The mathematical anti-atomism of Plato's cosmology II. –The 'khora' in Plato's *Timaeus*

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1. Presentation

According to Plato's staging, it takes place the next day after the *Republic*, where Socrates talked about the possibility to build a (as good as possible) city.
The principal speaker is Timeaus which gave its name to the title. We know nothing about him, except he was from Locri (or Locrus), a city in the south of Italia, part of *Magna Graeca*. Since the only important known school there is the Pythagoreans, it is reasonable to understand he represents some 'generic' Pythagorean, so that either Plato certainly did not wants to make fun of the Pythagoreans, but paid tribute to some of his sources.
After an introduction by Socrates, it begins with a first account seemingly completely unrelated: the 'war of worlds' between the huge kingdom of Atlantis and the Athenians, left to their own by the other Greeks, as in the Persian wars. What is easily forgotten through Athenian victories is the end, the tsunami that destroyed Atlantis, but also all the Athenian army which was there at this time. Maybe Plato wants to show the power of nature in respect with human power and the necessity to study the nature even for human affairs.

2. A 'plausible myth' or a 'reasonable discourse'?

Since the dialogue begins by what is evidently a myth, it is all too natural to see all the text as some myth. Actually, it is usually said it is a 'plausible myth', since this is what Timaeus himself says three times in his account (59c, 68d, 69b) about his account: an '*eikôs muthos*', with is translated by a 'plausible myth' but means also a 'myth about the perceptible things'. It is in line with the first Greek cosmogony, Hesiod's *Theogony* (8th century BCE, one of the oldest Greek text), which is more about gods' life than about the nature. For instance David Sedley 'uncovers a 'a remarkably deep isomorphism' between the two texts' (Hesiod's Theogony and Plato's Timaeus, in *Plato and Hesiod*, G. Boys-Stones and J. Haubold ed., Oxford Univ. Press, 2009).
This is such an accepted claim, that it is rarely noted that in front of the three times uses of '*eikôs muthos*', there are 16 times '*eikôs logos*' i.e. a rational discourse or even a rational explanation about the phenomena (30b, 34c, 44d, 48c, 48d, 49b, 53b, 55d, 56a, 56d, 57d, 59c, 68d, 72d, 90e). Thus, it is impossible to argue for the freedom to change what we decide to be absurd, or that the text either has not to be taken seriously or needs to be interpreted symbolically. Such claims appear even more implausible, when we consider the importance of scientific, in particular mathematical discourses for the creation of the universe.

3. The Intelligent Design (ID)

As seen in the first lecture, all the thinkers of the nature from Thales departed from the poets by considering a nature without any intervention of gods. And this is even truer (if possible) for the atomists, in particular Democritus. As a matter of fact, Socrates blames Anaxagoras (first part of 5th century BCE) because his nature did not need any intellect to work and everything was explained by mechanical cause instead of intelligent ones (*Phaedo* 97c).
Thus, when Plato in the *Timaeus* assign to a '*dêmiourgos*' the creation of the world, he seems to return to the old poets, and actually to write a story for young children, another reason to consider the text as a 'plausible myth'.
This does not escape to the early Christians, and it explains maybe why they respected Plato's books. But it is also one reason why the modern philosophers and scientists, fighting superstitions and religions, despised Plato and respected the atomists.

However, Plato's '*dêmiourgos*' is certainly much different from the all-powerful creator as described in the sacred scriptures. Namely,

- he is not able to create the world by himself
- he has to look to some models to do it.
- Last but certainly not least, his name itself means in Greek a 'civil servant' but mostly a 'craftman', and as a matter of fact, he has to negotiate with the 'necessity' ('*anagkê*') and Timaeus describes him as working on some piece of iron using his hammer as a blacksmith (35b-c), or as a potter on some pieces of clay (73e) or even a peasant planting some seeds (41e, 42a, 73b, c), all tasks and professions despised in the social hierarchy in the Antiquity. He is certainly far from the 'Lord', the 'King' described in the sacred scriptures or in the medieval literature about the life of the saints. Such a devaluation is all the more important since it is only one of three things presents at the same time and from all eternity: the Forms or Ideas and the puzzling '*khôra*'.

