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Definition, Power(s) and Divisions in Plato
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The Mathematical Milieu of Socrates and Plato

I begin with a straightforward instance of using the history of ancient Greek mathematics to solve a little puzzle for readers of Plato's dialogues. My text is *Lysis*, and the setting is a brand new wrestling school where several adolescent boys are discussing friendship with Socrates. Addressing two cousins, Menexenus and Ctesippus, Socrates asks a simple question, "Which of you two is older?" to which Menexenus replies, "We argue about that."[1] These days, one's exact date of birth is known, making such a response puzzling, but, for Athenians, age was counted in annual cohorts: every child born in a single solar year, measured from midsummer to midsummer, was the same age, one, because the Greeks had no zero. Even one was not an ἀριθμός in Greek mathematics, but it was numerical in the sense that it was required to perform arithmetic operations. The *need* for these elementary facts—cohorts, zero—reappears in other dialogues that emphasize the importance of an *age* requirement for a variety of civic responsibilities: citizenship status, military duty, marriage, eligibility to serve on the *boulē* or be elected *stratēgos*. The text of Pseudo-Aristotle's *Constitution of Athens* says Athenian youths reached the age of majority at eighteen; but commentators often say seventeen, a puzzle. Having been one at birth, a youth said to be eighteen was in fact seventeen by the arithmetic of our time.

Such facts about Athenian history, biography, and mathematics can be crucial to our understanding of Plato's philosophical contributions—but that is not the majority view these days. The usual view is that Plato was merely an interested bystander to the mathematics of his day so, in the first part of my talk, I will describe the controversies that account for this mistaken attitude, and review what we know of mathematicians of the time. In the second and much shorter part, I will pull together some of the reasons for granting Plato the status of knowledgeable participant in the mathematics of the Academy. As David Fowler had realized by the time he wrote the *second* edition his *Mathematics of Plato's Academy*, 1999, Plato had "detailed knowledge of important characteristics and problems of technical mathematics, and … he could … communicate on equal terms with that remarkable group of creative mathematicians who seem to have dominated, if not comprised, the group of friends and associates that assembled round him."[2] While I do not know the degree to which Plato may have been an innovator in mathematics, the extent of his systematic *use* of the mathematics of his day deserves the attention it is now receiving from participants in this conference. I am grateful to Professors Ofman and Brisson for their efforts at bringing all of us together for this most promising conference.

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[1] 207b8–c2, Burnet text, Lombardo translation.
[2] Fowler 1999: 8, 104.

Part 1.
How it transpired that Plato is denied his proper status in relation to the history of mathematics.

My explanation is a story with three episodes, an ancient one that has not completely subsided, another that makes matters worse in the nineteenth century, and a third from the twentieth century that has proved especially intractable among philosophers.[3]

Episode 1: The standard account of ancient mathematics, which endured through the Renaissance and into the twentieth century, is succinctly told by Fowler:

> The early Pythagoreans based their mathematics on commensurable magnitudes (or on rational numbers, or on common fractions m/n), but their discovery of the phenomenon of incommensurability (or the irrationality of the square root of 2) showed that this was inadequate. This provoked problems in the foundation of mathematics that were not resolved before the discovery of the proportion theory that we find in Book V of Euclid's *Elements.*[4]

Because Fowler realized the problems with the ancient account between the two editions of his book, he emphasizes three ways in which the standard account was misleading.[5]

First, it held that Greek geometry was a de-arithmetized version of Babylonian arithmetized geometry. The latter seemed "more normal" to moderns in the West, but the study of books 2 and 10 of Euclid's *Elements* proves that cannot be so. The geometrical approach was independent, non-arithmetized; thus incommensurability was *not* a foundational crisis in Greek mathematics. Remarking on "the supposed effects of the discovery of incommensurability," Fowler says "Far from being a period of crisis and confusion, the early fourth century was an extraordinary period of creativity, especially in Plato's circle; we have no historical evidence for any of the postulated difficulties of a 'foundation crisis.'"[6]

Second, Greek arithmetic had no notion of common fractions as previously thought, but proceeded by parts so, for example, "one ninth of 2 is $1/6^{th} + 1/18^{th}$." Third, the Greek notion of 'proportion' (*a* is to *b* as *c* is to *d*) differed from the notion of 'ratio'; and there were at least three competing definitions of 'ratio': from music theory, astronomy, and mathematics.

There is also physical evidence to undermine the standard account of late mathematical development. Before the middle of the sixth century—that is, a hundred years before Socrates was born—architectural drawings were exact:[7] not writing with a stylus on a wax-coated slate, but precisely planed marble blocks used in the construction of public

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[3] I was asked to talk about the role of biography and history in the interpretation of mathematical passages in Plato for this meeting, an area that I researched for my 2002 book, thus much of my text in part I is adapted from that source.
[4] Fowler 1999: 356. The point is crucial to new claims that distinguish his second edition from the 1987 one.
[5] My account of what is significant about Fowler's explanation is adapted from Fernando Gouvêa's review, 1999.
[6] Fowler 1999: 362, citing Freudenthal 1966 and Knorr 1975: 306–12.
[7] Coulton 1976; Haselberger 1985; Senseney 2011.

buildings, ships, and houses.[8] That physical fact confirms the manipulation of mathematics in advanced craftsmanship, but it does not tell us when formal proof was introduced for what was already intuited by working mathematicians—for example, that the intersection of two straight lines produces equal angles. The timing of the introduction of formal proof has been a sticking point in debates about the early evolution of Greek mathematics. Where mathematicians dominate the history of their subject, proof is moved back toward the sixth century, sometimes all the way back to Thales. Where philosophers dominate, proof is moved forward to the fourth century—a disparity of some two hundred years that gives mistaken pride of place to the mathematicians of Plato's Academy.

