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On Random Attractors for Mixing Type Systems*

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Dedicated to I. M. Gelfand on the occasion of his 90th birthday

Abstract. The paper deals with infinite-dimensional random dynamical systems. Under the condition that the system in question is of mixing type and possesses a random compact attracting set, we show that the support of the unique invariant measure is the minimal random point attractor. The results obtained apply to the randomly forced 2D Navier–Stokes system.

Key words: invariant measure, mixing type system, random attractor, stationary measure, 2D Navier–Stokes equations.

Introduction

This paper deals with random dynamical systems (RDS)

$$\varphi_k\colon H \to H, \quad H \ni u \mapsto \varphi_k u, \quad k \geqslant 0, \tag{0.1}$$

on a Polish** space H. Here the φ_k are random transformations (that is, $\varphi_k = \varphi_k(\omega)$, where ω is a random parameter). As functions of k, these transformations are assumed to have independent increments. Usually the time will be discrete (i.e., $k \in \mathbb{Z}_+$); however, RDS with continuous time will also be briefly discussed in the context of stochastic partial differential equations.

Many features of long-time behavior of the trajectories of (0.1) are described by random attractors of this RDS. Of many possible definitions of random attractors (e.g., see [3, 5, 6]), we choose the following: *a compact random set $\mathscr{A}_\omega$ is called a random attractor if all trajectories $\varphi_k(\omega)u$ of* (0.1) *converge to $\mathscr{A}_\omega$ in probability.* See Sec. 1.2 for the precise definition and a discussion of it.

The RDS (0.1) defines a Markov chain in H with transition function

$$P_k(u,\Gamma) = \mathbf{P}\{\omega : \varphi_k(\omega)u \in \Gamma\}, \tag{0.2}$$

where Γ is a Borel subset in H. The long-time behavior of this process is described to some extent by its stationary measures. Recall that a probability Borel measure μ on H is said to be *stationary* if $\mu(\Gamma) = \int_H P_k(u,\Gamma)\mu(du)$ for every $k \geqslant 0$ and every Borel set Γ. For systems in question, every stationary measure μ admits a Markov disintegration:

$$\mu(\Gamma) = \mathbf{E}\,\mu_\omega(\Gamma).$$

Here $\omega \mapsto \mu_\omega$ is a measure-valued map measurable with respect to the past, i.e., with respect to the σ-algebra generated by the random transformations $\varphi_k(\theta_{-m}\omega)$, where $k \geqslant m \geqslant 0$ and θ_n is the corresponding measure-preserving shift in the probability space; see [3, 6] and Sec. 1.1. It is known that

$$\operatorname{supp}\mu_\omega \subset \mathscr{A}_\omega \quad \text{a.s.}, \tag{0.3}$$

where $\mathscr{A}_\omega$ is an arbitrary random attractor; see [7] and Sec. 1.2.

The main result of this paper is Theorem 2.4, which states that the support of the disintegration of the unique stationary measure for the discrete time RDS (0.1) is its minimal random attractor

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**A metric space is said to be *Polish* if it is complete and separable.

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provided that the system satisfies some nonrestrictive compactness condition and is of mixing type in the sense that

$$\mathbf{E}\, f(\varphi_k(\omega)u) \to \int_H f(u)\mu(du) \quad \text{as } k \to +\infty \tag{0.4}$$

for every bounded continuous function $f\colon H \to \mathbb{R}$, where $u \in H$ is an arbitrary initial point and μ is the stationary measure. In other words, under the preceding conditions we have the equality in (0.3), where $\mathscr{A}_\omega$ is the minimal random attractor. The proof is based on an ergodic type theorem for the dynamical system $\{\Theta_k\}$ defined on the phase space $\Omega \times H$ as the skew product of θ_k and φ_k:

$$\Theta_k(\omega, u) = (\theta_k\omega, \varphi_k(\omega)u), \qquad k \geqslant 0;$$

see Theorem 2.3.

In Sec. 4 we consider the randomly forced 2D Navier–Stokes equations

$$\dot{u} - \nu\Delta u + (u, \nabla)u + \nabla p = \eta(t, x), \qquad \operatorname{div} u = 0, \tag{0.5}$$

where $u = u(t, x)$ is the velocity field, $p = p(t, x)$ is the pressure, and $\eta(t, x)$ is a random external force. The equations are supplemented by the Dirichlet or periodic boundary conditions. The random force η is smooth in x, while as a function of t it is either a kick force (then (0.5) defines a discrete-time RDS) or a white force (then it defines a continuous-time RDS). In both cases, the RDS satisfies the compactness condition. Imposing some nonrestrictive nondegeneracy assumption, we find from the results in [13] or [14], respectively, that the RDS satisfies the mixing type condition as well. Therefore, the abstract Theorems 2.3 and 2.4 apply to system (0.5) both for the kick and white forces. Accordingly, the support of the Markov disintegration for the unique stationary measure defines the minimal random attractor for (0.5) (see Theorems 4.1 and 4.2), and functionals depending on both the solution u and the corresponding forces satisfy a theorem of ergodic type (see Theorem 4.3).

