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Eulerian Limit for 2D Navier–Stokes Equation and Damped/Driven KdV Equation as Its Model

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Dedicated to Vladimir Igorevich Arnold
on his 70th birthday

Abstract—We discuss the inviscid limits for the randomly forced 2D Navier–Stokes equation (NSE) and the damped/driven KdV equation. The former describes the space-periodic 2D turbulence in terms of a special class of solutions for the free Euler equation, and we view the latter as its model. We review and revise recent results on the inviscid limit for the perturbed KdV and use them to suggest a setup which could be used to make a next step in the study of the inviscid limit of 2D NSE. The proposed approach is based on an ergodic hypothesis for the flow of the 2D Euler equation on iso-integral surfaces. It invokes a Whitham equation for the 2D Navier–Stokes equation, written in terms of the ergodic measures.

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INTRODUCTION

We consider the 2D Navier–Stokes equation (NSE) under the periodic boundary conditions. The equation is perturbed by a Gaussian random force which is smooth in the space variable, while as a function of time it is a white noise. We are interested in the inviscid limit for the NSE, i.e., in the behaviour of its solutions when the viscosity goes to zero. It is not hard to see that in order to have a limit of order one the force should be proportional to the square root of the viscosity (see [9, Section 10.3]). Accordingly, we consider the following equation:

$$\begin{gathered} \dot u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \sqrt{\nu}\,\eta(t,x), \qquad 0<\nu\le 1,\\ \operatorname{div} u = 0, \qquad u = u(t,x)\in\mathbb{R}^2, \qquad p = p(t,x), \qquad x\in\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z}^2). \end{gathered} \tag{1}$$

It is assumed that $\int u\,dx \equiv \int \eta\,dx \equiv 0$ and that the force η is divergence-free and is non-degenerate being interpreted as a random process in a function space (see Section 1). It is known that equation (1) defines a Markov process in the function space

$$\mathcal{H} = \left\{ u(x)\in L^2(\mathbb{T}^2;\mathbb{R}^2) \;\middle|\; \operatorname{div} u = 0,\ \int_{\mathbb{T}^2} u\,dx = 0 \right\}$$

and that this process has a unique stationary measure μ_ν. This is a probability Borel measure in $\mathcal{H}$ which attracts distributions of all solutions for (1). Let $u_\nu(t,x)$ be a corresponding stationary solution, i.e.,

$$\mathcal{D}u_\nu(t) \equiv \mu_\nu.$$

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It is proved in [8] (see also [9]) that when $\nu \to 0$ along a subsequence, the random field $u_\nu(t,x)$ converges in distribution to a non-trivial limit $U(t,x)$ (a priori depending on the subsequence), which is a random field, stationary in t, H^2-smooth in x, and such that each its realisation $U(t,x)$ satisfies the free Euler equation (i.e., equation (1) with $\nu = 0$). Accordingly, the marginal distribution $\mu_0 = \mathcal{D}U(0)$ is an invariant measure for the Euler equation. The process U is called an *Eulerian limit*. See Theorem 1.1 below.

The estimates for the stationary solutions u_ν (see (1.1) below) imply that $\mathbf{E}\int |u_\nu(t,x)|^2 dx \sim 1$ for all ν. That is, the characteristic size of the solution u_ν remains ~ 1 when $\nu \to 0$. Since the characteristic space scale is also ~ 1, the Reynolds number of u_ν grows as ν^{-1} when ν decays to zero. Hence, the Eulerian limits $U(t,x)$ describe the transition to turbulence for space-periodic 2D flows that are stationary in time.[1] So the study of the Eulerian limits is equivalent (at least, is closely related) to the study of 2D turbulence that is stationary in time and periodic in space.

The Euler equation under the periodic boundary conditions has an infinite-dimensional integral of motion $I = I(u)$, where $u(\cdot) \mapsto I(u(\cdot))$ is a map with values in a certain metric space B. In Theorem 1.2 we show that the measure $\mu_0 = \mathcal{D}U(0)$ may be disintegrated as[2]

$$\mu_0 = \int_B \gamma_{\mathbf{b}}\, d\lambda(\mathbf{b}). \tag{2}$$

Here $\lambda = I \circ \mu_0$ is the image of the measure μ_0 under the map I and $\gamma_{\mathbf{b}}$, $\mathbf{b} \in B$, is a measure on the iso-integral set $\{I(u) = \mathbf{b}\}$, invariant for the Euler flow.