One of the reasons for the disciplinary disparity is that mathematicians have been generally more open to the notion that, in non-arithmetized geometry, diagrams can themselves *be* proofs.[9] Our earliest texts use the term διάγραμμα for 'diagram' and 'proof' interchangeably, and both Plato and Aristotle continue that practice.[10] The *Meno* and the *Theaetetus* illustrate geometrical proofs by the diagrammatic method. We might want to connect the notion of diagramming, drawing a diagram, to the notion of construction as Plato describes it at *Philebus* 56a–e.[11] Discussing measurement generally, Socrates says that "harmonies are found not by measurement but by the hit and miss of training … and observing vibrating strings," so music, "medicine, agriculture, navigation, and strategy" are all inferior to the building trade, which owes its "superior level of craftsmanship over other disciplines to its frequent use of measures and instruments, which give it high accuracy." Socrates credits the precise constructions to the straightedge, compass, stonemason's rule (which is actually a second type of compass), plum line, and carpenter's square.[12] The Greek text of the passage is included as Appendix I.

I concede that 'superiority' and 'high accuracy' are not the equivalents of proof, that they do not rise to the standard of apodicticity; and Socrates in *Republic* 7 expresses doubts about the methods of the mathematicians of his day. However, having just distinguished precise methods of construction from those that are haphazard, Socrates distinguishes arithmetic into more and less precise types, corresponding to abstract and concrete calculation: summing 5 + 7, say, compared to combining herds of cattle. I'll return to that mathematical distinction in Part 2.

The second episode that has worked to diminish Plato's mathematical stature is

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[8] Artmann (1994: 18) discusses sources including *Philebus* 56b.

[9] Diagrammatic *proofs* (not mere illustrations) have reemerged: cf. Brown 1999 and 2004. But philosophers have tiptoed; mathematicians have crowded the internet with examples. See Nelson 1993 and the burgeoning internet collection for the Mathematical Association of America.

[10] Fowler 1999: 33. Caveing (1996: 282): "…according to Vogt, 'Theodoros' lesson' was divided into two parts of which the geometrical one answers to the verb ἔγραφε [147d3], and the arithmetical one to the verb ἀποφαίνων: on the one hand mere constructions of lines, on the other logical proofs. But, according to classical Greek syntax, if a verb in the indicative mode is accompanied with another in the participle, the two ideas are linked, and the main one is borne by the participle, while the other points out only a modality of the action. So Plato means 'Theodoros proved by means of geometrical constructions…', that is the drawing of lines is part of the proof itself." Caveing is excellent for the history of the controversies.

[11] 56a3–e3, Burnet text, Dorothea Frede translation.

[12] At b9–c2 (κανόνι γὰρ οἶμαι καὶ τόρνῳ χρῆται καὶ διαβήτῃ καὶ στάθμῃ καί τινι προσαγωγίῳ κεκομψευμένῳ), both τόρνος and διαβήτης are normally translated 'compass', the former having been a string tied to a fixed point, the latter more like our own compasses, with two "outstretched legs"—LSJ).

described by Marcus Giaquinto:[13] In the nineteenth century, mathematicians were excited by their explorations into the limits of infinite processes that resisted visual imagination, and, as a result, geometrical intuition came to be regarded as suspect, and visual understanding was thought to be in conflict with analytic truths. There followed a period when Athenian mathematics was denigrated for its basis in geometry, reinforcing the ancient account that I called 'episode 1'. For an influential example, Burnet in 1911 deprecates Socrates's use of a diagram in the slave's lesson in *Meno*, calling it "opposed to … the process of good inquiry" and Malcolm Brown cites the note with approval.[14] That is the same Malcolm Brown who first realized and pointed out that, in the *Meno*, the slave's first two attempts to double the area of the square are arithmetized, but that Socrates switches to geometry with the slave's third attempt, saying, "If you don't want to count it up, just show me on the diagram." Fowler calls it "the only arithmetized passage … anywhere up to Archimedes and beyond."[15] The scene is admittedly fishy—as Aristotle understood—because Socrates and Meno know what the slave does not, that the hypotenuse is incommensurable with the side, so any round-number reply is bound to be wrong.[16] A host of arithmetized proofs for the theorem of Theodorus of Cyrene appeared in the heyday of episode 2—despite the clear text of the dialogue that Theodorus performed the proof by means of geometrical constructions.[17] Luckily—or naturally—the search for secure foundations for axiomatic systems spawned conflicting schools later in the twentieth century,[18] with Kurt Gödel most especially, so the dogma—that visual representations conflicted with analytic truths—did finally subside.

Working mathematicians were mostly not seduced by the theory of a two-hundred-year dark age, but some philosophers and historians of mathematics were, and episode three, in the twentieth century, cements the ancient and nineteenth-century episodes with a simple failure to pay attention to the history and biography that Plato regularly deploys. A Platonic character whose actual biography can aid our understanding is the Athenian mathematician, Theaetetus of Sunium, son of Euphronius.[19] In his case, bungling the history and biography misled Platonic scholarship and the history of mathematics as well.

