Theory of Ratios in Euclid's *Elements* Book V revisited

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Introduction

The book V of Euclid's *Element* contains the most celebrated theory of ancient Greek mathematics, a general theory of ratios. There is a huge literature about its definition 5 and it has been intensively studied till nowadays by historians of mathematics. In mathematics, it is not so common for a definition, not a theorem, to be at the foundation of a global theory which, according to some mathematicians, almost contains Dedekind's construction of real numbers written more than 2000 years afterward. But there is a pitfall, the difficulty to understand and explain the three lines of its statement.

In the first part of this article we show the definition can be understood in a very natural and simple way and its origin is traceable to early Greek arithmetic. In the second part, we tackled the principal difficulties found and raised after the Antiquity by mathematicians and/or in modern times by historians of science. We show how this analysis solves most of these difficulties which are therefore due not to Euclid but to the modern exegeses. First and foremost it solves the most puzzling riddle, the absence of any criticisms till the end of the Antiquity.

Finally our reconstruction is helpful to settle the highly disputed debate concerning the hypothetical existence of general theory(ies) of ratios prior the one in the *Elements*. For lack of space we will develop this point in a next article[1].

Without other explicit reference, the translations from Euclid's *Elements* are excerpted from Richard Fitzpatrick's ([Euclid2007]).

[1] The origin of the present article is a course given at Bologna in September 2013 (cf. [Ofman2013]).

Part I. A reinterpretation

i. The definition

As T. Heath noted the book V of the *Elements* is founded on a definition, the famous definition 5[2]. This very definition was **criticized** till modern times[3]. And not without reasons: it appears at once excessively obscure, absurdly verbose and extremely counter-intuitive[4], whereas the arithmetical notion of 'ratio' of integers ('ἀριθμοί') seems so natural. A literal translation in modern terms is not easily understandable and moreover there are several possible logical statements accounting for it. And T. Heath has to admit to have to compromise 'between an attempted literal translation and the more one expanded version' ([Euclid1908], II, p. 120). The result is as follows (ib., p. 114):

Definition 5. *Magnitudes are said to be* ***in the same ratio*** (' 'Εν τῷ αὐτῷ λόγῳ'), *the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.*

Not unreasonably the translator thinks such a definition needs some defense. For this purpose he uses a model given by De Morgan[5], 'the best defense and explanation of the definition (…) seen' (ib., p. 121-122)[6], which however needs almost three pages and a diagram for its explanations.

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[2] 'The Euclidean definition is regularly appealed to in book V, as the criterion of magnitudes being in ratio, and the use of it would appear to constitute the *whole essence* of the new general theory of ratio' ([Euclid1908], II, p. 121; our italics).

[3] Roughly speaking till the 19th century and the definition of real numbers by Dedekind (cf. [Euclid1908], II, p. 121; [Knorr2001], p. 124) many authors thought necessary to give a reformed definition (for instance Galileo [Galilei1890], VIII, p. 349–362 or Clavius in his commentary of Euclid's *Elements*). It was not restricted to the occidental world, and even much earlier we find critics in the eastern world (cf. [Rash-Vaha1999], p. 340-346 for instance). Since then, it was succeeded by the opposite point of view, the definition being even considered as almost identical to Dedekind's definition itself (cf. for instance [Heath1921]. For a more recent claim, cf. R. P. Langlands' lecture at the Princeton Ins. Adv. Studies, November 26th 1999; in line http://www.math.duke.edu/langlands/Part1.pdf). Though most historians reject such a position as totally anachronistic (nevertheless cf. *infra*, note 22). Cf. also [Caveing1998], p. 268.

[4] Even for mathematicians as for instance D. Fowler who claims: 'Definition 5 (…) as a description, is almost impenetrable' adding 'though its latent power and scope are enormous.' ([Fowler1979], p. 813).

[5] Heath refers to some articles of De Morgan in the *Penny Cyclopedia*, 19, 1841.

[6] It is re-used by B. Vitrac with the same praises on its pedagogical character ([Euclide1991], p. 45).

One of the difficulties for a translation into modern symbolism is it can be accounted through several different (more or less equivalent) logical statements[7]. It is probably the reason why T. Heath contrary to what he used to do, does not give such an interpretation here.

The usual account in modern symbolism is as follows (using the '/' for the relation of 'ratio between magnitudes'[8]):

Let *a, b, c, d* be magnitudes; to say $a/b = c/d$ i.e. to **define** the equality[9] between 'a/b' and 'c/d' (keeping in mind the precise meaning of what 'are' the ratios 'a/b' and 'c/d' is not known) means:

for any (positive) integers m and n we have simultaneously[10]:

$ma > nb$ and $mc > nd$
or
$ma = nb$ and $mc = nd$
or
$ma < nb$ and $mc < nd$.

[7] It results partly from the non-independency of the different order relations but also from the (illogical) mathematicians' habit to use the same terms for different meanings depending of the context. Even in contemporary mathematics, the sufficient condition ('if') is used as the equivalence condition ('if and only if') in many cases, especially in definitions. We will follow for that matter this tradition later in this article.

[8] Modern historians of mathematics tend to avoid the symbols '/' (and '=' as well) in order to prevent any anachronism. For the clarity of the presentation for the modern reader we will continue to use modern symbolism. Nevertheless, as we will see, the terminological difference is not only the result of some conceptual differences between ancient and modern mathematics, but it has a deeper (purely) mathematical sense. We will also use the symbol '≔' to emphasize the equality of the identity is a definition (for instance '2x ≔ x + x' means not only the equality of '2x' with 'x + x', but it is the very definition of '2x').

[9] Euclid does not define the **equality** of two 'ratios' and he does not use the term 'equal' ('ἴσος'), probably because it would imply ratios are some quantities while it is a relation (definition V.3). He says either 'to be in' ('ἐν τῷ αὐτῷ λόγῳ') or 'to have' the same ratio or to form a ratio ('ανάλόγον'). Thus the vocabulary is the one of the relation and not the quantity as 'ποια σχέσις' (translated by Heath as 'a sort of relation') implies. Aristotle's opposition between the 'categories' ('κατηγορίαι') of 'quantity' ('πόσον' which regroups integers as well as magnitudes), of 'relation' and of 'quality' ('ποίοτης') does not mean Euclid is following Aristotle's lead (cf. [Mueller1991], p. 91) for, as the Stagirite himself emphasizes, this opposition was a common opinion in ancient Greece (*Categories*, 6, 6a30-36). In this paragraph as in the next one where we freely make use of modern notions, we may use the terms as 'equal' and 'equality' for ratios as well.

[10] T. Heath rejects this term (as translation of the Greek word 'ἅμα') in favor of 'alike' because time is certainly not involved ([Euclid1908], II, p. 120). This word has however a mathematical meaning without any temporal connotations so that it is well suitable here (see also [Euclide1991], II, p. 41, note 28).

ii. A modern perspective

Even in such a symbolic modern writing[11], it is not so easy to understand the definition[12]. The conditions of the previous paragraph:

$ma > nb$ and $mc > nd$
or
$ma = nb$ and $mc = nd$
or
$ma < nb$ and $mc < nd$

can be rewritten as follows: for any (positive) integers m and n we have simultaneously:

$m/n > b/a$ and $m/n > d/c$
or
$m/n = b/a$ and $m/n = d/c$
or
$m/n < b/a$ and $m/n < d/c$.

Then forgetting any formal and historical precautions, the above conditions may be re-written as follows:

$b/a = d/c$ if for any (positive) 'rational number' α, we have:

($\alpha > b/a$ and $\alpha > d/c$) or ($\alpha = b/a$ and $\alpha = d/c$) or ($\alpha < b/a$ and $\alpha < d/c$).

The middle identity has a special significance since it is the case of commensurability[13]. So it can be put apart and we get:

- either both the ratios (a/b) and (c/d) are equal to α (*)
- or restating negatively the first and third conditions and leaving aside the cases of equality settled by (*), we get: there is no rational number α such that

($a/b > \alpha$ or $c/d > \alpha$) and ($a/b < \alpha$ or $c/d < \alpha$)[14],

and by distributivity of the operator 'and' in relation to 'or', we finally obtain:

($a/b > \alpha$ and $c/d < \alpha$) or ($c/d > \alpha$ and $a/b < \alpha$). (**).