Nevertheless, Timeaus' '*dêmiourgos*' is certainly a god, and it is called as such several time in Timaeus' account, for example, about the creation of the universe, he says: 'For since God desired to make it resemble most closely that intelligible Creature' ('ὁ θεὸς ὁμοιῶσαι βουληθεὶς ζῷον ἓν ὁρατόν', 30d).
He is even certainly a powerful one, since he is able to create other gods (40a) and all other creatures in the universe. An important consequence which was strongly criticized in the Antiquity is since everything which is born has to die, the universe itself may be destroyed. The only reason why this will not happen is that its creature, the '*dêmiourgos*' is good, and as such want only what is good, thus he does not want its destruction.

4. The Four elements

i) Plato and the four elements

The Greek cosmology at least in the fifth century considered all the natural things to be composed from the four elements (or Empedocles' 'roots'): fire, air, water and earth. The first being the lightest, the last the heaviest.

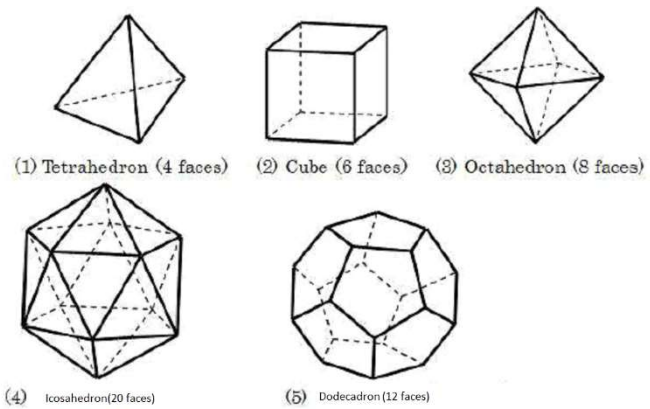
Plato is used to start many of his reasoning from old and probably well-accepted notions or stories. It is the clearest in his use of myths and other popular stories he changes to make easier and faster to understand his discourses. So Plato starts from the four elements and a world without any void as in Empedocles' world. But when his predecessors considered them as some physical foundational 'elements', for Plato they were mathematical figures.

5. The mathematics behind the four elements

i) The mathematical correspondance

As a matter of fact, each elements was an agglomerate of extremely small particles, since they were invisible, and corresponding to one regular solids (it is the origin of the so-called 'platonic solids').

- Fire is composed by invisible particles of tetrahedrons,
- air of octahedrons,
- water of icosahedrons,
- and finally earth of cubes.



From an incorrect picture on internet

As you see, one regular solid is missing, the dodecahedron, that is reserved for something else (55c).

ii) A mathematical atomism?

Such a presentation would mean that Plato was very close from the Atomists. The principal difference would be, while Democritus especially but also Epicurus and Lucretius, considered either atoms with an infinite number or an very large, but finite, number of shapes, and their components were physical objects, Plato considered instead four geometrical shapes. This is one reason that many modern

commentators of Plato speaks on *Timeaus*' cosmogony as a 'mathematical atomism'. Namely Aristotle criticizes several times both Plato (or the Platonists) and the atomists together. As a matter he is even more critical of the former than the latter, for he says that the atomists, at least, propose a physical construction for the world, while Plato (or the Platonists) did not even presents a theory of nature (a physics) but a mathematical theory.

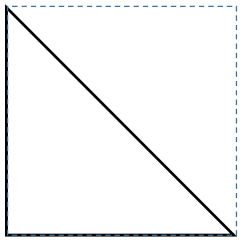
iii) A mathematical anti-atomism

But this is **not** Plato's construction. The four polyhedrons are certainly not 'atoms' but neither even 'elements' in the mathematical (Euclidean) sense. The elements from where everything, including the four elements, are made, are pure geometrical figures. They are:

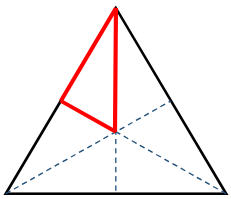
right-angle triangles.