We know exactly when the frame of the *Theaetetus* is set: in the spring of 391, in the house of Euclides in Megara, where Terpsion is visiting. Euclides has just returned from the harbor, where he has encountered Theaetetus—wounded and dying of dysentery—being carried home to Sunium from the Corinthian War. Euclides says that Socrates, himself now dead, had been deeply impressed by Theaetetus and had told him about a conversation he had had with the promising youth following a geometry lesson conducted by Theodorus. The conversation occurred shortly before Socrates's trial and execution in 399.[20] The

[13] 2007: 3–8. I rely here on Echterling's retelling.
[14] See Burnet's note to *Phaedo* 73a7; and Brown's approval at 1971: 204n.
[15] Fowler's *ad hoc* translation of *Meno* 84a1 at 1999: 366n12; Burnet text.
[16] Aristotle, *Prior Analytics* 41^a 26–30.
[17] See note 10.
[18] Giaquinto (2007: 6) notes the phases: (i) Carnap's conventionalism measured "convenience and truthfulness; there is neither need nor possibility of establishing the axioms true and the rules valid." (ii) Quine's holistic empiricism trumped conventionalism but did not distinguish math and science: "Even professional mathematicians must await the verdicts of empirical science before they can justifiably assert the truth of their mathematical beliefs." And Gödel (1964) reasserted intuitionism.
[19] Biographical material is adapted from Nails 2002.
[20] 142c5–6.

conversation was so memorable that Euclides took notes that he revised at his leisure, later confirming the details with Socrates on visits to Athens.[21] Terpsion wants to hear it, so a slave of Euclides reads the scroll that constitutes the dialogue's main action.

At that time, Theaetetus was *meirakion*, but on the young side, for he is not fully grown;[22] and Socrates says to the geometry master, Theodorus, "Well, you can see that all here are 'little boys' except you."[23] The little boy or adolescent, Theaetetus, died from his war wounds just eight years after that geometry lesson. No doubt Theaetetus was a great mathematician, some of his work codified in books 10 and 13 of Euclid's *Elements*. The first scholium to book 13 of Euclid states (to give the most famous example), that Theaetetus added the octahedron and icosahedron to the Pythagoreans' cube, pyramid, and dodecahedron for the total of five regular solids of the *Timaeus*.[24] However—as often happens with well-documented individuals from the ancient world—other mathematical discoveries whose authors are uncertain may later have been ascribed to the known Theaetetus. Historically, he was also credited with the two means of the *Timaeus*,[25] the mean of the *Parmenides*,[26] incommensurability in the *Meno* and *Theaetetus*,[27] rational and irrational cube roots in the *Theaetetus*,[28] and even continuous quantities. The provenance of these attributions is less certain; Pappus, citing Aristotle's student Eudemus, is an important source for crediting these various advances to Theaetetus himself.[29] It is perhaps understandable that scholars began to question whether Theaetetus could have accomplished all of that by 391.

In 1914, Eva Sachs produced a dissertation arguing that he could not. She found a later battle in Corinth, a famous one in 369, and attached Theaetetus's death to that one, giving him thirty years to move mathematics forward. Her argument was immediately and eagerly accepted: thirty years seemed more realistic than eight for so many discoveries. A second reason for the enthusiasm was that a later date neatly fit the then-growing developmentalist movement. From the eighteenth century, philosophers had a strong desire to make the *Theaetetus* the threshold for Plato's abandonment of forms as he "developed" philosophically and turned to the issues introduced in the *Statesman* and *Sophist*, dialogues with dramatic dates after the *Theaetetus* and with an overlap of characters.[30] Historians of mathematics were pleased to be able to locate and date ancient mathematical developments within the Academy itself, assuming a date of 387 for its establishment.[31] In the English-speaking world, two philosophers were key. Myles Burnyeat was one: he was attracted by a third

[21] 143a2–5.
[22] 155b7–c1.
[23] 168d8. Duke et al. text, Rowe translation.
[24] *Timaeus* 54d–55c.
[25] *Timaeus* 31b–32b.
[26] *Parmenides* 154b–d.
[27] *Theaetetus* 147d.
[28] *Theaetetus* 148a.
[29] Pappus 63–64. Eudemus *via* Pappus and Proclus is the chief primary source for the litany of mathematicians I mention briefly below.
[30] As John McDowell points out in the notes to his 1973 translation of *Theaetetus*, it is no good supposing that when Socrates reneges (183c) on his promise to discuss Parmenides (181a–b), he is merely postponing the discussion to the *Sophist.* The discussion of Parmenides in that dialogue is on a different subject.
[31] I have argued (2002: 248) that the date is too early, giving 383 instead—but that is irrelevant to Theaetetus; he was dead in either case.

reason: the poignant idea of Plato's writing a moving memorial dialogue for an Academic colleague with whom he had associated for nearly twenty years—and others were likewise moved. According to the other, Gregory Vlastos,[32] after writing his Socratic dialogues, Plato discovered advanced mathematics and was significantly changed by it; the encounter with the mathematics of Theaetetus marked Plato's philosophical turning point—a completely incredible view, given what we know of the education of Athenian males. It would be difficult to overemphasize the degree to which Socrates's generation was already immersed in the visual and spatial thinking involved in geometrical proof. But episode three, in short, eclipsed early Greek mathematics and neglected other mathematicians in the orbit of Socrates and Plato.