Notation. Let (H, d) be a Polish space, let $C_b(H)$ be the space of bounded continuous functions on H equipped with the norm $\sup_{u\in H} |f(u)|$, and let $L(H)$ be the space of functions $f\colon H \to \mathbb{R}$ such that

$$\|f\|_{L(H)} = \sup_{u\in H} |f(u)| + \sup_{u,v\in H} \frac{|f(u) - f(v)|}{|u - v|} < \infty.$$

If $(\Omega, \mathscr{F}, \mathbf{P})$ is a probability space and $\mathscr{F}'$ is a sub-σ-algebra, then by $\mathbb{L}(H, \mathscr{F}')$ we denote the set of functions $F(\omega, u)\colon \Omega \times H \to \mathbb{R}$ that are $\mathscr{F}'$-measurable in ω for any given $u \in H$ and satisfy the condition

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \|F(\omega, \cdot\,)\|_{L(H)} < \infty. \tag{0.6}$$

For $u \in H$ and $A \subset H$, we define the distance between u and A as

$$d(u, A) = \inf_{v\in A} d(u, v).$$

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1. Preliminaries

In this section, we recall some basic notions of the theory of random dynamical systems (RDS) and state a few results that will be used later. We mainly follow the book [3]. To simplify the presentation, we confine ourselves to the case of discrete time.

1.1. Random dynamical systems. Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space, let $\theta_k\colon \Omega \to \Omega$, $k \in \mathbb{Z}$, be a group of measure preserving transformations of Ω, and let H be a Polish space equipped with a metric d and the Borel σ-algebra $\mathscr{B}_H$.

Definition 1.1. A (*continuous*) *random dynamical system over* θ_k is a family of mappings $\varphi_k(\omega)\colon H \to H$, where $k \in \mathbb{Z}_+$ and $\omega \in \Omega$, with the following properties.

(i) *Measurability.* The mapping $(\omega, u) \mapsto \varphi_k(\omega)u$ of the space $\Omega \times H$ equipped with the σ-algebra $\mathscr{F} \otimes \mathscr{B}_H$ into H is measurable for each $k \geqslant 0$.

(ii) *Continuity.* For any $\omega \in \Omega$ and $k \geqslant 0$, the mapping $\varphi_k(\omega)$ is continuous.

(iii) *Cocycle property.* For each $\omega \in \Omega$, we have

$$\varphi_0(\omega) = \mathrm{Id}_H, \quad \varphi_{k+l}(\omega) = \varphi_k(\theta_l \omega) \circ \varphi_l(\omega) \quad \text{for all } k, l \geqslant 0. \tag{1.1}$$

For any integers $m \leqslant n$, by $\mathscr{F}_{[m,n]}$ we denote the σ-algebra generated by the family of H-valued random variables $\varphi_k(\theta_{m-1}\omega)u$, where $u \in H$ and $k = 0, \dots, n-m+1$. We extend this notation to the case in which $m = -\infty$ and/or $n = +\infty$ by setting $\mathscr{F}_{[-\infty,n]} = \sigma\{\mathscr{F}_{[m,n]}, m \leqslant n\}$ and similarly for $\mathscr{F}_{[m,+\infty]}$ and $\mathscr{F}_{[-\infty,+\infty]}$. A straightforward verification shows that

$$\theta_k^{-1}\mathscr{F}_{[m,n]} = \mathscr{F}_{[m+k,n+k]} \quad \text{for all } m \leqslant n \text{ and } k. \tag{1.2}$$

The σ-algebras $\mathscr{F}^- = \mathscr{F}_{[-\infty,0]}$ and $\mathscr{F}^+ = \mathscr{F}_{[1,+\infty]}$ are called the *past* and the *future* of $\varphi_k(\omega)$.

Let $\mathscr{P}_{\mathbf{P}}$ be the set of probability measures on $(\Omega \times H, \mathscr{F} \otimes \mathscr{B}_H)$ whose projections on Ω coincide with $\mathbf{P}$. It is well known (see [3, Sec. 1.4]) that each measure $\mathfrak{M} \in \mathscr{P}_{\mathbf{P}}$ admits a unique *disintegration* $\omega \mapsto \mu_\omega$, which is a random variable ranging in the space of measures such that

$$\mathfrak{M}(\Gamma) = \int_\Omega \int_H I_\Gamma(\omega, u)\mu_\omega(du)\mathbf{P}(d\omega) \quad \forall\, \Gamma \in \mathscr{F} \otimes \mathscr{B}_H,$$

where I_Γ is the indicator function of Γ.

Given an RDS $\varphi_k(\omega)$ over θ_k, we introduce the following semigroup of measurable mappings on $\Omega \times H$:

$$\Theta_k(\omega, u) = (\theta_k \omega, \varphi_k(\omega)u), \qquad k \geqslant 0.$$

The semigroup Θ_k is called the *skew product* of θ_k and $\varphi_k(\omega)$. A measure $\mathfrak{M} \in \mathscr{P}_{\mathbf{P}}$ is said to be *invariant for* Θ_k if $\Theta_k(\mathfrak{M}) = \mathfrak{M}$ (that is, $\mathfrak{M}(\Theta_k^{-1}(\Gamma)) = \mathfrak{M}(\Gamma)$ for every $\Gamma \in \mathscr{F} \otimes \mathscr{B}_H$). By Theorem 1.4.5 in [3], a measure $\mathfrak{M} \in \mathscr{P}_{\mathbf{P}}$ is invariant if and only if its disintegration μ_ω satisfies the following relation for $\mathbf{P}$-almost all $\omega \in \Omega$:

$$\varphi_k(\omega)\mu_\omega = \mu_{\theta_k \omega} \quad \text{for all } k \geqslant 0.$$

The set of all invariant measures for Θ_k will be denoted by $\mathscr{I}_{\mathbf{P}}(\varphi)$.

Definition 1.2. An invariant measure $\mathfrak{M} \in \mathscr{I}_{\mathbf{P}}(\varphi)$ is said to be *Markov* if its disintegration μ_ω is measurable with respect to the past $\mathscr{F}^-$. The set of such measures will be denoted by $\mathscr{I}_{\mathbf{P},\mathscr{F}^-}(\varphi)$.

We now proceed to the important class of *RDS with independent increments* (also called *white noise RDS*).

