^{1/6}}\backslash\{0\}) \hookrightarrow \pi_{1,\text{ét}}(\mathbb{A}^1_t\backslash\{0\}) \simeq \widehat{\mathbb{Z}}$$

**About $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$.**

**About elliptic curves.** $S = \mathrm{Spec}\,\mathbb{C}$. Let $\Lambda$ be a lattice in $\mathbb{C}$ and consider $\Lambda$ acting freely by translation on $\mathbb{A}^1_{\mathbb{C}}$. Then we would like to think that the algebraic space $\mathbb{A}^1_{\mathbb{C}}/\Lambda$ is represented by the elliptic curve $\mathbb{C}/\Lambda$. If that were the case then the morphism of algebraic spaces $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}/\Lambda$ should correspond to a morphism of schemes $\mathbb{C} \to \mathbb{C}/\Lambda$ which is not the case ! The reduction modulo $\Lambda$ is transcendental. However the analytification functor from the category of finite type algebraic spaces over $\mathbb{C}$ to the category of analytic spaces over $\mathbb{C}$ restricts to an equivalence on proper algebraic spaces. And indeed $\mathbb{A}^1_{\mathbb{C}}/\Lambda$ is not proper.

# 7 Références.

[1] *Algebraic Spaces and Stacks*, M.Olsson, AMS (2016)

[2] *A Primer on Mapping Class Groups*, B.Farb, D.Margalit, (PMS-49), Princeton University Press, (2012)