I pause to note, though this is not the place to argue at length, that the *Theaetetus* does not abandon the forms. Not at all. Forms are discussed at 185–86 as objects of knowledge—not of perception—"being and not-being, likeness and unlikeness, same and different, also things being one or having some number; … beautiful and ugly, good and bad … hardness … softness."[33] Further, Socrates urges Theaetetus to give an account of 'knowledge', *imitating* the one he had just given about the powers.[34] The significance of that is the approval it seems to demonstrate for linking the philosophical and mathematical methods. Moreover, the language of Socrates's dream, near the end of the dialogue, introduces terminology technical to mathematicians of the time.[35] Although some of the terms—*prōta, stoicheia, gnōstai,* and *rhētai*—are also philosophical terms,[36] it is implausible that Plato was unaware that the passage could double as mathematics. As Glenn Morrow pointed out, although the dream passage is introduced as an alternative formulation of Theaetetus's third definition of 'knowledge' it is "strangely enough … ignored in the refutation of that third proposal."[37] I am not claiming anything profound in this aside: only that the dialogue's focus on mathematics was not limited to Theodorus's geometry lesson, and that some mathematicals were candidate-forms.

What is important for my argument is to see that all three episodes supported the mistaken view that the mathematical achievements and discoveries attributed to Theaetetus were made during Plato's time, not Socrates's or even earlier—a claim based in part on the false premise that Theaetetus died in 369. That year for the death of Theaetetus raises four problems that are together insuperable, neatly enumerated by Holger Thesleff.[38] (i) Athens was *not* mustering 46-year-old academics for hoplite combat in 369, (ii) Theaetetus' skillful soldiering was far more likely to have been exhibited when he was of military age, 24, than at 46. (iii) Euclides's 30-kilometer walk, from which he has just returned as the dialogue's frame begins, is more likely for a man of 59 than for a man of 81. (iv) Socrates's remark that

[32] Vlastos 1991 is a consolidation of his views, but they had been appearing in seminars and articles since the 1980s.
[33] For forms, 185c9–11, 186a9, b2–3.
[34] 148d, referring back to 147e: "Just as you collected them, many as they are, in one class (ἑνὶ εἴδει), try, in the same way, to find one account (ἑνὶ λόγῳ) by which to speak of the many kinds of knowledge." McDowell (1973: 116) takes Socrates to be serious.
[35] 201e–202c … and central to the problems of Euclid's book X—the "cross of mathematicians" (Fowler 1999: 163).
[36] Others include *aloga, agnōsta, aisthēta, doxastai, sullabai, and alētheia doxa.*
[37] Morrow 1970: 309–33.
[38] The enumeration is explicit in Thesleff 1990: 149–50. For the martial skill and the comment on whether Theaetetus will live, see 142b–d.

seems prophetic to Euclides and Terpsion, that Theaetetus "would become someone to reckon with" if he lived to grow up, is appropriately applied to a man who dies before reaching 30, but not to one who reaches 46. Theaetetus died, rather, in 391, and the mathematics that fascinated Plato had been long established by then. The erroneous notion that Plato was a late-learner of mathematics enabled translators and commentators—when they failed to understand Plato's text—to call his mathematical illustrations merely playful, or imaginative jokes, or errors in his thinking, or—in some cases—outright dishonesty, ignoble lies.

A different way of opposing the idea that Plato learned mathematics late in life is to look at such evidence as we have of mathematicians in the milieu of Socrates and Plato. There are two stumbling blocks to that effort that I cannot overcome, given the current state of the physical record: (i) the extent to which persons called 'Pythagoreans' may have been *mathematikoi* as opposed to *akousmatikoi*; and (ii) the fact that some ancient reputations for mathematics are based on the titles of scrolls known only from sources decades or centuries after the deaths of their authors. Scores of mathematicians are named and quoted, but unverifiable. These are very wide error bars indeed, so a dose of salt will be needed to make the litany of mathematicians palatable.

I sketch the most famous ones, in support of the view that we should attribute to the sixth century the notion of diagrammatic proof and developments in the early history of Greek mathematics. In the deep background is Thales of Miletus, in the early sixth century. It is not so much what he demonstrated as that he demonstrated—that is, proved—two theorems that have been important in the mathematical tradition (as opposed to Aristotle's and other philosophers' interest in claims about the ubiquity of water). According to the tradition, "Thales' Theorem" proves that the diameter of a circle always subtends a right angle to any point on the circumference. Its converse enables us to find the center of any circle with a right-angled object such as the handy device called a 'carpenter's square' (or gnomon) that Socrates mentions in the *Philebus*. Second, Aristotle's student, Eudemus, attributes the "Intercept Theorem" to Thales, according to which, if two intersecting lines are intercepted by a pair of parallels, the line segments thereby produced are proportional. Both theorems had been in practical use since the eighth century in Egypt, and in the building trades locally, but the crux was Thales's proofs.