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[11] It is not (cannot be) identical to Euclid's expression, if only because there are other possible (logically equivalent or not) writings of this statement, for instance by replacing the operator 'and' by the implication '⇒' or the equivalence '⇔' (cf. for example the proof of the proposition V.11; see also [Euclid1908], II, p. 159, [Euclide1991], II, note 52, p. 91). It is an argument used by translators to stay as close as possible from the original text, but sometimes at the cost of the clarity (for this problem as tackled by T. Heath, cf. [Euclid1908], p. 120). And it is especially problematic in such a case as the book V whose mathematical analysis is far from being easy and where Euclid uses different logical expressions (not always logically equivalent) for the same statement in the proofs of the propositions.

[12] For a modern opinion, cf. B. Vitrac complaining of the 'unintuitive character of these definitions [V.5 to V.7]' ([Euclide1991], II, p. 47).

[13] Till the Part II, we may consider the definition of commensurability given by equality to a ratio of integers i.e. a/b is commensurable if there exist m and n integers such that $a/b = m/n$ i.e. $na = mb$.

[14] Since the cases '$x/y < \alpha$', '$x/y = \alpha$', '$x/y > \alpha$' are mutually exclusive.

From (*) and (**) we get: the equality of the ratios $a/b = c/d$ means:

- either there is a rational number α such that $a/b = \alpha$ and $c/d = \alpha$
- or there is no rational number separating a/b and c/d.

Since between two different rational numbers there is always another one separating them, the first condition is a particular case of the second. And we obtain:

Two ratios[15] are equal if there is no rational number separating them.

[15] We need to emphasize we do not know what the general ratios are. But we know one thing, in the language used in this paragraph, the rational numbers are such ratios (in other words they form a subset inside the general ratios; cf. also *infra*, notes 22 and 58).

iii. The reinterpretation of definition V.5

To formulate the above definition in a wording closer to the one in Euclid, we need to replace modern 'rational numbers' by 'ratio of integers'. Then to avoid too many notations, we will use, like in the previous paragraph, the same symbols '>' and '=' for the order and equality on magnitudes as well as for ratios. There is no ambiguity since magnitudes and ratios are different objects.[16].
We will also distinguish between some modern symbols or notions not found in Euclid (or in ancient Greek mathematics) from the ones in the book V (or elsewhere in the *Elements*).

1) Firstly, as in the previous paragraph, for any **general ratio** a/b and any **ratio of integers** m/n we put (definition):

   $a/b > m/n$ if $na > mb$ (and of course $a/b < m/n$ when $na < mb$),
   $a/b = m/n$ if $na = mb$ and $a/b \neq m/n$ if $na \neq mb$.

2) To compare two general ratios a/b and c/d according to book V (definition 5 and 7), we will use Euclid's wording and say a/b is 'the same' (or has 'the same ratio') as c/d (or still a, b, c, d are 'analogous') if they satisfy the condition of definition V.5. We will say a/b is 'greater than' c/d if they satisfy the condition V.7 and (even if it is not explicitly defined by Euclid) a/b is 'less than' c/d if c/d is 'greater than' a/b.

As we see the relations defined in 1) and 2) are certainly different for they are not defined on the same objects. While the second is defined on any couple of general ratios (i.e. ratios of homogeneous magnitudes) the first is defined only on couples formed by a general ratio and a ratio of integers.

Now as previously if a/b and c/d are two general ratios, we will say that a/b and c/d are **separated**[17] by a ratio of integers m/n (or m/n is 'between' a/b and c/d) if the following inequalities hold:
$a/b > m/n > c/d$ or $c/d > m/n > a/b$.

Obviously the property of separation for rational numbers is replaced by: for any ratios of integers m/n and p/q such that m/n and p/q are unequal (i.e. $m/n \neq p/q$[18]) there exists another ratio of integers r/s separating m/n and p/q[19].

Therefore according to the previous paragraph, the definition V.5 means simply:

two ratios of magnitudes are the same if **there is no ratio of integers separating them**[20]**.**

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[16] It is one of the many 'abuses of language' permeating all the mathematics.
[17] Between integers Euclid uses the term 'ἐμπίπτειν' ('falling in') as for instance in propositions 8, 9, 10 in book VIII.
[18] Meaning by definition $mq \neq np$.
[19] This is an arithmetical property. As such it does not depend of the general theory of ratios as given in book V, but of theory in the so-called arithmetical books (books VII to IX of Euclid's *Elements*). In terms of fractions it is easy to obtain it as follows: multiplying (if needed) both n and m (respectively q and p) by the same integer, we may assume $n = q$ and they are even. Let $r = (m+p)/2$ be the arithmetical mean of m and p and $s = p = q$. Since $m/n \neq p/q$, we have $m \neq p$ and either $p > r > m$ or $m > r > p$, so that either $p/q > r/s > m/n$ or $m/n > r/s > p/q$. Thus in both cases r/s separates m/n and p/q. The case of ratios is a little trickier and we will return to the question in §II.i.
[20] A particular situation can be found in [Caveing1998], p. 226-227. Once again we have to be careful when using modern notions since a 'rational number' is certainly different from a 'ratio of integers'. The former is a **class** containing an infinite

Along the same lines, the definition 7 gives the meaning for a ratio to be 'greater than' another:

Definition 7: *And when for equal multiples the multiple of the first exceeds the multiple of the second, and the multiple of the third does not exceed the multiple of the fourth, then the first is said to have a greater ratio to the second than the third to the fourth*.

Following the above analysis, its meaning is as evident as the previous one:

two ratios are such that one is 'greater than' the other if there is a ratio of integers between these first two ratios.

On a drawing, it may be represented as follows: If two ratios of magnitudes (having a ratio i.e. satisfying definitions 3 and 4!) are different, there is a ratios of integer *m/n*, such that[21]:

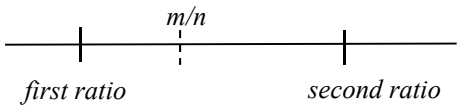

Figure 1

In this case, the second ratio is 'greater than' the first. And if no such ratios of integers exist, the two given ratios are the same.

In other words, the fundamental idea of the theory of ratios in Euclid's book V is the possibility for any general ratio to be **approximated by ratios of integers**[22]. The separation property is the fundamental property for the general theory of ratios. The seemingly 'impenetrable' theory of ratios, in particular the definitions V.5 and V.7, has then a very simple interpretation.

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number of the latters. Thus, even in modern mathematics, the definition of a rational number, *a fortiori* of an arbitrary number (in the broad sense of 'number') is not so simple.

[21] In modern terms, *m/n* defines or belongs to the class α which is the rational number of the previous paragraph (cf. *supra*, note 20).

[22] In modern terms the meaning is as follows: The positive rational numbers are dense in the set of the ratios of magnitudes. The latter is not the **set** of the real (positive) numbers, if only because magnitudes, and *a fortiori* their ratios, are not defined/constructed in Euclid's *Elements*. But such an indetermination is already true for the 'magnitudes': we know certainly some kinds of magnitudes (for instance 'lengths'), some of their properties (for instance their ability to be infinitely doubled), but there is no general construction. For instance the notion of transcendental (i.e. non-algebraic) magnitudes is defined much later and a proof of their existence was given by Liouville in the middle of the 19th century. The theory of book V is applied to indeterminate sets of indeterminate things, the 'homogeneous magnitudes' and their 'ratios', not to a defined set as the set of the real numbers.

iv. The consistency of the interpretation.

An immediate question on the above reconstruction is it is begging the question: to define the 'sameness' (and 'inequalities') of two ratios of magnitudes, we used in (**) both the very equalities and the inequalities we want to define. The answer is as follows:

a) The definitions 5-7 of book V form a set.

b) The definition requires as a prerequisite[23] the construction for the integers (as done in book VII).

c) According to the definition given in the previous paragraph the comparison between a ratio of magnitudes and a ratio of integers is defined as a comparison of multiples of magnitudes: if *a* and *b* are two (comparable i.e. homogeneous) magnitudes and *m* and *n* two (positive) integers, to say *a/b* is 'greater than' ('same than', 'less than') *m/n* means (by definition) the magnitude *na* is greater (equal, less) than the magnitude *mb*, and this has a sense by definition 4 (hence the importance to this definition).

