MATHEMATICS IN ANCIENT GREECE
FROM THE 6th TO THE 4th CENTURY BCE
FROM PYTHAGORAS TO EUCLID

History of mathematics, Bologna, October 2013

Part II

Logic, Reasoning and origins of the irrationality

I. Geometry and Greek mathematics.

We have seen two features of Greek mathematics at his origins.

- The first was the treatment of number using geometric figures, so we may say the mathematicians worked mainly in geometry used even in arithmetic.
- The second one was the use in geometry of a fundamental tool called the 'application of area' (cf. part I, §vii.2 note 19) consisting to study plane figures to treat linear problems, i.e. using surfaces to treat problems concerning lines, or in modern terms, adding a dimension to solve the problem.

It may be the first marked difference with previous mathematics, e.g. Egyptian and Babylonian mathematics: the importance given to geometry instead of arithmetic and computations.
These early mathematics were extremely efficient for computing surfaces and volumes (making sometimes some mistakes, as in the Babylonian computation of the volume of the cone or the pyramid), but they contain few purely geometrical results.
And so we may be extremely caution concerning their alleged knowledge about some kind of Pythagoras' theorem. Moreover, it is almost impossible to understand for common sense how even a limited knowledge of its formula would not have been followed by other geometrical results. In 1000 years, these extremely subtle mathematicians would have been unable to obtain such an immediate consequence as the result of the 'geometric mean', or at least a good understanding of Pythagoras' theorem itself. We have to remember that all the results of the 13 books of Euclid's Elements are supposed to have been found in less than three centuries (and even much less if the Pythagorean legacy is discarded, as some modern historians do).

But even if the importance of Geometry was a characteristic of Greek mathematics, it was essentially concerned with concrete objects as lines, figures, volumes. Thus, the proofs can be done and verified step by step on drawings.
For instance, in the second proof of Pythagoras' theorem, we could cut papyri and verify one by one the different steps of the proof. As such, it was not so different from the computation of some volumes, as pyramids or cones, or even the

approximation of π by the Babylonians and the Egyptians. As we saw, all the proofs could be done exclusively on figures almost without a word. Diagrams were not only a help for the proof, they were the proof itself with some consequences concerning the rigor according to Proclus (cf. part I, §vii.2) or Archimedes who recognizes two steps in the demonstration of the volume of the pyramid: firstly Democritus who gave its result using some unknown means, and the rigorous proof established later by Eudoxus (Archimedes, *The Method*, p. 13).

In this sense, mathematics will agree with the definition of science as 'not other thing than perception', as Theaetetus defines it in agreement with Protagoras view (Plato, *Theaeteus*, 151e2-3)

In our next study, we will enter in a new field, where the view is only a mean, where we will suppose something to be false we know to be true, where there is the necessity to draw wrong drawings instead of as exact as possible (e.g. proof of the proposition 2 of book III in Euclid's *Elements*), and finally where the proof has to be done exclusively by reasoning.

It is the **field of the impossible or of the absurd**. The demonstration by the '**absurd**' or the '**impossible**' reasoning, the so-called '***reductio ad absurdum'***, is a kind of proof where the purpose of the mathematician consists to conclude to an **impossible** or **absurd** statement , knowing perfectly well it is impossible or absurd.

II. Some well-known points of old logic.

It concerns the so-called proof '*reductio ad absurdum*' or the demonstration 'by impossible' ('ἀδύνατον') or 'by absurd' ('ἄτοπον'). Instead of proving something is true, its purpose is to prove some statement is false. It consists to put a hypothesis, knowing it is false, and to deduce from it something is impossible. According to Aristotle it has two applications.

1. In rhetoric or dialectic, it consists to prove the opponent is wrong by deducing something 'absurd' from one or several of his claims. In mathematics it consists to prove some assertion by deducing impossibility from its negation.

2. This kind of proof nevertheless is a form of demonstration less convincing than the direct one, maybe because it supposes 2 postulates not needed in the other case:

i) The '**principle of non-contradiction**'. It means that, under some precautions, it is impossible for a statement to be both true and false: '*It is impossible for the same thing to belong and not to belong at the same time to the same thing and in the same respect" (with the appropriate qualifications)*' (*Metaphysics*, IV, 3, 1005b19–20).
This principle is so fundamental it is impossible to think without it:
'*all men who are demonstrating anything refer back to this as an ultimate belief; for it is by nature the starting-point of all the other axioms as well.*' (ib., 1005b34).

This principle appears earlier in Plato's works:
'*The same thing clearly cannot act or be acted upon in the same part or in relation to the same thing at the same time, in contrary ways*" (*Republic*, 436b).