Definition 1.3. We say that an RDS $\varphi_k(\omega)$ has *independent increments* if its past and future are independent.

It follows from (1.2) that $\varphi_k(\omega)$ has independent increments if and only if the σ-algebras $\mathscr{F}_{[m,n]}$ and $\mathscr{F}_{[m',n']}$ are independent for any disjoint (finite or infinite) intervals $[m,n]$ and $[m',n']$.

For each RDS $\varphi_k(\omega)$ with independent increments, the set of random sequences $\{\varphi_k(\,\cdot\,)u, k \geqslant 0\}$, $u \in H$, is a family of Markov chains with respect to the filtration $\mathscr{F}_k = \theta_k^{-1}\mathscr{F}^-$. The corresponding transition function $P_k(u,\Gamma)$ has the form (0.2), and the Markov operators associated with P_k are given by the formulas

$$\mathfrak{P}_k f(u) = \int_H P_k(u,dv)f(v), \qquad \mathfrak{P}_k^*\mu(\Gamma) = \int_H P_k(u,\Gamma)\mu(du),$$

where $f \in C_b(H)$ and $\mu \in \mathscr{P}(H)$. We recall that $\mu \in \mathscr{P}(H)$ is called a *stationary measure* for the Markov family if $\mathfrak{P}_1^*\mu = \mu$. The set of such measures will be denoted by $\mathscr{S}_\varphi$. The following important result is established (for different situations) in [6, 15, 16].

Proposition 1.4. *Let $\varphi_k(\omega)$ be an RDS with independent increments. Then there is a one-to-one correspondence between Markov invariant measures $\mathscr{I}_{\mathbf{P},\mathscr{F}^-}(\varphi)$ for the skew product Θ_k and the stationary measures $\mathscr{S}_\varphi$ for the associated Markov family. Namely, if $\mu \in \mathscr{S}_\varphi$, then the limit*

$$\mu_\omega = \lim_{k\to+\infty} \varphi_k(\theta_{-k}\omega)\mu \tag{1.3}$$

exists in the $*$-weak topology almost surely and gives the disintegration of a Markov invariant measure $\mathfrak{M}$. Conversely, if $\mathfrak{M} \in \mathscr{I}_{\mathbf{P},\mathscr{F}^-}$ is a Markov invariant measure and μ_ω is its disintegration, then $\mu = \mathbf{E}\,\mu_\omega$ is a stationary measure for the Markov family.

1.2. Point attractors. Let $\{\varphi_k(\omega)\}$ be an RDS in a Polish space H over $\{\theta_k\}$ as above. A family of subsets $\mathscr{A}_\omega$, $\omega \in \Omega$, is called a *random compact* (*closed*) *set* if $\mathscr{A}_\omega$ is compact (closed) for almost all ω and $\Omega_U := \{\omega \in \Omega : \mathscr{A}_\omega \cap U \neq \varnothing\} \in \mathscr{F}$ for every open set $U \subset H$. A random compact set $\mathscr{A}_\omega$ is said to be *measurable with respect to a sub-σ-algebra* $\mathscr{F}' \subset \mathscr{F}$ if $\Omega_U \in \mathscr{F}'$ for any open set $U \subset H$.

Definition 1.5. A random compact set $\mathscr{A}_\omega$ is called a *random point attractor* (*in the sense of convergence in probability*) if for each $u \in H$ the sequence of random variables $d(\varphi_k(\omega)u, \mathscr{A}_{\theta_k\omega})$ converges to zero in probability, i.e.,

$$\lim_{k\to+\infty} \mathbf{P}\{d(\varphi_k(\omega)u, \mathscr{A}_{\theta_k\omega}) > \delta\} = 0 \tag{1.4}$$

for each $\delta > 0$. A random point attractor $\mathscr{A}_\omega$ is said to be *minimal* if for any other random point attractor $\mathscr{A}'_\omega$ we have $\mathscr{A}_\omega \subset \mathscr{A}'_\omega$ for almost all ω.

It is clear that a minimal random point attractor is unique (if it exists); i.e., if $\mathscr{A}_\omega$ and $\mathscr{A}'_\omega$ are two minimal random attractors, then $\mathscr{A}_\omega = \mathscr{A}'_\omega$ almost surely. Since θ_k is a measure-preserving transformation, it follows that (1.4) is equivalent to

$$\lim_{k\to+\infty} d(\varphi_k(\theta_{-k}\omega)u, \mathscr{A}_\omega) = 0, \tag{1.5}$$

where the limit is understood in the sense of convergence in probability. This type of convergence of a trajectory to a random set (the initial data are specified at time $-k$, $k \to \infty$, and the distance is evaluated at time zero) is normally used to define random attractors. We prefer the "forward" definition (1.4), which seems to be more natural. We note that of various types of random attractors considered in modern mathematical literature, the one in Definition 1.5 is the smallest; cf. [3, 5, 6].

If we replace relation (1.4) in Definition 1.5 by the condition that (1.5) holds for all $u \in H$ and $\omega \in \Omega_0$, where $\Omega_0 \in \mathscr{F}$ is a set of full measure independent of u, then we obtain the definition of a *random point attractor in the sense of almost sure convergence*. Since almost sure convergence implies convergence in probability, it follows that the resulting attractor also satisfies (1.4). In what follows, we mainly deal with random point attractors in the sense of convergence in probability; therefore, they will simply be called *random attractors*.

The following proposition is a straightforward consequence of Theorems 3.4 and 4.3 and Remark 3.5(iii) in [7].