   However we have now two definitions for greater and 'equal/same': one from the definition V.5, the other defined above. Since there is no such dualism in Euclid's theory, we have to check under which (unexpressed) hypotheses both are equivalent[24].

[23] As we will see in the second part, it is required for historical rather than for logical reasons.
[24] Since they are not defined on the same objects, we have to consider the objects common to both of them.

v. Around magnitudes

Nevertheless there is still a difficulty since there is no definition of 'magnitudes' ('μεγέθη') in the *Elements*.
In the absence of any hint, the most efficient way is to consider all the necessary properties required in definitions V.1 to V.4 as an abstract definition i.e. these conditions to be also sufficient
Since magnitudes do need to verify them, the largest extension of the theory is attained in this way[25].
As already stated by Aristotle, the fundamental rule in mathematics is the maximal generalization of any result. A scientific statement establishing a result has to embrace exactly all the things verifying the result[26]. And obviously Aristotle here is not reporting his own opinion, but the standard view of the mathematicians of his time. Thus what matters in mathematics are not so much the definitions but the properties used in the proofs.
The conditions in definitions V.1-4 are not about any (couples of) magnitudes but only the ones which 'can be said to have a ratio in relation to one another' ('λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται') or 'magnitudes-in-ratio'. Euclid calls them 'of the same kind' ('ὁμογενῶν') and they are usually translated as 'homogeneous magnitudes'.

The two first conditions give the meaning for a magnitude 'to be part' ('μέρος ἐστὶ μέγεθος μεγεθους', definition V.1) or conversely to be a 'multiple' ('πολλαπλάσιον', definition V.2) of another one.
Both conditions suppose the possibility any magnitude can be multiplied by integers i.e. of iterative additions.
The third one is about the meaning of the term 'ratio' (cf. *supra*, note 22).
The definition V.4 is certainly the most important condition for homogeneous magnitudes: for any two such magnitudes, each of them multiplied by a large enough integer will exceed the other. The definition entails therefore the possibility to compare two homogeneous magnitudes, in modern terms to define an order on them.
Integers clearly verify this condition. Moreover Euclid uses in the two first definitions the same terms he uses for integers in the first of the 'arithmetical' books (book VII) emphasizing they verify all the needed conditions.
Therefore under the previous hypothesis of an abstract operational definition for magnitudes, it seems integers should be understood as a kind of 'magnitudes', at least inside the theory elaborated in book V.

There are however two considerable difficulties with this claim. First the proof of the proposition 18 uses the existence of a fourth proportional, a property generally not true for

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[25] Depending of the context some other conditions may be added, cf. *supra*, note 39.
[26] For instance, it would be wrong to prove for isosceles triangles the sum of their angles is equal to two right angles, since it is true for any triangles (*Posterior Analytics*, I, 4, 73b34-74a4).

integers. Secondly in ancient Greek mathematics integers were considered as discrete objects opposed to continuous magnitudes[27]. Moreover both objections are consequence of the fundamental opposition between arithmetic and geometry.

Concerning the first difficulty, since the fourth proportional property appears only in one place in book V, and it seems to be alternative proofs which do not need, the demonstration was suspected as a modification done by posterior editors of the *Elements*[28]. It may also be argued the same property is independently proved for the integers in book VII (proposition 12), so that Euclid did not bother to give it again.
On the problem of the opposition of integers and magnitudes, it may be argued the connection between the theory in books VII and V was considered as evident by ancient Greek mathematicians who were using geometrical representations of integers[29].

As a matter of fact, these solutions are not really convincing. Firstly because the real difficulty in the proof of proposition V.18 arises when the four magnitudes are not together homogeneous, but only the first and the second the third and the fourth, a case which never happens for integers. Hence even if the proposition V.18 is a copy of the proposition VII.12, the proofs will necessarily differ. Whether the proof is genuine or interpolated, the problem remains open.
To argue integers are subsumed under magnitudes does not either solve the second difficulty for it does not answer the question why Euclid considers necessary to give two different theories, one for magnitudes, another for integers. Moreover why would Euclid use the term 'magnitude', almost synonymous with 'continuous' in ancient Greek mathematics, to include a theory done exclusively for integers? Lastly the usual most difficult question, why do we find no criticisms of Euclid's construction in the Antiquity?
As a matter of fact, all these objections are directed against the common assumption there are two different theories of ratios in the *Elements*, one included in the other. In the second part of this text we will see this interpretation is highly problematic though common among historians of mathematics who, when not openly critical of Euclid's presentation, have to admit their perplexity[30].

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[27] For instance Aristotle in *Physics*, 6, 231a4.
[28] [Euclide1991], II, p. 109-111.
[29] For instance Thomas Heath writes: 'It could not escaped [Euclid] numbers fall under the conception of magnitudes' ([Euclid1908], II, p. 113).
[30] For instance Heath admits there is a riddle here for him, since Euclid (and we should add the Greek geometers of the Antiquity) could hardly have missed the necessity of establishing the unicity of these supposedly different theories ([Euclid1908], II, p. 126).

Part II. Moderns against Ancients

In the first part we considered the theory of ratios in Euclid book V as an autonomous theory, the purpose being to understand it. In this second part we will be more concerned by its connections with the rest of the *Elements*.

i. Preliminaries for the theory of ratios

First we will consider the arithmetical results on ratios which are needed for the general theory according to the analysis done in the first part.

a) To relate the construction in the previous paragraphs to Euclid's theory in book VII we need both the definitions of the equality and the inequality for ratios of integers. The equality is given a little indirectly in the definition VII.20[31]. Four integers *m, n, p* and *q* are 'analogous' ('ἀριθμοὶ ἀνάλογόν εἰσιν') or 'proportional' or 'in the same ratio' (''Εν τῷ αὐτῷ λόγῳ') or the ratios *m/n* and *p/q* 'are the same' when 'the first is the same multiple, or the same part, or the same parts, of the second that the third (is) of the fourth'[32]. An equivalent condition is given in proposition VII.19:

   The ratios *m/n* and *p/q* are equal if and only if they verify the equality: $mq = np$.

   According to the interpretation of the definition VII.5 given in the previous paragraph, we need the following analogous condition for the inequality:
   The ratio *m/n* is 'greater than' the ratio *p/q* if and only if they verify the inequality: $mq > np$, which can be written in modern symbolism:
   $m/n > p/q$ if and only if $mq > np$ (1).

   The symbols '>' and '=' (evidently which are not found in Euclid) for relations between ratios of integers are the same as the ones defined in §I.iii for relations between general ratios and ratios of integers. Thus, since any ratio of integers is a general ratio, the situation is a particular case of the one in §I.iii.

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[31] Numbered VII.21 by Vitrac ([Euclid1991], II, p. 262.

[32] ''Αριθμοὶ ἀνάλογόν εἰσιν, ὅταν ὁ πρῶτος τοῦ δευτέρου καὶ ὁ τρίτος τοῦ τετάρτου ἰσάκις ᾖ πολλαπλάσιος ἢ τὸ αὐτὸ μέρος ἢ τὰ αὐτὰ μέρη ὦσιν.'. The definition needs some explanations which are given in previous one in the same book, essentially the definitions 3, 4 and 5. In particular to be a part (respectively some parts) of an integer means to be a divisor of it (respectively a multiple of such a divisor). An example for the first (respectively second, third case) is 24 to 6 (respectively 2 to 6, 4 to 6). The definition does not say anything when the first number is strictly greater but not a multiple of the second, as 7 to 6. So that it seems clear the ratio has to be decomposed in the sum of an integer plus 'a part' or 'several parts'. For instance, 7/6 is to be considered the addition of 1 and a part of 6. We find this kind of constructions in early texts as Plato's *Parmenides* (cf. *infra*, notes 33 and 34), and even in some Mesopotamian and Egyptian texts (as the Rhind papyrus) more than thousand years earlier (cf. [Clagett1999], II, part 2; see also [Itard1962], in particular p. 16-23). The book VII is the first of the 'arithmetical books' of the *Elements* (cf. supra note 19) and while the theory of general ratios is founded on definition V.5, the theory for integers is based on definition VII.20, which uses the wording of fractions as opposed to the language of ratios. In the proofs Euclid uses both notions (cf. [Euclid1908], II, p. 312).

b) We will re-examine the property of separation for ratios of integers (already studied in §§I.ii-iv):
Let m/n and p/q be two ratios of integers, if $m/n \neq p/q$ there is another one separating them (and the converse).