The Eulerian limit Theorem 1.1 supports the popular claim that the 2D Euler equation describes 2D turbulence, while the measure μ_0 and its disintegration (2) specify this claim. Accordingly, the properties of solutions for the Euler equation are relevant for 2D turbulence if they correspond to a set of vector fields u in $\mathcal{H}$ of positive μ_0-measure. For example, it is known that some steady solutions of the Euler equation are its Lyapunov-stable equilibria (see [2, Addendum 2] and [1]). They are relevant for 2D turbulence if the corresponding set of values of the vector integral I has positive λ-measure.

The task to study the limiting measure μ_0 and the measures $\gamma_{\mathbf{b}}$ and λ is (very) difficult. To develop corresponding intuition, in Section 2 we discuss as a model for (1) the damped/driven KdV equation under the periodic boundary conditions:

$$\begin{gathered} \dot u - \nu u_{xx} + u_{xxx} - 6uu_x = \sqrt{\nu}\,\eta(t,x), \\ x \in \mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}, \qquad \int u\,dx \equiv \int \eta\,dx \equiv 0. \end{gathered} \tag{3}$$

This equation is obtained by replacing the Euler equation in the NSE (1) by the KdV equation $\dot u + u_{xxx} - 6uu_x = 0$, i.e., by replacing one Hamiltonian PDE with infinitely many integrals of motion by another. The inviscid limit for equation (3) is studied in [7]. It is shown there that the limiting solutions are stationary processes $U(t,x)$ formed by smooth solutions of KdV. In this case the disintegration (2) simplifies significantly: now the measures $\gamma_{\mathbf{b}}$ are the Haar measures in infinite-dimensional tori, and λ is a measure on the octant $\mathbb{R}^\infty_+$, which is a stationary measure for an SDE obtained as the Whitham averaging of equation (3). See Theorem 2.1 in Section 2.

In Section 3 we use the results for the inviscid limit in equation (3) as a pilot model to study the disintegration (2), and suggest three assertions describing the objects involved there.

Notations. $\mathcal{P}(M)$ denotes the set of probability Borel measures on a metric space M; $\mathcal{D}\xi$ denotes the distribution of a random variable ξ; $f \circ \mu$ stands for the image of a measure μ under a map f; and the symbol $\rightharpoonup$ indicates $*$-weak convergence of Borel measures.

[1] This kind of 2D turbulence is usually modeled by the 2D NSE stirred by a stationary random force.

[2] In Theorem 1.2 the result is stated in an equivalent form, where the Euler equation is written in terms of vorticity.

1. THE EULERIAN LIMIT AND ITS VORTICITY

We first specify the force η in the 2D NSE (1). Let $(e_s,\ s\in\mathbb{Z}^2\setminus\{0\})$ be the standard trigonometric basis of H:

$$e_s(x)=\frac{\sin(sx)}{\sqrt{2}\pi|s|}\begin{bmatrix}-s_2\\ s_1\end{bmatrix}\qquad\text{or}\qquad e_s(x)=\frac{\cos(sx)}{\sqrt{2}\pi|s|}\begin{bmatrix}-s_2\\ s_1\end{bmatrix},$$

depending on whether $s_1+s_2\delta_{s_1,0}>0$ or $s_1+s_2\delta_{s_1,0}<0$. The force η is

$$\eta(t,x)=\frac{d}{dt}\zeta(t,x),\qquad \zeta(t,x)=\sum_{s\in\mathbb{Z}^2\setminus\{0\}}b_s\beta_s(t)e_s(x),$$

where $\{b_s\}$ is a set of real constants satisfying

$$b_s=b_{-s}\neq 0\quad\forall\, s,\qquad \sum|s|^2b_s^2<\infty$$

and $\{\beta_s(t)\}$ are standard independent Wiener processes.

Let μ_ν be the stationary measure for (1) and $u_\nu(t,x)$ be a corresponding solution stationary in time. It is known to be stationary (= homogeneous) in x, and a straightforward application of Ito's formula to $\|u_\nu(t)\|_0^2$ and $\|u_\nu(t)\|_1^2$ implies that

$$\mathbf{E}\|u_\nu(t)\|_1^2\equiv\frac{1}{2}B_0,\qquad \mathbf{E}\|u_\nu(t)\|_2^2\equiv\frac{1}{2}B_1,\tag{1.1}$$

where we denote $B_l=\sum|s|^{2l}b_s^2$ for $l\in\mathbb{R}$ (note that $B_0,B_1<\infty$ by assumption); see, e.g., [9].

The theorem below describes what happens to the stationary solutions $u_\nu(t,x)$ as $\nu\to 0$. There we denote by $\mathcal{H}^l$, $l\geq 0$, the Sobolev space $\mathcal{H}\cap H^l(\mathbb{T}^2;\mathbb{R}^2)$ with the norm

$$\|u\|_l=\left(\int\left((-\Delta)^{l/2}u(x)\right)^2dx\right)^{1/2}.\tag{1.2}$$

For a proof of the theorem see [8, 9].