More specifically **two types** of such triangles:

Isosceles right-angles triangles and **one sixth of equilateral triangles** (55b-c).



isosceles right-angle triangle



scalene right-angle triangle

Plato's structural triangles

Why two different kinds of triangles? Because among the four first regular solids, all their faces are formed from equilateral triangles except the cube.

I will not discuss why Plato uses this kind of triangles rather other possibilities, in particular from the Antiquity to nowadays, commentators find especially puzzling the use of these scalene triangles instead of simply half-equilateral triangles.

iv) Geometrical consequences for the four physical elements

A few pages previously, when the four classical elements were first presented, they were said able to be 'passing on to one another in an unbroken circle the gift of birth' (49c). Chances are that it was the orthodox point of view of the thinkers of the nature, since it is also the one Aristotle supports after criticizing Plato. Aristotle's criticism is about the

change done by Timaeus after giving the correspondence between geometrical polyhedrons and physical elements. It is not exact that the elements can transform one into another. Namely the element 'earth' can never change into another element and change is possible only among the three other elements. The element 'earth' will stay such an element, while for instance the element 'fire' can be transformed into 'air', 'water' but not 'earth'.

The reason of such strange but absolute behavior between elements is not physical but mathematical. Since an isosceles triangle cannot be transformed into a scalene one, and conversely, it is impossible for a cube that each of its 6 faces are composed by 2 isosceles triangles to change into any polyhedron whose faces are equilateral triangles formed by six scalene triangles (cf. above figure).

Conversely since the faces of an tetrahedron (particle of fire), octahedron (particle of air) and icosahedron (particle of water) are respectively given by 4, 8 and 20 identical equilateral triangles, they can be transformed on into the others, according to the following laws:

2 particles of fire will give 1 particle of air

5 particles of fire will give 1 particle of water

3 particles of fire + 1 particle of air will give 1 particle of water

1 particle of water will give 5 particles of fire or 1 particle of fire and 2 particles of air and so on.

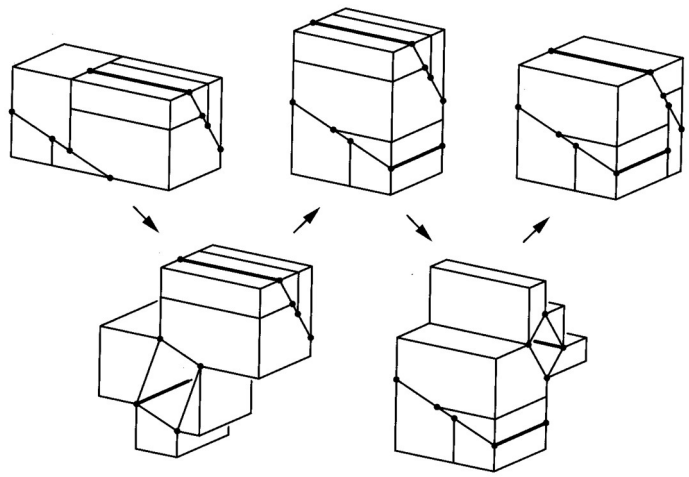
Thus, the laws of physics are determined by the laws of geometry.

Moreover, it is fundamental to understand that the behavior of the elements is determined not by its interior but only by its exterior i.e. its faces. In particular its volume is irrelevant when we want to study its physical properties. Thus, the problem to keep the same volume through the above 'laws of transformation' are irrelevant. Actually, this is impossible since the volume of the four polyhedrons of edge a are respectively:

Cube : a^3, Tetrahedron: $(\sqrt{2}/12)a^3$, Octahedron: $(\sqrt{2}/3)a^3$, and finally for the Icosahedron: $5((3 + \sqrt{5})/12)a^3$.

v) An excursus through the Hilbert's third problem (courtesy of Pr Salvatore Coen)

Before my first lecture, Salvatore Coen told me about this problem, I completely forgot (if I ever knew it). The question asked by Hilbert at the famous 2nd International Congress of Mathematicians in Paris (Sorbonne) in 1900 was the following: is it possible to 'dissect' any polyhedron into another one with the same volume (the original question was a little more complicated).