It follows from (1): let $m/n > p/q$ and let h be an integer such that $2mq > h > 2np$. From (1) we get immediately: $2mq/2nq > h/2nq$ and $h/2nq > 2np/2nq$ so that: (proposition VII.18) $m/n > h/2nq > p/q$ and $h/2nq$ separates m/n and p/q. And evidently the same goes for $p/q > m/n$.

Without using (1), this is also an immediate consequence of the property that the sum (or difference when it exists) of two ratios of integers is still a ratio of integers[33], since it is then enough to take their arithmetic mean i.e. half of their sum.

There is still a slightly different way for obtaining the result through an elementary property known by ancient Greek mathematicians. Indeed it is reported by the old Parmenides in Plato's eponymous book: the ratio of a given integer to an integer t is 'greater than' its ratio to any integer greater than t (i.e. in modern terms, for any integer k: if $u > t$ then $k/t > k/u$). Moreover the argument which is given in the wording of fractions seems to assume as evident the much simpler inverse property: if one integer is 'greater than' another, its ratio to any integer k is greater than the ratio of the second to k (i.e. for $u > t$ then $u/k > t/k$)[34].
Let h be the 'smallest number which n and q measure' (in modern terms the LCM or Least Common Multiple of n and q) as obtained in proposition VII.34, so that $h = ni = qj$[35]. From proposition VII.17 we get:
$m/n = 2mi/2ni = 2mi/2h$ and $p/q = 2pj/2qj = 2pj/2h$ (1').
Moreover since m/n and p/q are different, we have: $2mi \neq 2pj$.

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[33] It is evident when 'ratios of integers' are replaced by 'fractions of integers'. In the *Elements* the predominant language is the one of ratios i.e. of relations, not of fractions (cf. *supra*, note 9). Therefore some historians doubt the ancient Greek geometers would consider on ratios such operations as the addition or the subtraction, though they (including Euclid) did add and subtract angles (which are also defined as relations, cf. definition I.8). However without textual testimonies concerning the case of ratios, the arguments can only be indirect and the discussion is still open (cf. for instance [Knorr1982], [Vitrac1992], [Fowler1999], …). Nevertheless cf. hereinafter and in particular next note.
[34] As a matter of fact the problem is about time hence it seems to concern general magnitudes (for instance in book VI of his *Physics*, Aristotle gives several long and detailed proofs for the continuity of time, showing firstly 'nothing that is continuous can be composed of indivisibles' (231a24)). However the language used by Parmenides is the language of fractions with terms like 'to add' ('προτίθημειν') and 'fraction, portion' ('μόριον'). Hence the issue concerns actually ratios of integers or of commensurable magnitudes rather than general ratios. Furthermore the argument is founded on the addition and subtraction of 'parts', and it is considered as obvious that the order between two integers is the same as the order of their ratios to any given integer (i.e. if $m > p$ then for any n, $m/n > p/n$). In the text, Parmenides is discussing the consequences of the supposition that the one exists, or inversely, does not exist. He then asks (rhetorically) a very young Athenian (named Aristotle, but certainly not the Stagirite) 'if we add an equal to a greater and to a less time, will the greater differ from the less by the same or by a smaller **fraction**?' ('τῷ ἴσῳ μορίῳ **(…)** ἢ σμικροτέρῳ;'), the answer is immediately 'σμικροτέρω' ('by a smaller fraction') (154d2-4). David Fowler attaches an immense importance to this statement, maybe somewhat too much: he claimed to have based on it his (highly conjectural) theory for a pre-Eudoxean theory of ratios founded on the '*anthyphairesis*' ([Fowler1999], p. 41-50). We will consider in a next article this question, and more generally the question of early (if any) ratios theories before the one given by Euclid (whose Eudoxus is generally credited).
[35] Or even simply $h = nq$.

Let $r = mi + pj$ be the arithmetical mean of $2mi$ and $2pj$. Since $2mi$ and $2pj$ are different, we have either $2mi > r > 2pj$ or $2pj > r > 2mi$ so that it separates them. Thus the above elementary property (from Plato's *Parmenides*') entails $r/2h$ separates $2mi/2h$ and $2pj/2h$ i.e. (from (1') above) it separates m/n and p/q.

Let us prove its converse: if there is a ratio r/s separating m/n and p/q then we have: $m/n \neq p/q$.
For any integer k, t, u, v, w: $t/u > v/w$ entails (conservation of inequalities by product[36]) $kt/u > kv/w$.
Hence let r/s, m/n and p/q be multiplied by snq, then $snqr/s$ separates $snqm/n$ and $snqp/q$.
By commutativity (and associativity) $snqr/s = s(rnq)/s$, $snqm/n = n(msq)/n$ and $snqp/q = q(spn)/q$.
By proposition VII.17 we get: $s(rnq)/s = rnq$, $n(msq)/n = msq$ and $q(spn)/q = spn$ hence rnq separates msq and spn so that $smq \neq spn$. Hence (obviously or proposition VII.17) $mq \neq pn$ which entails (proposition VII.19) $m/n \neq p/q$.

As we see, the property of separation for ratios of integers (trivial for fractions[37]) is easy to obtain through very elementary methods.

[36] As a matter of fact, the implication follows immediately from the formula (1) since $mp > nq$ is equivalent to $(km)p > (kn)p$. But from proposition VII.9 we have: $km/n = k(m/n)$ and $kp/q = k(p/q)$ and the property of conservation of inequalities by products follows immediately in the case of fractions, so that it was certainly known early by ancient Greek mathematicians.
[37] Cf. *supra*, note 33.

ii. The separation property for the 'mixed' theory of ratios

There have been a lot of criticisms about inconsistencies in Euclid's theories of ratios, since there are two constructions of ratios. One in book V for magnitudes, the other in book VII for integers (cf. next paragraph). As a matter of fact, there is another question: their consistency with the 'mixed' one, when it is about the comparison of a general ratio and a ratio of integers. We will show here some results useful for this question, in particular a generalization of the **separation property** done in the paragraphs I.ii-iii *supra*.

1. First, for the consistency of Euclid's theory of ratios, we will show two 'simple' and 'evident' properties[38] on magnitudes, obvious on integers (cf. *infra*, §II.v). They are used to show the consistency of the constructions in books VII and V.

   For any magnitudes x and y and any integers m, n the following equivalences hold:

   $m > n$ is equivalent to $mx > nx$ (P1)
   and
   $x > y$ is equivalent to $mx > my$ (P2)

   It is a consequence of the order on the magnitudes is a total order[39] and of the foundational statements, the 'κοιναὶ ἔννοιαι' (the 'common notions') at the very beginning of the *Elements*.
   More precisely the 5th common notion states[40]: 'the whole [is] 'greater than' the part.' ('Καὶ τὸ ὅλον τοῦ μέρους μεῖζόν.').

   According to this common notion and Euclid's actual use of the notion of order, its meaning is as follows: a thing is 'greater than' another (its part) if there is a third one (another part) so that added to the second it is equal to the first[41].

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[38] Heath's words, cf. *infra*, II.iii.1.

[39] It is not explicitly said in the definition any magnitude may be compared to any other, i.e. the order on the magnitudes is a total order. It is only required that for any couple of magnitudes there is a couple of integers such that the first (respectively the second) magnitude multiplied by the first (respectively the second) integer is 'greater than' the second (respectively the first). However this property of total order is used without further explanation in the book V as for instance in the proof of proposition 10: 'I say that A is greater than B. For if not, A is surely either equal to or less than B' ('Εἰ γὰρ μή, ἤτοι ἴσον ἐστὶν ἢ μεῖζον.'). Assuming the order is total, we get an important property: any ratio is comparable to any ratios of integers (cf. *infra*, note 43). For (by definition, cf. *supra*, §ii) $x/y > m/n$ (respectively $x/y = m/n$, $x/y < m/n$) means $nx > my$ (respectively $nx = my$, $nx < my$).
For the order on ratios there are even less details. But since the relation 'less than' is not explicitly defined, it is reasonable to understand it as the relation 'complementary' to greater-or-equal, which implies any ratio is either greater, equal or less than another, so that the order is indeed total. Moreover as showed by B. Vitrac ([Euclide1991], II, p. 131-132) the property of the fourth proportional used in the proof of proposition V.18 entails a total order on ratios (nevertheless cf. the end of this paragraph). Anyway, we do not use this last property here.