Proposition 1.6. (i) *Let $\varphi_k(\omega)$ be an RDS with independent increments. Suppose that there exists a random compact set $\mathscr{K}_\omega$ attracting the trajectories of $\varphi_k(\omega)$ in the following sense: there exists a set $\Omega_0 \in \mathscr{F}$ such that $\mathbf{P}(\Omega_0) = 1$ and*

$$\lim_{k\to+\infty} d(\varphi_k(\theta_{-k}\omega)u, \mathscr{K}_\omega) = 0 \quad \textit{for any } \omega \in \Omega_0,\ u \in H. \tag{1.6}$$

Then $\varphi_k(\omega)$ possesses a random attractor $\mathscr{A}_\omega$ that is measurable with respect to the past $\mathscr{F}^-$.

(ii) *For each Markov invariant measure $\mathfrak{M} \in \mathscr{I}_{\varphi,\mathscr{F}^-}$, its disintegration μ_ω is supported by each random attractor $\mathscr{A}'_\omega$; i.e., $\mu_\omega(\mathscr{A}'_\omega) = 1$ almost surely.*

Outline of proof. (i) As shown in [7], under the assumptions of the proposition the RDS $\varphi_k(\omega)$ possesses a random point attractor $\mathscr{A}_\omega$ in the sense of almost sure convergence. Since θ_k is a measure-preserving transformation, we conclude that (1.5) holds for $\mathscr{A}_\omega$. The construction implies that $\mathscr{A}_\omega$ is measurable with respect to the past.

(ii) Let $\mathscr{A}'_\omega$ be an arbitrary random attractor. To show that the disintegration of each invariant measure $\mathfrak{M} \in \mathscr{I}_{\varphi,\mathscr{F}^-}$ is supported in $\mathscr{A}'_\omega$, it suffices to observe that (1.4) implies the almost sure convergence (1.5) along an appropriate subsequence $k = k_n$ and repeat the argument in [7, Theorem 4.3].

2. Main Results

As before, by $(\Omega, \mathscr{F}, \mathbf{P})$ we denote a probability space and by H a Polish space equipped with a metric d and the Borel σ-algebra $\mathscr{B}_H$. Let $\{\varphi_k(\omega)\}$ be an RDS in H over a measure-preserving group of transformations θ_k. We introduce the following two hypotheses.

Condition 2.1. *Mixing.* The Markov family $\{\varphi_k(\omega)u\}$ is a system of mixing type in the following sense: it has a unique stationary measure μ, and

$$\mathfrak{P}_k f(u) = \mathbf{E}\, f(\varphi_k(\,\cdot\,)u) \to (\mu, f) = \int_H f(u)\mu(du) \quad \text{as } k \to \infty \tag{2.1}$$

for each $f \in L(H)$ and each initial point $u \in H$.

Condition 2.2. *Compactness.* There is a random compact set attracting the trajectories of $\varphi_k(\omega)$ (in the sense specified in Proposition 1.6). Moreover, for any $u \in H$ and $\varepsilon > 0$ there exists an $\Omega_\varepsilon \in \mathscr{F}$, a compact set $K_\varepsilon \subset H$, and an integer $k_\varepsilon = k_\varepsilon(u) \geqslant 1$ such that $\mathbf{P}(\Omega_\varepsilon) \geqslant 1 - \varepsilon$ and

$$\varphi_k(\theta_{-k}\omega)u \in K_\varepsilon \quad \text{for } \omega \in \Omega_\varepsilon,\ k \geqslant k_\varepsilon. \tag{2.2}$$

Let $\mu \in \mathscr{S}_\varphi$ be the unique stationary measure for the Markov semigroup $\mathfrak{P}_k^*$, and let $\mathfrak{M} \in \mathscr{I}_{\varphi,\mathscr{F}^-}$ be the corresponding Markov invariant measure for the skew product Θ_k (see Proposition 1.4).

Theorem 2.3. *Suppose that Condition* 2.1 *is satisfied. Then for each function* $F(\omega, u) \in \mathbb{L}(H, \mathscr{F}^-)$ *we have*

$$\mathbf{E}\, F(\Theta_k(\,\cdot\,, u)) \to (\mathfrak{M}, F) = \int_\Omega \int_H F(\omega, u)\mu_\omega(du)\mathbf{P}(d\omega) \quad \textit{as } k \to \infty, \tag{2.3}$$

where $u \in H$ *is an arbitrary initial point.*

We now discuss the relationship between invariant measures and random attractors. We denote the disintegration of $\mathfrak{M}$ by μ_ω and set

$$\mathscr{A}_\omega = \begin{cases} \operatorname{supp} \mu_\omega, & \omega \in \Omega_0, \\ H, & \omega \notin \Omega_0, \end{cases} \tag{2.4}$$

where $\Omega_0 \in \mathscr{F}$ is a set of full measure on which the limit (1.3) exists. By Corollary 1.6.5 in [3], $\mathscr{A}_\omega$ is a random closed set. Moreover, it follows from (1.3) that $\mathscr{A}_\omega$ is measurable with respect to $\mathscr{F}^-$.

Theorem 2.4. *Suppose that Conditions* 2.1 *and* 2.2 *are satisfied. Then* $\mathscr{A}_\omega$ *is a minimal random point attractor.*

The proofs of these theorems are given in Sec. 3. We now discuss a class of RDS satisfying Conditions 2.1 and 2.2.