Theorem 1.1. *Any sequence* $\widetilde{\nu}_j\to 0$ *contains a subsequence* $\nu_j\to 0$ *such that*

$$\mathcal{D}u_{\nu_j}(\cdot)\rightharpoonup\mathcal{D}U(\cdot)\qquad in\quad \mathcal{P}\big(C(0,\infty;\mathcal{H}^1)\big).$$

The limiting process $U(t)\in\mathcal{H}^1$, $U(t)=U(t,x)$, *is stationary in* t *and in* x. *Moreover, the following assertions hold*:

1. (a) *Its every trajectory* $U(t,x)$ *is such that*

$$U(\cdot)\in L_{2\,\mathrm{loc}}(0,\infty;\mathcal{H}^2),\qquad \dot U(\cdot)\in L_{1\,\mathrm{loc}}(0,\infty;\mathcal{H}^1);$$

(b) *it satisfies the free Euler equation*

$$\dot u(t,x)+(u\cdot\nabla)u+\nabla p=0,\qquad \operatorname{div}u=0;\tag{1.3}$$

(c) $\|U(t)\|_0$ *and* $\|U(t)\|_1$ *are time-independent quantities. If* g *is a bounded continuous function, then*

$$\int_{\mathbb{T}^2}g(\operatorname{rot}U(t,x))\,dx\tag{1.4}$$

is also a time-independent quantity.

2. *For each* $t\geq 0$ *we have* $\mathbf{E}\|U(t)\|_1^2=\frac12 B_0$, $\mathbf{E}\|U(t)\|_2^2\leq\frac12 B_1$ *and* $\mathbf{E}\exp\big(\sigma\|U(t)\|_1^2\big)\leq C$ *for some* $\sigma>0$ *and* $C\geq 1$.

Due to assertion 1(b), the measure $\mu_0 = \mathcal{D}U(0)$ is invariant for the Euler equation. By assertion 2 it is supported by the space $\mathcal{H}^2$ and is not the δ-measure at the origin.

Below we denote by $H^l = H^l(\mathbb{T}^d)$ the Sobolev space of functions with zero mean value on the torus $\mathbb{T}^d$, $d = 1$ or 2. The norm in this space is denoted by $\|\cdot\|_l$, i.e., as the norm in $\mathcal{H}^l$, and is defined as in (1.2).

Let us write the Euler equation (1.3) in terms of the vorticity $\xi(t,x) = \operatorname{rot} u(t,x) = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$:

$$\dot{\xi} + (u\cdot\nabla)\xi = 0, \qquad u = \nabla^{\perp}(-\Delta)^{-1}\xi. \tag{1.5}$$

Here $\nabla^{\perp} = (\partial/\partial x_2, -\partial/\partial x_1)^t$ and Δ is the Laplacian operating on functions on $\mathbb{T}^2$ with zero mean value. By Theorem 1.1, $V = \operatorname{rot} U$ satisfies (1.5) for every value of the random parameter. Now we show that $V(t)$ belongs to a certain function space K where (1.5), supplemented by an initial condition $\xi(0) = \xi_0 \in K$, has a unique solution, continuously depending on ξ_0. To define this space, we first set

$$\mathcal{K} = \bigl\{u \in L_{2\,\mathrm{loc}}(\mathbb{R};\mathcal{H}^2) \mid \dot{u} \in L_{1\,\mathrm{loc}}(\mathbb{R};\mathcal{H}^1)\bigr\}$$

with the topology of uniform convergence on bounded time intervals. This is a Polish space (i.e., a complete separable metric space). Next we define $\widetilde{\mathcal{K}}$ as the set of solutions for (1.3) belonging to $\mathcal{K}$. This is a closed subset of $\mathcal{K}$ and so is also a Polish space. The group of flow-maps of the Euler equation acts on $\widetilde{\mathcal{K}}$ by time shifts which are its continuous homeomorphisms. Now consider the continuous map

$$\pi\colon \widetilde{\mathcal{K}} \to H^0(\mathbb{T}^2), \qquad u(t,x) \mapsto \operatorname{rot} u(0,x).$$

Due to a Yudovich-type uniqueness theorem (see Lemma 3.5 in [8]), π is an embedding. We set

$$K = \pi(\widetilde{\mathcal{K}})$$

and provide K with the distance induced from $\widetilde{\mathcal{K}}$. It makes K a Polish space (we do not know if K is a linear space or not, i.e., whether it is invariant with respect to the usual linear operations). It follows from the results of [3] that $H^2(\mathbb{T}^2) \subset K$. So

$$H^2(\mathbb{T}^2) \subset K \subset H^0(\mathbb{T}^2), \tag{1.6}$$

where the embeddings are continuous.