[40] There are some discussions about its authenticity. According to Proclus, it is genuine, but some modern commentators, as Tannery and Heath (cf. [Euclid1908], I, p. 232) disagree. It does not matter much here: it was so commonly used that its interpolation would mean Euclid thought it did not even need to be explicitly written.

[41] It is not possible to use directly either the common notion I.5 for the inequalities between ratios as given in book V (definition 7), or the first common notions about the general equality ('if equal things are added to equal things then the wholes are equal', and so on). It must first to be proved they verify respectively the properties of an order and of an equality. This will be done later in §II.v.

Let us return to the above properties (P1) and (P2). From the 5th common notion, they are evident indeed since $m > n$ and $x > y$ is equivalent to: there exist p (integer) and z (magnitude) such that : $m = n + p$ and $x = y + z$.

The definition of '>' entails immediately its 'transitivity': let r, s, t be any integers (respectively magnitudes). By definition:
$r > s$ and $s > t$ means there exists u, v integers (respectively magnitudes) such that $r = s + u$ and $s = t + v$ so that $r = (t + v) + u$ = (associativity) $t + (v + u)$ so that (by definition of '>'): $r > t$.

Let us prove now the properties (P1) and (P2): $mx = (n + p)x = nx + px$ and $mx = m(y + z) = my + mz$[42], which once again by the 5th common notion is equivalent to: $mx > nx$ (which is (P1)) and $mx > my$[43] (which is (P2)).

In the same way we obtain the invariance of '>' by addition or subtraction:
For u and z integers (respectively magnitudes):
$u > z$ entails (by definition) there exists an integer (respectively a magnitude) t such that $u = z + t$.
Let w be an integer (respectively a magnitude):
$u + w$ = (common notion I.2[44]) $(z + t) + w$ = (associativity and commutativity of the addition) $(z + w) + t >$ (definition of '>') $z + w$ (P3).
Moreover if $z > w$ (which entails by 'transitivity': $u > w$), since we have:
$u = (u - w) + w = ((z + t) - w) + w)$ = (associativity and commutativity of the addition) $((z - w) + t) + w$,
we get (common notion I.3[45]):
$u - w = (z - w) + t >$ (definition of '>') $z - w$ (P4).

2. Since 'greater than' (respectively 'same than') is defined between two general ratios and '>' (respectively '=') between two ratios such that one of them is a ratio of integers, we need to compare both relations on couples of ratios whose one (at least) is a ratio of integers[46]. So let a/b be a general ratio and m/n a ratio of integers.

   In the same way as above, it is possible to show a **separation result**:
   For a/b a general ratio and m/n a ratio of integers such that $m/n > a/b$ (respectively $a/b > m/n$), there exists a ratio of integers p/q separating them.

   Let us suppose $m/n > a/b$ i.e. (definition) $mb > na$.
   From 1. let x be a magnitude such that $mb = na + x$ and k an integer such that (definition V.4) $kx > nb$, thus:
   $k(mb) = k(na) + kx > k(na) + nb$ (1).

[42] Since the multiplication by an integer is a sequence of successive additions, the distributivity of the multiplication in relation to the addition (of integers or magnitudes) is trivial (whatever the meaning of the addition) and freely used by Euclid.
[43] Here for the converse, we need the property of **total** order on the magnitudes (cf. also *supra*, note 39).
[44] Common notion I.3: 'And if equal things are added to equal things then the wholes are equal.'
[45] Common notion I.3: 'And if equal things are subtracted from equal things then the remainder are equal.'
[46] Cf. *supra*, note 24.

Let p and q be integers such that: $p + 1 = km$ and $q = kn$.
We have then:

a) $km = p + 1 >$ (definition of '>') p so that $km/q = km/kn =$ (proposition VII.17) m/n

and since $km = p + 1$ we get:
$km/q = (p+ 1)/q =$ (proposition VII.5) $p/q + 1/q >$ (définition of '>') p/q thus:
$m/n > p/q$ (2).

b) $(km)b = (p + 1)b =$ (distributivity) $pb + b >$ (from (1)) $k(na) + nb =$ (associativity of the addition) $(k(na) + (n - 1)b) + b$ so that (from (P4) $pb > k(na) + (n - 1)b > kna = qa$ so that ('transitivity' of '>'): $pb > kna$ i.e. (definition):
$p/q > a/b$ (3).

From (2) and (3) we have: $m/n > p/q > a/b$ and by symmetry we will obtain an analogous result when $a/b > m/n$ i.e the separation results hold between ratios of magnitudes and ratios of integers.

iii. The consistency of the theory of ratios

Let us now prove the implication: ‘greater than’ entails ‘>’.

By definition (cf. *supra*, §iii):
a/b ‘greater than’ *m/n* means: there exists a ratio of integers *p/q* such that:
$a/b > p/q$ and $p/q > m/n$ i.e. (definition) $qa > pb$ and $np > qm$.

Multiplying the first inequality by the integer *n* and the second by the magnitude *b* (this has a sense since *pn*, *mq* are integers), we get:

(by property (P1)) $n(qa) > n(pb)$ and (by property (P2)) $(np)b > (qm)b$
so that by associativity[47] and commutativity[48] of the multiplication on integers[49]:

$(nq)a > (np)b$ and $(np)b > (qm)b$, thus[50]: $(nq)a > (qm)b$ hence: $q(na) > q(mb)$.

Thus (by (P1)) we get: $na > mb$ i.e. (definition): $a/b > m/n$.

By symmetry we get as well: *m/n* ‘greater than’ *a/b* entails $m/n > a/b$.

3. For the converse we have to prove the relation ‘>’ entails ‘greater than’. So let us suppose: $a/b > m/n$.
   If *a/b* is not commensurable, by definition V.5 (§iii), the ratios *a/b* and *m/n* are not the same. If both are commensurable (*ib*. and note 19) there exists a ratio of integers *p/q* separating *a/b* and *m/n* and *a/b* is (by definition) ‘greater than’ *m/n*. By symmetry we get as well: $m/n > a/b$ entails *m/n* ‘greater than’ *a/b*.
   Hence the converse is true so that both relations ‘>’ and ‘greater than’ are equivalent (for the comparison between general ratios and ratios of integers).

4. Let us now see what happens with the relations ‘=’ and ‘the same as’. If $a/b = m/n$ both are commensurable and by definition V.5 (*supra*, §iii) *a/b* and *m/n* have the same ratio. Conversely, if the ratios *a/b* and *m/n* are ‘the same’ then (by definition) either $a/b = m/n$ (and the converse is true) or there is no ratio of integers separating *a/b* and *m/n*, which entails (*supra*, §iii) $a/b = m/n$.

So if the magnitudes verify the properties (P1) and (P2)[51], the relations ‘greater than’ (respectively ‘the same) of definition V.5 and ‘>’ (respectively ‘=’) are equivalent (as relations between a **general ratio** and a **ratio of integers**).

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[47] The associativity of the addition and multiplication is used by Euclid without even to be mentioned.
[48] It is proved in proposition 16 of book VII.
[49] There is no need to prove the commutativity of the multiplication of a magnitude by an integer, since the only possible sense is the iterated addition of the magnitude. Thus the ‘commutativity’ is a definition of convenience: $zm := mz := z + ... + z$ (*m*-times).
[50] The problems about the relation '>' do not concern integers or magnitudes on which it is known to define an order and even a total order (cf. *supra*, note 39 and note 43), but ratios. The ‘transitivity’ of relations of order is freely used by Euclid as well as the ancient Greeks.

Remark. These results of consistency are of no use for the understanding of the general theory of ratios *per se*, nevertheless they help to understand why there are not two theories, one for couples of the general ratios, another for mixed couples.
It does not mean a demonstration was needed to prove the consistency between these theories. To the contrary, the absence of any such proof demonstrates all these properties were well known before the elaboration of the general theory of ratios which was built from them.
In the second part, we will see the same unity for the general theory and the theory of ratios of integers, and the same remark would apply to the absence of proof binding them.