Example 2.5. *Randomly forced dynamical systems.* Let H be a Hilbert space with norm $|\cdot|$ and an orthonormal basis $\{e_j\}$, and let P_N be the orthogonal projection onto the subspace $H_N \subset H$ generated by $e_1, \dots, e_N$. Suppose that a continuous operator $S\colon H \to H$ satisfies the following two conditions:

$$|S(u)| \leqslant q|u| \quad \text{for } u \in H, \tag{2.5}$$

$$|\mathsf{P}_N(S(u) - S(v))| \leqslant \tfrac{1}{2}|u - v| \quad \text{for } |u| \vee |v| \leqslant R, \tag{2.6}$$

where $q < 1$ is a constant independent of u, $R > 0$ is an arbitrary constant, and $N \geqslant 1$ is an integer depending only on R. Consider the RDS generated by the equation

$$u_k = S(u_{k-1}) + \eta_k, \tag{2.7}$$

where $k \in \mathbb{Z}$ and η_k is a sequence of i.i.d. H-valued random variables. If the distribution χ of the random variables η_k is compactly supported, then Condition 2.2 is satisfied. If, moreover, χ is sufficiently nondegenerate (in the sense of [13]), then the Markov family corresponding to (2.7) is of mixing type. Thus, under the above hypotheses, the support of the unique invariant measure is a random point attractor. We note that Equation (4.3) below, which corresponds to the kick-forced Navier–Stokes system, satisfies (2.5) and (2.6); see [13].

Conditions 2.1 and 2.2 are also satisfied for a large class of unbounded kicks η_k (see (2.7)). We do not dwell upon that case.

Remark 2.6. Theorems 2.3 and 2.4 remain valid for RDS $\varphi_t(\omega)$ with continuous time $t \geqslant 0$. In this case, we assume that $\varphi_t(\omega)u$ is continuous with respect to (t,u) for any given $\omega \in \Omega$ and that Conditions 2.1 and 2.2 hold with k replaced by t. Reformulation of the above results for continuous time is rather obvious, and therefore we do not give detailed statements.

3. Proofs

3.1. Proof of Theorem 2.3. *Step* 1. We first assume that $F(\omega,u) \in \mathbb{L}(H, \mathscr{F}_{[-\ell,0]})$, where $\ell \geqslant 0$ is an integer. Since θ_k is a measure-preserving transformation, we have

$$p_k(u) := \mathbf{E}\, F(\theta_k\omega, \varphi_k(\omega)u) = \mathbf{E}\, F(\omega, \varphi_k(\theta_{-k}\omega)u) = \mathbf{E}\,\mathbf{E}\{F(\omega, \varphi_k(\theta_{-k}\omega)u) \mid \mathscr{F}_{[1-m,0]}\}$$

for each $m \geqslant 1$. By the cocycle property (see (1.1)),

$$\varphi_k(\theta_{-k}\omega) = \varphi_m(\theta_{-m}\omega)\varphi_{k-m}(\theta_{-k}\omega), \qquad m \leqslant k.$$

Hence, by setting $F_m(\omega,u) = F(\omega, \varphi_m(\theta_{-m}\omega)u)$, we obtain

$$p_k(u) = \mathbf{E}\,\mathbf{E}\{F_m(\omega, \varphi_{k-m}(\theta_{-k}\omega)u) \mid \mathscr{F}_{[1-m,0]}\} \tag{3.1}$$

for each $m \leqslant k$. We now note that $F_m \in \mathbb{L}(H, \mathscr{F}_{[1-m,0]})$ for $m \geqslant \ell+1$. Since $\varphi_{k-m}(\theta_{-k}\omega)u$ is measurable with respect to $\mathscr{F}_{[1-k,-m]}$ and since the σ-algebras $\mathscr{F}_{[1-m,0]}$ and $\mathscr{F}_{[1-k,-m]}$ are independent, it follows from (3.1) that

$$p_k(u) = \mathbf{E}\,\mathbf{E}'\{F_m(\omega, \varphi_{k-m}(\theta_{-k}\omega')u)\} = \mathbf{E}\,(\mathfrak{P}_{k-m}F_m)(\omega,u), \tag{3.2}$$

where $\ell+1 \leqslant m \leqslant k$ and $\mathbf{E}'$ stands for the expectation with respect to ω'. In view of Condition 2.1 and the Lebesgue theorem, the right-hand side of (3.2) tends to $\mathbf{E}\,(\mu, F_m(\omega,\,\cdot\,))$ as $k \to +\infty$ for each $m \geqslant \ell+1$. Recalling the definition of F_m, we see that

$$(\mu, F_m(\omega,\,\cdot\,)) = (\varphi_m(\theta_{-m}\omega)\mu, F(\omega,\,\cdot\,)) \to (\mu_\omega, F(\omega,\,\cdot\,)) \quad \text{as } m \to \infty,$$

where we have used Proposition 1.4. It follows that

$$\lim_{k\to+\infty} p_k(u) = \mathbf{E}\,(\mu_\omega, F(\omega,\,\cdot\,)),$$

which coincides with (2.3).

Step 2. We now show that (2.3) holds for functions of the form $F(\omega,u) = f(u)g(\omega)$, where $f \in L(H)$ and g is a bounded $\mathscr{F}^-$-measurable function. To this end, we use a version of the monotone class theorem (see [2, Theorem 3.3]).

We take some $f \in L(H)$ and denote the set of bounded $\mathscr{F}^-$-measurable functions g for which the convergence (2.3) holds with $F = fg$ by $\mathscr{H}$. It is clear that $\mathscr{H}$ is a linear space containing the constant functions. Moreover, as was shown at Step 1, it contains all bounded functions measurable with respect to $\mathscr{F}_{[-\ell,0]}$ for some $\ell \geqslant 0$. Since the union of $\mathscr{F}_{[-\ell,0]}$, $\ell \geqslant 0$, generates $\mathscr{F}^-$, we see that the desired assertion will be proved as soon as we establish the following property: if $g_n \in \mathscr{H}$ is an increasing sequence of nonnegative functions such that $g = \sup g_n$ is bounded, then $g \in \mathscr{H}$.