In European literature,[39] Pythagoras of Samos, in the later sixth century, is credited with discovering irrational magnitudes—the forerunner of what becomes the Pythagorean theorem: the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides—and several other theorems, for example, that the sum of the interior angles of a triangle equal two right angles. More important for the Platonic tradition is the general claim we find in Aristotle's brief account of Pythagoras,[40] that reality itself is mathematical in nature and can therefore be measured: numbers are first principles. For the Platonic tradition, Burnyeat takes up that theme when he explains "why mathematics is good for the soul," why mathematics must be studied as preparation for dialectic.[41] Nevertheless,

[39] Heath 1921. For an attempt to dismiss Pythagoras's having been a mathematician at all, see work by Carl Huffman.
[40] *Metaphysics* A 985b23–986a2.
[41] Burnyeat 2000.

the fact that we have nothing written by Pythagoras complicates any proper historical account.

Plato sets the main action of the *Parmenides* in 450, when a youthful Socrates is discussing philosophy with Parmenides and Zeno—and not for the first time. Although we have no evidence that Zeno's paradoxes of time, space, and motion were taught to Athenian schoolboys, I would nevertheless count Zeno as a mathematician *qua* logician who had an effect on philosophical debates long before Plato. It was Zeno, after all, who demonstrated the limitations of Pythagorean pebble-arithmetic, generating paradoxes by juxtaposing the Parmenidean continuum against Pythagorean numbers conceived as discrete.

Diogenes says that Socrates's older contemporary, Protagoras,[42] wrote an *On Mathematics.* If so, it is lost. But Protagoras' friend, Theodorus, *was* a historical mathema–tician whose discoveries had been made by about 440.[43] In Plato's dialogue, Theodorus draws a diagram to address the issue of incommensurability by demonstrating what we would now call irrational numbers[44] at a time when both the concept and theorem necessary to prove similar rectangles by the method of anthyphairesis were available to him, and long before the Academy was established. Theodorus' demonstration is interpolated in the text of Euclid at 10.117.[45] Continuing with the generation preceding Socrates', there is also Bryson of Heraclea Pontica, who thought he had squared the circle.[46]

I come now to a few of the rough contemporaries of *Socrates*, bearing in mind that Socrates lived nearly two generations before Plato, whose command of mathematics is at issue in the three episodes. (i) Hippocrates of Chios[47] was first to compile an *Elements* (Στοιχεῖα) of geometry—the systematic introduction of definitions and axioms, to which he also contributed the method of reduction.[48] But he was a practicing mathematician as well, working on the major puzzles of his era: squaring the circle, and duplicating the cube. Toward squaring the circle, Hippocrates devised a step that would later help to solve the problem, though it is impossible to solve with compass and straightedge alone. (ii) Philolaus of Croton was a prominent Pythagorean and had more claim to the label 'mathematician' than most.[49] Fragments of his *On Nature* describe all things as composed of limiters and limiteds structured by mathematical principles. He shows up in Plato as the Pythagorean whom Simmias and Cebes of Thebes, and Echecrates of Phlius, have recently been frequenting at the time of Socrates' execution 'though the context of the *Phaedo* is metempsychosis. (iii) Hippias of Elis is introduced as a mathematician in the *Protagoras* and in the Platonic *Greater* and *Lesser Hippias* dialogues, so the weight of scholarship has attributed to Hippias the discovery of the curve called the *quadratrix*

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[42] ±490–420.
[43] Artmann 1994: 22.
[44] *Theaetetus* 147d.
[45] Thomas 1991: 110na.
[46] Circumscribe a square around a circle, and another square within the circle; the circle is then *intermediate* between the inscribed and the escribed squares (Heath 1921: 1.223–4).
[47] ±470–±410.
[48] Reduction involves recognizing that an intractable problem is an instance of some more general problem that can be solved more easily, solving the general problem, then applying the solution to the previously intractable one.
[49] ±470–±385.

for trisecting an angle,[50] which Dinostratus used later to square the circle. The *quadratrix* was the first curve to be plotted point by point using a sliding apparatus rather than a compass and straightedge. Far more famous among the Socratic contemporaries is (iv) Democritus of Abdera.[51] Morrow showed that the passage on hypothesis in Plato's *Meno* implies knowledge of conic sections,[52] though philosophers have generally put that discovery later, despite the evidence that Democritus knew that a cone holds one-third the volume of a cylinder with the same base and height.[53] I take this example as evidence that the results of mathematical investigations were not cabined geographically and that Plato knew (and embedded in his dialogues) what would have been recent mathematical developments.

We reach now Plato's contemporaries, though their lives overlapped his at both ends. From time to time, scholars speculate that the archaic Hekademia, which became the site of the Academy after Plato's return from his first voyage to Sicily, had already for some time been a gathering place for mathematical devotees. Pappus embeds some of Eudemus' late fourth century BCE account of the history of mathematics; Proclus's summary at the head of Euclid's *Elements* is probably based on Eudemus. The summary is chronological and supports the notion of a pre-Academy site for mathematicians in that it is only after naming several of them that Eudoxus of Cnidos is explicitly identified as an associate of Plato's Academy proper.