In symbolic logic it is written: $\vdash \neg (P \wedge \neg P)$ i.e. for any proposition P, we have:
$P \wedge \neg P$ false

ii) The '**principle of third excluded**' (or 'law of excluded middle' or '***tertium non datur***'): '*But on the other hand there cannot be an intermediate between contradictories, but of one subject we must either affirm or deny any one predicate.*' (*Metaphysics*, IV, 7).

In symbolic logic it is written: $\vdash (P \vee \neg P)$ i.e. for any proposition P, we have: $P \vee \neg P$ true.

iii) There is also another principle, certainly the first and most important one, needed in logic independently of the '*reductio ad absurdum*': **the principle of identity**.
Its purpose is to say how to use the identity '='.
It is not easy to give a formal definition of it, but it essentially means: 'P is *P* and is different from non-P'.

Aristotle explains it as the following: If a word had ‘*an infinite number of meanings, obviously reasoning would be impossible; for not to have one meaning is to have no meaning (…); for it is impossible to think of anything if we do not think of one thing; but if this is possible, one name might be assigned to this thing.*’ (*Metaphysics*, IV, 4).

Once again informally, this principle is found earlier in Plato’s dialogue *Theaetetus*:

‘Socrates. *Now in regard to sound and colour, you have, in the first place, this thought about both of them, that they both exist?*
Theaetetus. *Certainly*.
Socrates. *And that each is different from the other and the same as itself?*
Theaetetus. *Of course.*
Socrates. *And that both together are two and each separately is one?*’ (*Theaetetus*, 185a7-b3).

The meaning in modern terms is at least that the identity noted ‘=’ is reflexive and its opposite ‘≠’ is symmetric. But defining rigorously the ‘identity’ is not an easy task.

Concerning the ‘principle of third excluded’, as Aristotle remarks, there are some difficulties. For statement concerning the **future** or as we nowadays say ‘future contingent propositions’, it is not possible to affirm the truth or the falsity of the statement. As an illustration, he gives the following example: ‘Tomorrow a sea battle will take place’. Concerning this kind of statements, Aristotle writes:

‘***Everything must either be or not be****, whether in the present or in the future, but* ***it is not always possible to distinguish and state determinately which of these alternatives must necessarily come about****. Let me illustrate. A sea-fight must either take place to-morrow or not, but it is not necessary that it should take place to-morrow, neither is it necessary that it should not take place, yet it is necessary that it either should or should not take place to-morrow. Since propositions correspond with facts, it is evident that when in future events there is a real alternative, and a potentiality in contrary directions, the corresponding affirmation and denial have the same character. This is the case with regard to that which is not always existent or not always nonexistent. One of the two propositions in such instances must be true and the other false, but we cannot say determinately that this or that is false, but must leave the alternative undecided. One may indeed be more likely to be true than the other, but it cannot be either actually true or actually false. It is therefore plain that* ***it is not necessary that of an affirmation and a denial one should be true and the other false****. For in the case of that which exists potentially, but not actually, the rule which applies to that which exists actually does not hold good.*’ (*On Interpretation*, chap. 9, translation E. Edghill).
Since mathematics is not about time, the principle holds good for mathematical statements, and the ‘*reductio ad absurdum*’ is a correct form of demonstration.

Remark. It is important to distinguish between the '*reductio ad absurdum*' and contraposition. The contraposition of an implication: P implies Q is: non P implies non Q, and in logic both propositions,

$P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$

are equivalent[1].

[1] The contraposition is of course different from the '*reductio ad absurdum*', but is it a connection between both?

III. A new form of mathematics.

In the first part of this course, we saw deciding the question of the differentiation of Greek mathematics is extremely controversial. Many results the ancient Greeks attributed to Greek people, for instance Pythagoras' theorem, are now claimed to have been known in other civilizations much before having been proved rigorously by Greek mathematicians i.e. according to their concept of rigor.
Thus most historians may say that what differentiates Greek mathematics from others is this very concept of rigor. But
- as we saw in the previous paragraph, it is at least a controversial assessment since we have neither the first mathematical Greek texts, nor the ones of the other people. And when we have some texts of non-Greek mathematics, as for the Babylonians and the Egyptians, they are exercises or parts of lessons for future scribes, not for mathematicians.
- Moreover, we have seen that rigor is a relative concept, since for Proclus pre-Euclidian mathematicians were not really rigorous when they proved Pythagoras' theorem (cf. supra, part I, §vii.2)[2], rigor itself is more probably the result rather the cause of the differences between these mathematics. Furthermore many historians pointed out the elasticity of the idea of rigor, from the Antiquity to our time, showing what is considered non rigorous by one people (or at some period) is considered to be perfectly rigorous by another one (or at another period).
- Moreover rigor itself is not a satisfying answer as we could wonder why, at a certain time, Greek mathematicians would have suddenly demanded rigorous proofs, though previously like all the other mathematicians, they were completely satisfied by non-rigorous arguments.