Suppose that a sequence $\{g_n\} \subset \mathscr{H}$ satisfies the above conditions. Without loss of generality, we can assume that $0 \leqslant g, g_n \leqslant 1$. By Egorov's theorem, for each $\varepsilon > 0$ there exists an $\Omega_\varepsilon \in \mathscr{F}$ such that $\mathbf{P}(\Omega_\varepsilon) \geqslant 1-\varepsilon$ and

$$\lim_{k\to+\infty} \sup_{\omega\in\Omega_\varepsilon} |g_n(\omega) - g(\omega)| = 0.$$

It follows that for each $\varepsilon > 0$ there exists an integer $n_\varepsilon \geqslant 1$ such that $n_\varepsilon \to +\infty$ as $\varepsilon \to 0$ and

$$g_{n_\varepsilon}(\omega) \leqslant g(\omega) \leqslant g_{n_\varepsilon}(\omega) + \varepsilon + I_{\Omega_\varepsilon^c}(\omega) \quad \text{for all } \omega \in \Omega.$$

Multiplying this inequality by $f(\varphi_k(\theta_{-k}\omega)u)$, taking the expectation, passing to the limit as $k \to +\infty$, and using the estimate $\mathbf{P}(\Omega_\varepsilon^c) \leqslant \varepsilon$, we obtain

$$\begin{aligned}\mathbf{E}\{(\mu_\omega, f)g_{n_\varepsilon}(\omega)\} &\leqslant \liminf_{k\to+\infty} \mathbf{E}\{f(\varphi_k(\theta_{-k}\omega)u)g(\omega)\} \\ &\leqslant \limsup_{k\to+\infty} \mathbf{E}\{f(\varphi_k(\theta_{-k}\omega)u)g(\omega)\} \leqslant \mathbf{E}\{(\mu_\omega, f)g_{n_\varepsilon}(\omega)\} + 2\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and $\mathbf{E}\{(\mu_\omega, f)g_{n_\varepsilon}(\omega)\} \to \mathbf{E}\{(\mu_\omega, f)g(\omega)\}$ as $\varepsilon \to 0$ (by the monotone convergence theorem), we conclude that

$$\mathbf{E}\{f(\varphi_k(\omega)u)g(\theta_k\omega)\} = \mathbf{E}\{f(\varphi_k(\theta_{-k}\omega)u)g(\omega)\} \underset{k\to+\infty}{\longrightarrow} \mathbf{E}\{(\mu_\omega, f)g(\omega)\},$$

which means that $g \in \mathscr{H}$. This completes the proof of (2.3) for the case in which $F(\omega, u) = f(u)g(\omega)$.

Step 3. Now consider the general case. Let $F \in \mathbb{L}(H)$ be an arbitrary function such that $\|F(\omega, \cdot)\|_{L(H)} \leqslant 1$ for almost every $\omega \in \Omega$. For any $u \in H$ and $\varepsilon > 0$, we take an integer $k_\varepsilon(u) \geqslant 1$ and sets $\Omega_\varepsilon \in \mathscr{F}$ and $K_\varepsilon \Subset H$ for which (2.2) holds. By the Arzelà–Ascoli theorem, the unit ball $B_\varepsilon = \{f \in L(K_\varepsilon) : \|f\|_{L(H)} \leqslant 1\}$ is compact in the space $C_b(K_\varepsilon)$, and therefore, there exists a finite set $\{h_j\} \subset B_\varepsilon$ whose ε-neighborhood contains B_ε. It follows that B_ε can be covered by disjoint Borel sets $U_j \ni h_j$, $j = 1, \ldots, N$, whose diameters do not exceed 2ε. By $f_j \in L(H)$ we denote arbitrary extensions of h_j to H such that $\|f_j\|_{L(H)} \leqslant 2$. For instance, we can take

$$f_j(u) = \inf_{v\in K_\varepsilon} (h_j(v) + d(u, v) \wedge 1).$$

Consider the following approximation to F:

$$G_\varepsilon(\omega, u) = \sum_{j=1}^{N} f_j(u)g_j(\omega), \qquad g_j(\omega) = I_{U_j}(F_{K_\varepsilon}(\omega, \cdot)),$$

where $F_{K_\varepsilon}(\omega, u)$ is the restriction of F to $\Omega \times K_\varepsilon$. Since only one of the functions g_j can be nonzero, we have $\|G_\varepsilon(\omega, \cdot)\|_\infty \leqslant 2$. Therefore,

$$|G_\varepsilon(\omega, u) - F(\omega, u)| \leqslant 2\varepsilon + I_{K_\varepsilon^c}(u)(\|G_\varepsilon(\omega, \cdot)\|_\infty + \|F(\omega, \cdot)\|_\infty) \leqslant 2\varepsilon + 3I_{K_\varepsilon^c}(u) \tag{3.3}$$

for any $u \in H$ and almost every $\omega \in \Omega$, where we have used the inequality $\|F(\omega, \cdot)\|_\infty \leqslant 1$. We set

$$p_k(u) = \mathbf{E}\, F(\theta_k\omega, \varphi_k(\omega)u), \qquad p_k(u, \varepsilon) = \mathbf{E}\, G_\varepsilon(\theta_k\omega, \varphi_k(\omega)u).$$

It is clear that

$$|p_k(u) - (\mathfrak{M}, F)| \leqslant |p_k(u) - p_k(u, \varepsilon)| + |p_k(u, \varepsilon) - (\mathfrak{M}, G_\varepsilon)| + |(\mathfrak{M}, G_\varepsilon - F)|. \tag{3.4}$$