The "first generation" who apparently predated the formal Academy are (i) Leodamas of Thasos; (ii) Archytas of Tarentum, who reigned 366–350 BCE; (iii) Theaetetus—further evidence that Theaetetus was dead before the founding of the Academy; by my estimation, he had been dead about seven years. Although not named by Proclus, (iv) Archedemus of Syracuse (known from Plato's seventh letter) was a Pythagorean mathematician associated with Archytas. Boethius quotes Archytas' proof against the division of a superparticular ratio into equal parts;[54] and Eutocius, citing Eudemus, preserves Archytas' solution to doubling the area of a cube.[55] Porphyry's *Commentary on Ptolemy's Harmonics* gives Archytas' discussion of the three senses of 'mean': arithmetic, geometric, and harmonic (or subcontrary). Younger was (v) Neoclides, whose student (vi) Leon compiled an *Elements* and is sometimes credited with the knowledge of conic sections that would have been required to produce the passage on hypothesis in the *Meno.* Speusippus is *not* mentioned though Neoclides would have been about his age.

Mathematicians of Plato's age and younger are listed next, first among them (vii) Eudoxus—the first academician, strictly speaking, and a foremost mathematician of the fourth century.[56] The scholiast of book 5 of Euclid's *Elements*[57] credits Eudoxus with the development of the general theory of proportion. He also developed a method of approach to

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[50] Proclus, *Commentary on Euclid* 67.2.
[51] ≥460–±370.
[52] *Meno* 86e4–87b2.
[53] 1970: 313.
[54] *On Music* 3.11.
[55] *Commentary on Archimedes' Sphere and Cylinder* 84.12–88.2.
[56] ≤390–≥340.
[57] 280.1–9.

the limit, using inscribed polygons, that became the standard method for avoiding infinitesimals; Archimedes, *On the Sphere and Cylinder* cites Eudoxus' demonstrations, based on that method, that any pyramid (and any cone) is a third part of the prism (or the cylinder) having the same base and equal height;[58] as we have seen, Democritus and Neoclides have also been credited with the discovery—but of course it could have been discovered independently in that heady time for mathematics. Eudoxus' most stunning legacy is a geometrical model of the apparent motions of the sun, moon, and planets in homocentric spheres that was able to show retrograde motion[59] and was not overturned before Kepler. He shares credit with Archytas for developing a method for doubling the area of a cube, though the details of Eudoxus' role are obscure. Eudoxus had a school at Cyzicus that is sometimes said to have combined with Plato's Academy. Eudoxus is said to have been *scholarch* at the Academy during Plato's Sicilian absence in 366.[60]

Proclus's summary continues with what appear to be others of Eudoxus' generation (viii) Amyclas, (ix) Menaechmus and his brother (x) Dinostratus, (xi) Theudius of Magnesia-—who also produced an *Elements.* This is where Aristotle would have appeared if he had been making mathematical discoveries. Then the youngest age group, (xii) Hermotimus of Colophon and (xiii) Philip of Opus. Although Proclus provides additional details about the mathematical contributions of most of these men, I mean only to show the implausibility of the claim that fourth century mathematics was a time of crisis that left Plato confused.

Part 2. ***Some evidence that Plato was more mathematically astute than scholars have previously acknowledged: forms and method.***

I have so far argued that much of the mathematics Plato used in the dialogues, including incommensurability, was so well-established by the time of Socrates' youth that Athenian males would have been familiar with it in school. I do not call Plato a working mathematician, but I think he deployed mathematical proofs, *diagramma*, from an understanding more often associated with father Parmenides and some Pythagoreans that everything is subject to measurement, a claim with ontological and epistemological implications.

Terence Echterling's recent work on Plato's consistent use of mathematics—and diagrams in particular—in the *Republic, Meno, Theaetetus,* and *Timaeus* has increased my respect for Plato's project.[61] Echterling has shown that intermediate steps in constructing a figure often reveal connections not apparent in mere algebraic manipulation of symbols in the modern formal sense. So, for example, a divided line produced with a compass and straightedge requires an internal triangle that visibly resolves what scholars have assumed was either a mistake or a deliberate inconsistency in Plato's own account:[62] that the two

58 Preface to book 1.
59 Aristotle, *Metaphysics* Λ 8 1073b17–32.
60 Philochorus, fr. 223.
61 Echterling 2018, 2019.
62 Foley for a mistake (a contradiction); Smith for deliberate avoidance of perfection.

interior segments of the line are equal in length, yet that each segment exhibits greater clarity and truth than the one below. The triangle illustrates the greater area of the mathematicals than the sensible objects.

Platonic ontology is also implicated when mathematics is understood to be more than the manipulation of numbers, lines, and surfaces. Specifically, quoting Echterling, "if no mathematicals were forms for Plato himself, important criticisms by Aristotle would miss their mark completely, and that is implausible." The nub of that criticism is this, from *Metaphysics* A 6:

> ...besides sensible things and Forms [Plato] says there are the objects of mathematics, which occupy an intermediate position, differing from sensible things in being eternal and unchangeable, from Forms in that there are many alike, while the Form itself is in each case unique.[63]

In the Apostle translation, the expedient is adopted of distinguishing the two senses of *arithmos:* lower-case 'numbers' measured in units or discrete quantities versus uppercase 'Numbers' generated by the One and the Dyad. Kenneth Sayre's work likewise seeks to make Platonic philosophy compatible with Aristotle's testimony. Naturally, however, the Aristotelian criticisms continue to ignite controversy. Last week, in fact, the *Plato Journal* of the International Plato Society published seven new papers addressing those criticisms. Introducing the papers on mathematical intermediates, Nicholas Baima says,

> Despite Aristotle's testimony, no place in Plato's corpus do we find an explicit endorsement of intermediate objects. Scholars, thus, face a choice: they can either accept the testimony and mine the corpus for places where Plato might implicitly endorse such a thesis or they can argue that Aristotle was confused and that Plato doesn't think intermediate objects exist.