[51] These properties are needed for the consistency of Euclid's theory of ratios, cf. *infra*, §II.v.

iv. The riddle of the missing proof

Modern commentators consider there is a gap in Euclid's theory of ratios or more precisely theories of ratios for, they argue, there are two theories of ratios, one for integers in book VII another for magnitudes in book V[52]. Since integers cannot be considered as magnitudes for the former are discrete and the latter continuous[53], a proof of the compatibility of the two theories is needed[54]. It is all the more necessary for Euclid switches without any further explanations from one theory to the other (for instance in proposition X.5).

In its absence many of them, historians as well as mathematicians, consider their duty to complete Euclid's work, either by giving their own proof or by taking over an older one, as for instance Galileo and Clavius in the 16th century as well as Simson in the 18th, and so on[55]. Simson inserted directly his proof in his translation of the book V of Euclid's *Elements* (propositions C and D added to definition 5, [Simson1756], p. 128-129). Heath, followed by many other commentators, takes on Robert Simson's proof ([Heath 1908], III, p. 25)[56].

We will try to show there is neither a gap nor the need of a supplementary proof in Euclid's theory. Nevertheless they are right on one point: there is indeed a gap, though not inside Euclid's constructions but between ancient and modern readers of his treatise.

For instance, concerning the proposition 5 of Book V, while most modern commentators are extremely critical of the Euclidean proof, the Ancients considered it being at once evident and limpid. For instance, in his commentary of Euclid, the mathematician Pappus (4th century C.E.) writes there are several kinds of ratios of magnitudes, and one of them has to be 'understood in the sense that it denotes some such relation as exists between the numbers, all commensurable continuous quantities, for example, bearing, *as is evident*, a ratio to one another like that of a number to a number' ([Thomson1930], I, §6, p. 69, p. 7 in the original manuscript; our italics)[57].

According to part I, there is definitively a connection between the theories as given in book VII and V. But rather than two autonomous parallel theories, we may consider them as a copy by extension, the second (in book V) of the first (in book VII). It is not done through a supposed subsumption of integers under magnitudes, but of 'ratios of integers' under 'ratios of magnitudes' via the 'commensurable ratios'[58].

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[52] Cf. *supra*, §0.
[53] The parallelism, even partial, in these books makes it hard to maintain a strict disjunction between 'integers' and 'magnitudes' (cf. *infra*, note 68).
[54] For recent examples cf. [Fowler1979], p. 812-813; [Euclide1991], p. 266-267; cf. also *supra*, note 3.
[55] Cf. the commentary of Heath in [Euclid1908], II, p. 126 and also [Palmieri2001], p. 562
[56] Cf. nevertheless *supra*, note 30.
[57] There is some discussion about the authorship of this text however indubitably written by an expert in mathematics.
[58] Definition X.1: 'Those magnitudes measured by the same measure are said (to be) commensurable, but (those) of which no (magnitude) admits to be a common measure (are said to be) incommensurable'.

The definition of the ratios of magnitudes as presented in part I, appears indeed as an extension of the construction for integers. There is no need to prove both constructions are compatible since the second is the model for the first. It explains the oddness entailed by the usual point: we do not find any criticisms against this definition in the Antiquity. The reason is not the unlucky absence of precisely all the writings critical of the definition (and more generally of the whole theory of the book V) but that the readers of the *Elements* in Euclid's time and till late Antiquity, were still able to understand the foundations of the theory. *A contrario*, a lot of difficulties were raised afterwards, when the deep meaning of book V was forgotten.

Through the definitions of Book V, Euclid is able to get a theory of ratios of magnitudes as an autonomous and consistent construction independent of the one in Book VII. In other terms, by considering multiples of magnitudes instead of ratios, the theory is self-contained more concise and easier. Conversely it has definitively a negative aspect, at least for the moderns, the difficulty to understand its meaning as well as its origins[59]. For mathematicians post-Antiquity who had not a clue on the development of Greek mathematics, it appears as a black box, working extremely well but making no sense.

[59] A similar situation is found in many ancient texts, for instance in Plato's *Theaetetus* (147d4-7). The young Athenian Theaetetus gives an account on a lesson of his teacher Theodorus about the incommensurability of sides of squares of areas equal to some integers. This is usually transcribed in modern terms as a demonstration of the irrationality of the square root of some integers, beginning with 3 then 5 till 17. Since Plato does not precise the method used, modern commentators and historians consider it is a total enigma and consequently it is useless for the understanding of the rest of the text. But the exact opposite is true: if Plato does not care to give it, it is because the method was so well known by his contemporaries it was no use making it explicit, especially when the sequence of integers considered by Theodorus is correctly understood (cf. [Ofman2014], p. 76).

v. The riddle of the missing order

1. According to most commentators another proof is lacking. It is needed to show the alleged order in Euclid's general theory of ratios (i.e. given by the 'greater than') is actually an order. And similarly that 'sameness' has the properties required of an equality (or in modern terms it is an equivalence relation). So once again there is a logical gap, namely a proof is missing that for any triple of couples of homogeneous magnitudes (x,y), (z,w), (t,u), we have:

1) x/y 'greater than' z/w and z/w 'greater than' x/y is impossible
2) x/y 'greater than' z/w and z/w 'greater than' t/u entails x/y 'greater than' t/u

and

1') x/y is 'the same as' x/y
2') x/y 'is the same as' z/w entails z/w 'is the same as' x/y.

Therefore mathematicians and historians suggested once again some emendation, for instance Simson taken over by Heath who writes: 'there is no difficulty in proving this with the help of two simple *assumptions* which are *indeed obvious*' ([Euclid1908], II, p. 126, 130, our italics; [Simson1756]). These assumptions are essentially the properties (P1) and (P2) as defined in §I.iv:

for any integers m and n and any magnitudes x and y, the following equivalences holds:
$m > n \Leftrightarrow mx > nx$ (P1) and $x > y \Leftrightarrow mx > my$ (P2).

As seen in §I.iv the properties (P1) and (P2) are 'indeed obvious', but why? There is no need to establish them or to take them as 'axioms' as Simson did, since they can be considered as trivial consequences of the very notion of order in ancient Greek mathematics (*supra*, §I.iv). So there was no gap in Euclid construction and consequently no complaints were leveled in the Antiquity, even as very early commentators and mathematicians were hunting the slightest problem in the *Elements*.

2. First we prove for any (strictly positive) integers m, n, p, q and any homogeneous magnitudes a and b:

$m/n > a/b > p/q$[60] entails $m/n > p/q$ (*).

By definition (*supra*, I.iv) the first inequalities can be written: $mb > na$ and $qa > pb$.

By multiplying the first inequality by the integer q and the second by n, we get (from (P1))

[60] Here we use the symbol '>' as the relation defined in §I.3 between a general ratio and a ratio of integers.

$q(mb) > q(na)$ and $n(qa) > n(pb)$.

By associativity and commutativity[61] of the multiplication, we get:

$(qm)b > (qn)a$ and $(qn)a > (np)b$, hence[62]:

$(qm)b > (np)b$, and (from (P2)): $qm > np$.

As Heath pointed out several times in his commentary some supposed gaps in the proofs of book VII are easily solved using the language of fractions (as opposed to the language of ratios) used in the definitions VII.4 and VII.20[63] (*contra* Heiberg[64]).
Likewise in the language of fractions (i.e. 'part' and 'parts' of an integer, cf. *supra* note 34), it is indeed easy to apply the 5th common notion:

From the last inequality there exists an integer r such that $qm = np+r$.

Thus qm/nq = (by commutativity) qm/qn = (proposition VII.17) $m/n = (np + r)/nq$ = (proposition VII.5) $np/nq + r/nq$ = (proposition VII.17) $p/q + r/nq$ i.e. in a wording close from Plato's *Parmenides* 154d (cf. *supra*, note 34) m/n is equal to p/q plus some parts of nq so that (common notion I.5): $m/n > p/q$[65].