Let us estimate each term on the right-hand side in (3.4). Combining (2.2) with (3.3), we obtain

$$\begin{aligned}|p_k(u) - p_k(u, \varepsilon)| &\leqslant |\mathbf{E}\{F(\omega, \varphi_k(\theta_{-k}\omega)u) - G_\varepsilon(\omega, \varphi_k(\theta_{-k}\omega)u)\}| \\ &\leqslant 2\varepsilon + 3\mathbf{P}\{\varphi_k(\theta_{-k}\omega)u) \notin K_\varepsilon\} \leqslant 2\varepsilon + 3\mathbf{P}(\Omega_\varepsilon^c) \leqslant 5\varepsilon\end{aligned} \tag{3.5}$$

for $k \geqslant k_\varepsilon(u)$. Furthermore, the functions g_j are $\mathscr{F}^-$-measurable, and hence, by Step 2,

$$p_k(u, \varepsilon) \to (\mathfrak{M}, G_\varepsilon) \quad \text{as } k \to +\infty \tag{3.6}$$

for each given $\varepsilon > 0$. Finally, inequality (3.3) implies that

$$|(\mathfrak{M}, G_\varepsilon - F)| \leqslant 2\varepsilon + 3(\mathfrak{M}, I_{K_\varepsilon^c}) = 2\varepsilon + 3\mu(K_\varepsilon^c). \tag{3.7}$$

Since $\varepsilon > 0$ is arbitrary, it follows from (3.4)–(3.7) that the desired convergence (2.3) will be established once we show that $\mu(K_\varepsilon^c) \to 0$ as $\varepsilon \to 0$.

To this end, we note that

$$\mu(K_\varepsilon^c) = \int_H \mathbf{P}\{\varphi_k(\omega)u \notin K_\varepsilon\}\mu(du). \tag{3.8}$$

It follows from Condition 2.1 that

$$\limsup_{k\to+\infty} \mathbf{P}\{\varphi_k(\omega)u \notin K_\varepsilon\} = \limsup_{k\to+\infty} \mathbf{P}\{\varphi_k(\theta_{-k}\omega)u \notin K_\varepsilon\} \leqslant \varepsilon$$

for each given $u \in H$. Passing to the limit as $k \to +\infty$ in (3.8), we conclude that $\mu(K_\varepsilon^c) \leqslant \varepsilon$ for each $\varepsilon > 0$. This completes the proof of Theorem 2.3.

3.2. Proof of Theorem 2.4. We first show that the random compact set $\mathscr{A}_\omega$ is a random attractor. We take a $\delta \in (0,1)$ and consider the function

$$F(\omega, u) = 1 - \frac{d(u, \mathscr{A}_\omega)}{\delta} \wedge 1, \qquad u \in H,\ \omega \in \Omega.$$

We claim that $F \in \mathbb{L}(H, \mathscr{F}^-)$. Indeed, the definition of F implies that $F(\omega, u)$ is bounded and

$$|F(\omega, u) - F(\omega, v)| \leqslant \frac{d(u,v)}{\delta} \quad \text{for all } u, v \in H \text{ and } \omega \in \Omega.$$

Thus F satisfies (0.6). Since $\mathscr{A}_\omega$ is a random compact set measurable with respect to $\mathscr{F}^-$, we conclude that the random variable $\omega \to d(u, \mathscr{A}_\omega)$ is measurable with respect to $\mathscr{F}^-$ for each $u \in H$ (see Sec. 6.1 in [3]). This proves the desired properties of F.

Since $F(\omega, u) = 1$ for $u \in \mathscr{A}_\omega$, it follows that $(\mathfrak{M}, F) = 1$. By applying Theorem 2.3, we obtain

$$\mathbf{E}\, F(\theta_k\omega, \varphi_k(\omega)u) = 1 - \mathbf{E}\left(\frac{d(\varphi_k(\omega)u, \mathscr{A}_{\theta_k\omega})}{\delta} \wedge 1\right) \to (\mathfrak{M}, F) = 1,$$

that is,

$$p_k(u) := \mathbf{E}\left(\frac{d(\varphi_k(\omega)u, \mathscr{A}_{\theta_k\omega})}{\delta} \wedge 1\right) \to 0. \tag{3.9}$$

We now note that, by Chebyshev's inequality,

$$\mathbf{P}\{d(\varphi_k(\omega)u, \mathscr{A}_{\theta_k\omega}) > \delta\} \leqslant \frac{p_k(u)}{\delta}\,.$$

In view of (3.9), the right-hand side of this inequality tends to zero as $k \to +\infty$. This completes the proof of the fact that $\mathscr{A}_\omega$ is a random attractor.

To show that $\mathscr{A}_\omega$ is a minimal random attractor, it suffices to note that the invariant measure $\mathfrak{M} \in \mathscr{I}_{\mathbf{P},\mathscr{F}^-}$ is supported in every random attractor $\mathscr{A}'_\omega$, by Proposition 1.6. Therefore, $\operatorname{supp}\mu_\omega \subset \mathscr{A}'_\omega$ for almost every $\omega \in \Omega$.