Count me among those who do not attribute confusion to Aristotle, so by default I am a "miner of the corpus." Plato's division of numbers into concrete and abstract in the *Philebus* is something that Aristotle may have had in mind.

I suggest what one might call a further division of the divided line, one that acknowledges that numbers are not all alike and that some numbers are forms. Precedents for subdividing the forms are not absent from our inherited, vast literature. We might say that Parmenides was first, asking young Socrates about the just, the beautiful and the good; then about human, fire, and water; and finally about hair, mud, and dirt.[64] In *Republic* 6, Socrates dissuades Glaucon from continuing to work out all the ratios of the divided line because it would take too much time.[65] The guest from Elea in the *Sophist* deliberately sets aside hot and cold, justice and injustice, to discuss the most important kinds, the ones that naturally combine;[66] he says "[t]he people who came before us were thoughtless and lazy about

[63] Ross edition and translation of 987b14–17. Cf. A 6 987b27–28.
[64] *Parmenides* 130b7–d1.
[65] 534a5–8.
[66] *Sophist* 250a2 (hot-cold), 247b1–2 (in/justice), and 254b8–d2 for the most important ones.

dividing kinds into types, and so they never even tried to divide them."[67] Once one begins looking, one finds scholars saying that one form is of a different kind from another; or that a hierarchy of forms is suggested by some text. There have been other divisions as well, some based on philological considerations.[68]

I cannot claim that Plato was systematic in subdividing the forms, or that he was certain about them all. Some forms—motion and rest, sameness and difference, hot and cold, and other such pairs—are extremes of continua. Other forms, however, are mutually exclusive, for example, the odd and the even, equal and unequal, finite and infinite, primes. While the odd and the even are forms in the *Phaedo* at 103d–104b, and in the *Theaetetus* at 198a7–8, they are posits of mathematicians in the *Republic's* divided line passage.[69] The equal is paradigmatically a form with its own proof in the *Phaedo*.[70] The one (τὸ ἕν) is a candidate Platonic form in the *Parmenides* and *Sophist*;[71] and Lloyd Gerson cites Sextus Empiricus, for the similar claim that Plato distinguished two senses of 'one': first principle (πρώτη μόνας) and first number (τὸ ἐν τοῖς ἀριθμοῖς ἕν).[72]

While Socrates in the *Republic* distinguishes the inferior methods of geometers from the superior methods of dialecticians, if I am right in agreeing with Hugh Benson that the failing was in the practice—not the subject matter—of mathematics, then we are warranted in gathering some of the techniques of the mathematicians under the umbrella term 'dialectical method'—our most promising means of achieving such "pieces of knowledge"[73] as are possible for mortals. I take 'dialectical method' to be a flexible term applied to the dialogues. It is a bootstrapping method, a piecemeal method, the various techniques of which we use when we do not already have knowledge but desire it and seek it systematically.

Plato uses some of the mathematicians' methods often enough to ensure that we ought to take them as components of the dialectical method. Burnyeat has made much of the first: (i) the crucial relationship between definition in mathematics and philosophy.[74] Morrow has explored further similarities between the *elenchus* and the procedures of the mathematicians:

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[67] ἦ δῆλον δὴ χαλεπὸν ὄν, διότι τῆς τῶν γενῶν κατ' εἴδη διαιρέσεως παλαιά τις, ὡς ἔοικεν, ἀργία τοῖς ἔμπροσθεν καὶ ἀσύννους παρῆν, ὥστε μηδ' ἐπιχειρεῖν μηδένα διαιρεῖσθαι (*Sophist* 267d5–8, tr. White).

[68] Dan Devereux (1994) distinguishes "forms themselves" from "inherent forms" and notes that Socrates "argues that, in addition to the forms which always are and are unchanging, there are forms in us, which can flee or be destroyed when their opposite approaches (102b–103c)" at 72n19. Discussing forms in the *Phaedo,* he distinguishes ἰδέα for perishable entities such as heat and cold that might be described as immanent, from εἶδος for imperishable forms, which he takes to be separate from sensible particulars from the *Phaedo* onward. Holger Thesleff 1999, argues that, "all Ideas can be termed Forms, but relatively few Forms can be termed Ideas" (63). He identifies εἶδος as the preferred term for what he calls "conceptual Forms," which he takes to be "opposite to Ideas proper," with examples from the *Phaedo*: ἀνομοιότης and τὸ ἴσον, σμικρότης and μέγεθος. Thesleff is more concerned with the term ἰδέα, often accompanied by ἀυτὸ ὃ ἔστι, and reserved for such value terms as ἀρετή, τὸ καλόν, τὸ ἀγαθόν, and δικαιοσύνη. A third sort of form is denoted by γενός—Plato's term for 'type' or 'kind' in the *Sophist.* Thesleff concedes that this tripartition is not discussed or defended by Plato.

[69] What the mathematicians posit may well be forms; the problem is the attitude of the mathematicians, hence Benson's 2011 title, "The Problem is not Mathematics, but Mathematicians."