3. The symmetry (usually called 'parallelism') between the books V and VII in the *Elements* is rightly emphasized by commentators (for instance the two first definitions of book V in relation to the definitions 3, 4 and 5 of book VII; the propositions V.1-2 in relation to the proposition VII.5; the proposition V.6 and the propositions VII.7; the propositions V.16 and the proposition VII.13, and so on). Both are considered as theories of proportions, the latter for integers, the former for magnitudes. Surprisingly something seems to have escaped most of the exegetes. In spite of the closeness of books V and VII, there is an important point of divergence. Book VII is about proportions or as we have seen (*supra*, 0.a)) about proportionality (ἀνάλόγον) i.e. equalities of ratios. And while book V is definitively also about proportionality it is mostly about inequalities i.e. the 'greater than' as even the definition of equality (definition V.5) uses some inequalities[66].

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[61] Cf. *supra*, notes 47 and 48.
[62] By 'transitivity' of the order on the magnitudes (cf. *supra*, note **Erreur ! Signet non défini.**).
[63] The definition VII.20 indicates when four integers are proportional i.e. two ratios of integers are the same (cf. *supra*, II.0.a)).
[64] Heiberg, for his part, refers to some propositions of book V. He proceeds likewise in several cases when an analogous result is not given in Euclid's proof but is found in book V, as for instance in the proof of the fundamental propositions VII.14 or V.19. According to our analysis, it means begging the question.
[65] It is a possible proof for the formula (1) in 0.a), *supra*.
[66] In his treatise *On the Heavens*, Aristotle argues in the same way to prove the impossibility of an infinite body with a finite weight. And when he switches from ratios between commensurable magnitudes to incommensurable ones, the equality used in his argumentation is immediately replaced by an inequality (273b10-15).

As a matter of fact, there is no definition of 'greater than' or 'less than' in book VII or in the other 'arithmetical books'[67]. It is not easy indeed to understand such a situation if there are two parallel hence separated theories of proportions, one for magnitudes in book V, the other for integers in book VII[68]. Once again, the only reasonable explanation appears to be that book VII is, among other things, an introduction to the theory of ratios in book V[69].

4. For any integers m, n, p, q we have (by definition[70]):

$p/q > m/n$ if an only if $pn > mq$ (**).

This formula is nothing else than the corresponding formula for the equality given in the proposition VII.19:

$m/n = p/q$ if and only if $pn = mq$ (***).

Now from (**) and (***), it is evident '>' (respectively '=') has the properties 1) and 2) (respectively 1') and 2')) on ratios of integers[71]. Moreover the definition of 'less than' (symbol '<') is then evident, as well as it defines an order on these ratios[72].

5. We have now to show the relation 'greater than' on ratios of magnitudes (as defined in the definition V.7) is an order, hence to prove:

- x/y 'greater than' z/w and z/w 'greater than' x/y is impossible
- x/y 'greater than' z/w and z/w 'greater than' t/u entails x/y 'greater than' t/u.

a) By definition (§I.iii) x/y 'greater than' z/w and z/w 'greater than' x/y mean there exist integers m, n, p, q such that:

$x/y > m/n > z/w$ and $z/w > p/q > x/y$, thus:

$m/n > z/w > p/q$ and $p/q > x/y > m/n$.

From (*) (cf. 2.) we get:

$m/n > p/q$ and $p/q > m/n$

which is impossible since '>' is a (strict) order on the ratios of integers (cf. 3.).

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[67] It is the very absence of any means to compare the ratios of integers that has leaded us previously to the common notions in this paragraph as well as in §0, *supra*.

[68] For instance B. Vitrac sees here the need for an exception to understand what he calls the 'incomplete treatment' of the ratios in the case of 'integers' in relation to the case of magnitudes, to the point he is compelled to 'put aside the difference [between integers and magnitudes]' ([Euclide1991], II, p. 313).

[69] It does not mean integers are to be understood as some special magnitudes, but that their ratios, as 'commensurable ratios', should be considered as special ratios of magnitudes. This is formally established in proposition X.5.

[70] Cf. *supra*, 0.b) formula (1). It does not mean it is the origin of the order on ratios of integers (or on fractions depending of the understanding of the book VII). Even more probably the original order should have been rather founded on the common notion I.5 as done in I.iv *supra*. It is why we considered the case of a theory of ratios without it (cf. supra, note 36 and also next note).

[71] In modern terms, the relation 'greater than' is a relation of (strict) order, the 'same than' a relation of equivalence on the ratios of integers. Indeed it is clear that for any integer m, n, p, q, r, s we have: i) $mq > pn$ and $pn > mq$ is impossible ii) $mq > pn$ and $pn > rs$ entails $mq > rs$ i') $mn = mn$ ii') $mq = pn$ entails $pn = mq$. The symbols '=' and '>' are here the usual ones defined on the integers. It is not surprising these properties hold since in modern mathematics the formula (***) gives the relation of equivalence on the couples of integers defining the rational numbers, and (**) defines the (strict) order on these numbers, though the integers in the modern sense include both zero and the negative integers.

[72] There is of course no definition of an 'order' or an 'equivalence relation' in Euclid or even in the Antiquity since it was done at the end of the 19th century. But their properties were frequently used by ancient Greek mathematicians, Euclid included, so that they need to prove them, except of course the ones considered as evident.

b) The second inequalities (x/y 'greater than' z/w and z/w 'greater than' t/u) mean (by definition, §I.iii) there exist integers m, n, p, q such that:

$x/y > m/n > z/w$ and $z/w > p/q > t/u$
which entails:
$m/n > z/w > p/q$ so that (from (*)): $m/n > p/q$.
Thus we have:
$m/n > p/q$ and $p/q > t/u$
which gives by definition (*supra*, §I.iii):
m/n 'greater than' t/u which (from I.iv.3.) is equivalent to: $m/n > t/u$, so that:
$x/y > m/n > t/u$. Thus (by definition) we obtain: x/y 'greater than' t/u.

6. By definition V.5 (cf. *supra*, §I.iii), two general ratios are 'the same' if they are not separated by a ratio of integers, so that we get:

- x/y is 'the same as' x/y since otherwise there would be a ratio of integers m/n such that $x/y > m/n > x/y$ i.e. (by definition) $nx > my > nx$[73], and we get the condition 1').
- x/y is 'the same as' z/w entails z/w is 'the same as' x/y, since if a ratio separates x/y and z/w it separates also z/w and x/y, and we get the condition 2')).

As the relation of 'transitivity' is proved by Euclid in proposition V.11 the 'sameness' has all the required properties.

Remark. As in the final remark of §I.iv, we should not be fooled by the above (relatively) long demonstrations. They are not some missing links to the general theory of ratios in book V, since they belong exclusively to the arithmetical theory of ratios, certainly established long before the general one and used as a model for the latter.

[73] That is impossible since '>' is a strict order on magnitudes.

vi. The riddle of the missing definition

As we have seen, several basic terms are not explicitly defined in book V and not even in the whole *Elements*. It is also often said 'ratio' is not defined by Euclid. This is not exact since the definition 3 is just that: 'A ratio is a certain type of relation[74] in relation to *size* of two magnitudes of the same kind' ('λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ *πηλικότητά* ποια σχέσις.'). Admittedly, it is considered apocryphal by some historians[75]. Otherwise it is generally understood as a definition 'for ornament' and for no useful purpose, 'a metaphysical, rather than a mathematical definition'[76]. It may be linked to other such definitions in the *Element* as the one of the straight-line ('whatever lies evenly with points upon itself') or the point ('that of which there is no part') in book I. It is presented as mathematically useless and anyway not used in the *Elements*. However the definition V.3 appears to have at least one important role. It emphasizes the generality and the abstractness of a theory concerned with relations between sizes of the objects of the theory[77], though basic notions such as magnitudes and ratios are not defined, baffling the commentators[78].
One argument for a later interpolation of definition V.3 is the use of the term 'πηλικότης' not found elsewhere in the *Elements* except in a definition of book VI considered precisely as unauthentic (definition VI.5[79]). Moreover it is redundant with 'μέγετος', both terms meaning 'magnitude' or 'quantity'.

But this definition is found in all the manuscripts of the *Elements* hence it is difficult to reject its authenticity. Moreover 'πηλικότης' is an attribute of 'μέγετος', so that it is not redundant but brings some precision to the statement. For Heath it was given as a definition of 'convenience' without any real mathematical purpose[80].
He is certainly right about the needed distinction between the attribute ('πηλικότης') and the thing to which it belongs ('μέγετος') as well as a metamathematical use of the definition[81].