A contrario, it is a sure claim the reasoning by '*reductio ad absurdum*' is not found outside Greek mathematics.
Nowhere else than in ancient Greek mathematics, they were used as mathematical tools. There are of course some sentences of older authors which can be related to it[3] (for instance staying with Greek texts, cf. Homer, *Odysseus*, 16, 194 sq.) .
Nevertheless, it has nothing to do with the understanding of such a reasoning as a universal truth, as it is needed in mathematics.

Among the scholars working on classical Greece, there is a controversy on the origins of this form of thinking. For some of them, it has to be found in philosophy, for instance in the Eleatic school especially in Parmenides' texts, and from there it spread to mathematics. For others it was first used in mathematics, the philosophers using it afterwards.

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[2] But also as we saw, it does not mean these mathematicians were not rigorous, even according to Proclus, but they were not as much rigorous and convincing as in Euclid.
[3] To see the use of this kind of reasoning in everyday life, you could look to http://www.youtube.com/watch?v=ytWGiOuzpe4. Do you think it is really a 'reductio ad absurdum'?

As you can guess, most philosophers are in favor of the first choice, most mathematicians for the second one.

As a matter of fact the question is ill-formulated. Firstly because the use of such a kind of rhetorical reasoning is certainly very old and used in any language, either to show an evident mistake (as in Odysseus 16, 194 sq.) or to point out to a (supposed) mistake and to ridicule an adversary as in some rhetorical disputes (cf. supra, note 3).

The real question is when did it become a kind of reasoning which could be trusted at least as much as the direct reasoning? This statute was not restricted to mathematics, since it is presented by Plato as Socrates' fundamental technique.

The so-called Socrates' method or the '*elentic*' method (from the Greek word 'ἔλεγχος') is a method of refutation based on the '*reductio ad absurdum*':
It consist asking something to an interlocutor, adding other statements absolutely endorsed by him, then showing these premises imply something which is obviously wrong for the interlocutor or at least he cannot admit.
This kind of refutation, connected by many Socrates adversaries to electrical shocks, supposes a complete and unanimous trust in this form of reasoning. Thus it is extremely doubtful it could come exclusively from philosophy, since it would mean a general agreement between all the different schools of thinking in ancient Greece, agreement which would be almost impossible to attain, so much their rivalries were violent.

The only reasonable possibility has to be searched elsewhere, in something almost all Greeks agreed: the rational certitude of mathematics. So it is highly probable the acceptation of the '*reductio ad absudum*' as a trustworthy reasoning was the result, not the consequence, of its acceptance and use in mathematicd.

Thus the origins of the demonstration by '*reductio ad absurdum*' have indeed to be found in mathematics. But then there is a question: why were Greek mathematicians driven to this kind of proof, to the point Euclid used it as his preferred form of demonstration (Aristotle, and the philosophers who followed his logic, was much less enthusiastic as he considered this form as a second choice for scientific demonstration)?

The answer has to be searched in the oldest texts related to Greek mathematics we have, the books of Plato and Aristotle. In both, the origins of mathematical irrationality are connected to the incommensurability of the diagonal. In Aristotle's *Analytics*, which treats this question, it is even considered as the model of the '*reductio ad absurdum*'.

IV. The incommensurability in mathematics.

We come here to something completely new in mathematics. Certainly it had to be disturbing for the mathematicians of this time, and nevertheless it was a collateral consequence of Pythagoras' theorem, even if it was an extremely unexpected one. It opened a completely new field not only in mathematics but in the way of thinking, maybe 'against the will' of its discoverers as it happens often in mathematics. Retrospectively, maybe the most amazing thing is the cause of such an upheaval was such a uninspired thing, the study of the diagonal of the square.

To resume it in a few words according to authors of the late Antiquity (from 1^{st} century BCE to the 6^{th} CE), as they studied the diagonal of a square, the Pythagoreans discovered its ratio to the side of the square, cannot be equal to a ratio of two integers. In modern terms it means the square root of *2* is irrational.

According to many modern historians of mathematics from the end of the 19^{th} century to the mid of the 20^{th} century, this situation determined a deep crisis inside Greek mathematics i.e. Pythagorean mathematics. Not only it was something completely strange and unexpected, but it was against the very base of Pythagorean philosophy that 'all is the numbers' as Aristotle says about Pythagorean doctrine.
For instance, according to Paul Tannery:

'*the Pythagoreans started from the idea natural to any non-educated man, any length is necessarily commensurable to the unity. The discovery of the incommensurability of some lengths with each other, and first of all the diagonal of the square to its side, done either by the Master or his pupils, had to be a real* ***logical scandal****, a* ***formidable*** *stumbling block.*' (De la solution…in Mémoires scientifiques I, p. 268).