[70] 72e1–77a5.

[71] *Parmenides* 129d3, 5, 8–e1, and *Sophist* 251b7–8.

[72] Echterling 2018: 12–13 citing Gerson 2013, 121n95 who cites *Against the Dogmatists* 10.276.

[73] McDowell's translation, throughout the aviary section of the *Theaetetus*, for ἐπιστῆμαι; cf. the Rowe translation of *Symposium* 207e6, reserving 'branches of knowledge' or 'sciences' for μαθήματα. See Benson 2012 and 2015.

[74] Burnyeat 1990 and 2000.

(ii) Socrates insists on deductive implication, tracing the consequences of common opinions, even in practical matters; (iii) avoidance of contradiction; and (iv) methodical, sometimes tedious, demonstration.[75] Most philosophers have shied away from saying that (v) the method of hypothesis was another key way in which Socrates' practice was like that of the mathematicians, but it is explicitly introduced in the *Meno* as a geometer's method.[76]

In conclusion, there is much to gain from rejecting the strict divisions of the divided line—though I say that having been bedazzled by the line for most of my life. If we accept Plato's invitation to divide the line further by the same ratios, including the suggestion that numbers are of two sorts, Aristotle's description falls into place, even as he himself remains critical of Plato's approach. *Merci.*

Appendix I: *Philebus* 56a3–e3

ΣΩ. Οὐκοῦν μεστὴ μέν που μουσικὴ πρῶτον, τὸ σύμφωνον ἁρμόττουσα οὐ μέτρῳ ἀλλὰ μελέτης στοχασμῷ, καὶ σύμπασα αὐτῆς αὐλητική, τὸ μέτρον ἑκάστης χορδῆς τῷ στοχάζεσθαι φερομένης θηρεύουσα, ὥστε πολὺ μεμειγμένον ἔχειν τὸ μὴ σαφές, σμικρὸν δὲ τὸ βέβαιον.

ΠΡΩ. Ἀληθέστατα.

ΣΩ. Καὶ μὴν ἰατρικήν τε καὶ γεωργίαν καὶ κυβερνητικὴν καὶ στρατηγικὴν ὡσαύτως εὑρήσομεν ἐχούσας.

ΠΡΩ. Καὶ πάνυ γε.

ΣΩ. Τεκτονικὴν δέ γε οἶμαι πλείστοις μέτροις τε καὶ ὀργάνοις χρωμένην τὰ πολλὴν ἀκρίβειαν αὐτῇ πορίζοντα τεχνικωτέραν τῶν πολλῶν ἐπιστημῶν παρέχεται.

ΠΡΩ. Πῇ;

ΣΩ. Κατά τε ναυπηγίαν καὶ κατ' οἰκοδομίαν καὶ ἐν πολλοῖς ἄλλοις τῆς ξυλουργικῆς. κανόνι γὰρ οἶμαι καὶ τόρνῳ χρῆται καὶ διαβήτῃ καὶ στάθμῃ καί τινι προσαγωγίῳ κεκομψευμένῳ.

ΠΡΩ. Καὶ πάνυ γε, ὦ Σώκρατες, ὀρθῶς λέγεις.

ΣΩ. Θῶμεν τοίνυν διχῇ τὰς λεγομένας τέχνας, τὰς μὲν μουσικῇ συνεπομένας ἐν τοῖς ἔργοις ἐλάττονος ἀκριβείας μετισχούσας, τὰς δὲ τεκτονικῇ πλείονος.

ΠΡΩ. Κείσθω.

ΣΩ. Τούτων δὲ ταύτας ἀκριβεστάτας εἶναι τέχνας, ἃς νυνδὴ πρώτας εἴπομεν.

ΠΡΩ. Ἀριθμητικὴν φαίνῃ μοι λέγειν καὶ ὅσας μετὰ ταύτης τέχνας ἐφθέγξω νυνδή.

ΣΩ. Πάνυ μὲν οὖν. ἀλλ', ὦ Πρώταρχε, ἆρ' οὐ διττὰς αὖ καὶ ταύτας λεκτέον; ἢ πῶς;

ΠΡΩ. Ποίας δὴ λέγεις;

ΣΩ. Ἀριθμητικὴν πρῶτον ἆρ' οὐκ ἄλλην μέν τινα τὴν τῶν πολλῶν φατέον, ἄλλην δ' αὖ τὴν τῶν φιλοσοφούντων;

ΠΡΩ. Πῇ ποτε διορισάμενος οὖν ἄλλην, τὴν δὲ ἄλλην θείη τις ἂν ἀριθμητικήν;

ΣΩ. Οὐ σμικρὸς ὅρος, ὦ Πρώταρχε. οἱ μὲν γάρ που μονάδας ἀνίσους καταριθμοῦνται τῶν περὶ ἀριθμόν, οἷον στρατόπεδα δύο καὶ βοῦς δύο καὶ δύο τὰ σμικρότατα ἢ καὶ τὰ

[75] Morrow 1970: 319–20.
[76] *Meno* 86e4–87b2.

πάντων μέγιστα· οἱ δ' οὐκ ἄν ποτε αὐτοῖς συνακολουθήσειαν, εἰ μὴ μονάδα μονάδος ἑκάστης τῶν μυρίων μηδεμίαν ἄλλην ἄλλης διαφέρουσάν τις θήσει.

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