Nevertheless, it may also have an often overlooked purely mathematical meaning, given by its relative character emphasized by the presence of the term 'σχέσις' ('relation') connected to 'κατὰ πηλικότης' ('with respect of size') through 'ποια' ('certain type of'). So a paraphrase of it ought not to be only 'the ratio of two homogeneous magnitudes depends of both of the

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[74] In [Euclid2007], the term 'σχέσις' is translated by 'condition', but we prefer to follow Heath's translation which seems more appropriate here.
[75] For example Herman Hankel (*Zur Geschichte der Mathematik in Alterthum und Mittelalter*, 1874) or more recently Friedhelm Beckmann (Neue Gesichtspunkte zum 5. Buch Euklids, *Arch. Hist. Exact Sciences*, 4, 1-2, p. 27-28).
[76] Isaac Barrow, *Lectiones Cantabrigiae*, quoted and approved by Heath ([Euclid1908], II, p. 117).
[77] This approach may be linked to Galileo's works on the free fall of bodies, in modern terms uniformly accelerated movement, which relies intensively on Euclid's ratios theory. At the end of his life Galileo took again this theory, wanting to give a better and clearer definition of ratios of magnitudes than the one in definition 5 (cf. for instance [Ofman2008], p. 571 and also *supra* note 3). The absence of infinitesimal magnitudes entails the same difficulties for Galileo as the absence of real numbers for the Greek geometers.
[78] For instance D. Fowler writes: 'It is a curious, obvious, and unexplained fact that the *Elements* does not contain a precise definition of ratios, though the word *logos* is used frequently with this meaning.' ([Fowler1979], p. 812).
[79] It is not even given in [Euclid2007].
[80] [Euclid1908], II, p. 117.
[81] A contemporary interpretation for the term 'metaphysical' he uses in his text.

magnitudes', but also 'and conversely it depends only of them, more precisely of their sizes'. In other words it must be independent of anything else, first and foremost of the chosen unit. Hence the ratio ought not to change when the unit of measure is multiplied or divided by an integer and (when it has a sense) the following equality holds:

for any homogeneous magnitudes a and b and any (positive) integer m:

$ma/mb = a/b$.

It is proved in proposition V.15 for magnitudes and in proposition VII.17 for integers[82]. It gives immediately (when it has a sense) for any integer n[83]:

$(a/n)/(b/n) = a/b$.

Finally (when it has a sense) we obtain for any integers m and n:

$(ma/n)/(mb/n) = a/b$.

[82] Of course, the definition V.3 does not **prove** anything but it gives a condition for magnitudes to be 'in ratio'. The propositions V.15 and VII.17 show exactly that: the property of independence of the choice of the unit holds for both the theories of book V and of book VII. Since the unit in arithmetic is not supposed to be either divided or multiplied, this interpretation in valid in the framework of the theory of book V ('if anyone attempts to cut up the 'one' in argument, [the experts in this study] laugh at him and refuse to allow it; but if you mince it up, they multiply, always on guard lest the one should appear to be not one but a multiplicity of parts.', Plato's *Republic*, 525c). Incommensurability was early a place where arithmetic and geometry mixed as shown in the 'mathematical part' of Plato's *Theaetetus* (147d-148b, cf. also [Ofman2014], p. 76-78).

[83] Applying the first formula to $a' = na$ and $b' = bn$.

vii. The riddle of the missing integers

The last difficulty considered here is the puzzling use by Euclid of the definition 5 inside his demonstrations.
More precisely instead of taking all the couples of integers m and n to form all the 'equal multiples' of the homogeneous magnitudes, in agreement with the definition, he considers only couples of different ('ἄλλα') integers (i.e. $m \neq n$), which would be certainly considered a mistake according to (modern) logic. The most puzzling problem is that nothing in the proof would require such an exclusion and it would be exactly identical if all the couples of integers were the ones required by definition 5.
So the problem is not mathematical but linguistic. The statement of the definition 5 in the demonstrations as well as what Euclid actually proves for the 'sameness' differ from the statement of the original definition 5. The conditions stated and proved for the equality of general ratios are strictly weaker than these given in definition V.5. And nevertheless Euclid claims to have obtained through his proof the sought equality of ratios. It is certainly not a slip of the tongue, for exactly the same wording is repeated throughout the proofs of book V every time he uses this definition, beginning with the first one, the proof of proposition V.4.

It is possible to argue Euclid uses a standard sentence for brevity and does not actually mean what he says. His purpose is to emphasize a crucial point: the couple of integers (m,n) is not **necessarily** formed of the same integers i.e. m and n **can** be different. Without modern symbolism, he ought to use a long sentence in Greek, so that he prefers to **exclude** the case of equality i.e. the couples (m,m). In other words, he transforms a sufficient condition into a necessary one. It does not matter much since the proof works just as well for the 'forgotten' case of (m,m).
There is certainly some truth in this argument since, as we saw, the problem is not mathematical but purely linguistic. However it would be unreasonable to believe Euclid did not see the difference between what he says and its conclusion, and completely impossible no one realized it during the whole Antiquity. So the question stays unsolved: why did the ancient Greek mathematicians (including Euclid himself) see no problems here?

The answer is easy according to our analysis since it is evident both situations are trivially equivalent. According to the definition 5, two ratios x/y and z/w are equal if there are no ratios of integers separating them. But if the only such ratio is $m/m = 1/1$ (proposition VII.9 or for general ratios V.15), it entails $x/y = 1/1 = z/w$ since otherwise (by definition of the equality) there would be another ratio of integers p/q with $p \neq q$ separating either x/y and $1/1$ or $1/1$ and z/w, so that (by definition V.5) x/y is the same as z/w.
These statements were sufficiently obviously equivalent to be considered as identical. Even nowadays no mathematician would probably object to such a 'simplification'.

Conclusion

The above analysis of the definitions V.5 and V.7 dispels the surprise and the ensuing recurrent *logicist* criticism of the lack of demonstration connecting the 'supposed' two definitions of equality of ratios, respectively for magnitudes (definition V.5) and for integers (definition VII.20[84]). It also dispels several riddles in the usual interpretations of Euclid's theory of ratios.

There were never been two distinct but parallel theories of proportions. The theory of ratios of magnitudes is an extension of the theory of proportional integers of book VII which was well understood till the end of the Antiquity[85].

So far as these questions are concerned, in agreement with the ancient testimonies, there is neither a logical gap between these theories nor inconsistencies in Euclid's theory of ratios of magnitudes.

The most important ideas at the basis of the general theory of ratios are as follows:

i) Any ratio of homogeneous magnitudes (whatever they are) has to be 'approximated' in some way by ratios of integers.
ii) No ratio of magnitudes (and not only magnitudes themselves) is less than any ratio of integers (a form of the modern 'Archimedean property').

The condition ii) is given by the definition V.4. The condition i) does not suppose the possibility to compare two ratios in general, but any general ratio (i.e. a ratio of the homogeneous magnitudes) to any ratio of integers (cf. *supra*, §I.iii and note 39) i.e. extending in some way the property of (strict) separation for ratios of integers to general ratios[86]. It is the foundation of the so-called celebrated 'exhaustion method' used for instance by Euclid in book XII to prove the (surface) of 'two circles are as the square of their diameters' (proposition 2).

This question is closely related to the question of approximations of general ratios by ratios of integers (cf. *supra*, §I.ii and also note 22) and in particular to the *anthyphairesis* construction. It has not been considered here and will be discussed in a next article in connection with the problem of pre-Euclidean theories of ratios.

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[84] The definition gives directly the meaning of (four) proportional integers, without defining the equality of ratios. There is an analogous definition in the book V, the definition 6 (located between the ones we have considered). Some historians think it is inauthentic since superfluous given the previous definition, arguing it was added later to get a better symmetry between the theories of book V and VII.
[85] cf. *supra*, introduction and also note 3.
[86] It could come as a shock for the ones believing in a complete disjunction between the thinking of ancient and modern mathematicians, for many modern theories are built in this way. As a recent example, we may consider the definition of the derivative for generalized functions i.e. distributions, coming from a property of the product of differentiable functions i.e. the identity: $\int (fg)' = \int f'g + \int fg'$.

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