4. The Navier–Stokes Equations

In this section, we consider the randomly forced 2D Navier–Stokes system

$$\dot u - \nu\Delta u + (u, \nabla)u + \nabla p = \eta(t,x), \qquad \operatorname{div} u = 0. \tag{4.1}$$

The space variable x belongs either to a smooth bounded domain D, and then the boundary condition $u|_{\partial D} = 0$ is imposed, or to the torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, and then we assume that $\int u\,dx = \int \eta\,dx \equiv 0$. We are interested in the time evolution of the velocity field u (but not of the pressure p). Accordingly, we replace the force η by its divergence-free component (neglecting the gradient component) and assume in what follows that

$$\operatorname{div}\eta = 0.$$

We first consider the case in which the right-hand side η is a random kick force of the form

$$\eta(t,x) = \sum_{k\in\mathbb{Z}} \delta(t-k)\eta_k(x), \tag{4.2}$$

where the η_k are i.i.d. random fields as in [12, 13]. Let H be the Hilbert space of divergence-free vector fields in the domain in question satisfying the boundary conditions in the usual sense (e.g., see [1]). We normalize the solutions $u(t,x)$ of (4.1), (4.2), treated as random curves in H, to be right continuous. Then, by evaluating the solutions at integer times $t = k \in \mathbb{Z}_+$ and by setting $u_k = u(k, \cdot)$, we obtain the equation

$$u_k = S(u_{k-1}) + \eta_k. \tag{4.3}$$

Here S is the time-one shift along trajectories of the free Navier–Stokes system (4.1) (with $\eta \equiv 0$); see [12, 13] for details. Defining φ_k, $k \geqslant 0$, as the map sending $u \in H$ to the solution u_k of (4.3) equal to u at $t = 0$, we obtain an RDS of the form (0.1). One readily verifies that it satisfies the desired compactness condition (cf. [12, Sec. 2.2.1]). Moreover, if the distribution of the kicks η_k satisfies a nonrestrictive nondegeneracy assumption specified in [13], then the corresponding Markov chain in H has a unique stationary measure μ and condition (0.4) holds. Hence Theorem 2.4 applies, and we arrive at the following result:

Theorem 4.1. *If the kick force* (4.2) *satisfies the above conditions, then the support $\mathscr{A}_\omega$ of the Markov disintegration μ_ω of its unique stationary measure μ is a minimal random attractor of the RDS* (4.3). *Moreover, there is a deterministic constant $D = D_\nu$ such that the Hausdorff dimension of the set $\mathscr{A}_\omega$ does not exceed D for almost every ω.*

In Remark 2.6, we point out that Theorems 2.3 and 2.4 remain valid for a class of RDS with continuous time $t \geqslant 0$. This class includes the system describing the white-forced 2D Navier–Stokes equations, i.e., Eq. (4.1) with

$$\eta(t, x) = \frac{\partial}{\partial t}\zeta(t, x), \qquad \zeta(t, x) = \sum_{j=1}^{\infty} b_j \beta_j(t) e_j(x). \tag{4.4}$$

Here $\{e_j\}$ is the L^2-normalized trigonometric basis in H and $\{\beta_j\,, t \in \mathbb{R}\}$ is a family of independent standard Wiener processes. It is assumed that the real coefficients b_j decay faster than any negative degree of j:

$$|b_j| \leqslant C_m j^{-m} \quad \text{for all } j, m \geqslant 1,$$

so that $\eta(t, x)$ is almost surely smooth in x. Consider the space $\mathscr{H}$ of continuous curves $\xi\colon \mathbb{R} \to H$ such that $\xi(0) = 0$ equipped with the topology of uniform convergence on bounded intervals. Let $\mathscr{B}$ be the σ-algebra of Borel subsets of $\mathscr{H}$, $\{\theta_t\}$ the group of canonical shifts of $\mathscr{H}$ ($\theta_t \xi(s) = \xi(s+t) - \xi(t)$) and $\mathbf{P}$ the distribution of the process ζ in $\mathscr{H}$. We take $(\mathscr{H}, \mathscr{B}, \mathbf{P})$ for the probability space $(\Omega, \mathscr{F}, \mathbf{P})$. Then the Navier–Stokes system (4.1), (4.4) defines a continuous-time RDS over θ_t (see [3]) and a Markov process in H. This RDS satisfies the compactness condition; see [5, Sec. 3.1]. Moreover, it is shown in [14] (see also [4, 11]) that there exists an integer $N = N_\nu$ such that if

$$b_j \neq 0 \quad \text{for } 1 \leqslant j \leqslant N, \tag{4.5}$$

then the corresponding Markov process in H has a unique stationary measure μ and satisfies (0.4). Thus we arrive at the following result:

Theorem 4.2. *If* (4.5) *holds, then the white-forced 2D Navier–Stokes system* (4.1), (4.4) *has a unique stationary measure μ. The supports $\mathscr{A}_\omega$ of its Markov disintegration μ_ω define the minimal random attractor of the corresponding RDS in H. Moreover, there exists a deterministic constant $D = D_\nu$ such that the Hausdorff dimension of the set $\mathscr{A}_\omega$ does not exceed D for almost every ω.*

The fact that $\operatorname{supp}\mu_\omega$ has finite Hausdorff dimension in both discrete and continuous cases follows from the general results on upper bounds for the Hausdorff dimension of global random attractors (see [8–10]), since these attractors contain $\operatorname{supp}\mu_\omega$ (see Corollary 3.6 in [7]).

Consider the skew product system $\{\Theta_k\}$ corresponding to the RDS in question. By applying the continuous-time version of Theorem 2.3, we obtain the following result:

Theorem 4.3. *Let G be a bounded measurable functional on $H \times \mathscr{H}$ uniformly Lipschitz in the first variable and such that $G(u, \zeta(\cdot))$ depends only on $\{\zeta(s),\, s \leqslant 0\}$. Let $u(t)$ be the solution* *of* (4.1), (4.4) *equal to* u_0 *for* $t = 0$. *Then*

$$\mathbf{E}G(u(t), \theta_t\zeta) \to \mathbf{E}\int_H F(v,\zeta)\,\mu_\zeta(dv) \quad \text{as } t \to \infty$$

for each $u_0 \in H$ *provided that assumption* (4.5) *is satisfied.*

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