An example of a dissection (a parallelepipeds into a cube)
from Mircea Pitici, Geometric Dissections, 2008

The answer is known and it is negative. But more important, we know now it depends of an invariant called the 'Dehn invariant'. And this invariant is 0 for the cube and never 0 for the other regular solids, thus none of them can be dissected into the cube and conversely the cube cannot be dissected into any of them (there are many interesting problems about this question in Hartshorne's *Geometry: Euclid and Beyond*, Springer, 2000, especially chapter 5).

6. Discovering the 'space'

i) An absolute theoretical discovery about the perceptible things

At the beginning of his account, Timaeus has to distinguish two classical Platonist worlds:

- the real world composed of the eternal Ideas, or intelligible forms
- the world of things that appear and then disappear, that are copies of the formers.

In a second part, he argues about the need to consider these two worlds, as consequences of two kinds of knowledge, the sciences and the opinion ('*doxa*').

But then, a new world appears, he never spoke about before and will never afterwards. It is called the '*khôra*', the 'space' in Greek. It is introduced with a wealth of precautions and qualified as something which is almost impossible to believe, 'barely an object of belief' ('μόγις πιστόν', 52b).

It is impossible to conceive it using either our senses or our intellect, as either for a phenomenon or a rational thought. Thus, how is it possible to come to such conclusion? The answer is by a 'a kind of bastard reasoning' ('λογισμῷ τινι νόθῳ', ib.). Actually, theory itself is imposing the **necessity** of this third term.

Since there is two worlds and one of them is a copy of the other, another one in needed on that the first could leave its mark ('τυπωθέντα'), as a stylus on the surface of the wax.

ii) The elements and the 'space'

Let us now try to understand the connection of the latter with the 'laws of transformation' of the formers. Usually commentators, after just paying lip to the triangles as elements of our world (the world of phenomena), they forgot them and

consider the polyhedral as some atoms belonging to the 'space' (the '*khôra*'), and changing one into another when it is possible.

But it is not what the texts say. According to Timaeus, there can be no direct transformation from a 'polyhedron' into another. This needs two steps.

- The first one consists in the separation of all the faces of the polyhedron such that only the triangles (scalene or isosceles) are preserved. Namely these triangles never change nor appear or disappear, in explicit opposite way to the polyhedral.
- These triangles are then moving because the 'space' is always moving, and this is the reason why perceptible things appear always moving.
- Under some conditions, these triangles can or cannot bind together to form once again some elementary particles, according to the 'laws of transformation'. In particular, isosceles triangles have only two choices: either to stay as triangles or to form some cubes.

iii) The paradoxes

The reason so many scholars thought the need to emendate the text is certainly no without any reason. As a matter of fact, according to the very words of Timaeus, the part concerning the 'khôra' seems to be against all the intellectual evidence, without any hint to support it.

I will conclude by a brief abstract of some of them, many of them already noticed by Aristotle.

- The existence of planar figures, the right-angle triangle, living and moving in a 3-dimensional space, the 'khôra'
- The disappearing or appearance inside the 'khôra' of polyhedron without any change of shape and of volume of the 'khôra'.
- The change among different types of polyhedra according the 'laws of transformation' without any change in the volume and shape of the 'khôra' though these laws does not conserve the global volume and shape.
- The impossibility to get a paving of the space using any polyhedron except the cubes, while the 'khôra' does not contain any void.
- The impossibility to get a sphere by a (finite) association of polyhedral.
- The existence of different magnitudes of triangles, which means it is extremely difficult when in movement to bind together to form a polyhedron.
- The problem of the heavy and the light in Timaeus' account.
- The elements having some weight when the 'khôra' has absolutely no such quality.
- And so on.

I have not the time to consider if the geometrical approach, different from the classical one where all these above problems are insoluble may help to dissolve some or all of these paradoxes, but if could consider these questions in the discussion.