Due to what was said above, the Euler equation defines a group of continuous homeomorphisms

$$S_t\colon\ K \to K, \qquad t \in \mathbb{R}. \tag{1.7}$$

Theorem 1.1 shows that $U(\cdot) \in \widetilde{\mathcal{K}}$ for each value of the random parameter. Therefore, the measure

$$\theta = \mathcal{D}V(0) = \operatorname{rot} \circ \mu_0$$

is supported by K (i.e., $\theta(K) = 1$). Since $S_t \circ \theta = \mathcal{D}V(t)$ and $V(t)$ is a stationary process, θ is an invariant measure for the dynamical system (1.7). By the estimates in assertion 2 of Theorem 1.1, it is supported by the space $K \cap H^1(\mathbb{T}^2)$ and $\int \exp(\sigma\|v\|_0^2)\,\theta(dv) < \infty$.

Our next goal is to express assertion 1(c) of Theorem 1.1 in terms of the measure θ. Let us denote by $\mathcal{P}(\mathbb{R})$ the set of probability Borel measures on $\mathbb{R}$ furnished with the Lipschitz-dual distance

$$\operatorname{dist}(m', m'') = \sup_{f\in\mathcal{L}} \bigl|\langle m', f\rangle - \langle m'', f\rangle\bigr|,$$

where $\mathcal{L}$ is the set of all Lipschitz functions f on $\mathbb{R}$ such that $\mathrm{Lip}(f) \le 1$ and $|f| \le 1$. This is a Polish space, and the convergence with respect to the introduced distance is equivalent to the $*$-weak convergence of measures (see [5]). Due to (1.6) the map

$$M\colon\ K \to \mathcal{P}(\mathbb{R}), \qquad \xi \mapsto \xi \circ \big((2\pi)^{-2}dx\big),$$

is continuous. Indeed, for any $f \in \mathcal{L}$ we have

$$\big|\langle M\xi_1, f\rangle - \langle M\xi_2, f\rangle\big| = \left|\int \big(f(\xi_1(x)) - f(\xi_2(x))\big)(2\pi)^{-2}dx\right| \le \int |\xi_1 - \xi_2|(2\pi)^{-2}dx = o(1) \quad \text{as} \quad \mathrm{dist}(\xi_1, \xi_2) \to 0,$$

where we use (1.6) to get the last equality. Therefore, the map

$$\Psi\colon\ K \to \mathcal{P}(\mathbb{R}) \times \mathbb{R}_+ =: B, \qquad \xi(\cdot) \mapsto (M(\xi), \|\xi\|_{-1}),$$

is also continuous.

Repeating (say) the arguments in [8, 9] which prove assertion 1(c) of Theorem 1.1, we find that each trajectory $u(t, x)$ of (1.3) belonging to the space $\mathcal{K}$ satisfies $\Psi(u(t)) = \mathrm{const}$. Recalling the definition of the flow-maps S_t, we see that they commute with Ψ. That is,

$$S_t\colon\ K_{\mathbf{b}} \to K_{\mathbf{b}} \qquad \forall\, t \in \mathbb{R} \tag{1.8}$$

for every $\mathbf{b} \in B$. Here

$$K_{\mathbf{b}} = \Psi^{-1}(\mathbf{b}) \subset K, \qquad \mathbf{b} \in B.$$

Clearly, $K_{\mathbf{b}}$ is a closed subset of K, and it is not hard to see that it is not compact. We provide $K_{\mathbf{b}}$ with the induced topology.

Let us denote $\lambda = \Psi \circ \theta$. This is a measure on Borel subsets of the Polish space B.

Theorem 1.2. *There exists a family* $\{\theta_{\mathbf{b}},\ \mathbf{b} \in B\}$ *of measures on Borel subsets of* K *with the following properties*:

1. $\theta_{\mathbf{b}}(K_{\mathbf{b}}) = 1$ *for each* $\mathbf{b}$, *the function* $\mathbf{b} \mapsto \theta_{\mathbf{b}}(A)$ *is Borel-measurable on* B *for any Borel set* $A \subset K$, *and*

$$\theta(A \cap \Psi^{-1}(D)) = \int\limits_D \theta_{\mathbf{b}}(A)\, d\lambda(\mathbf{b}) \tag{1.9}$$

for any Borel set $D \subset B$.

2. *For* λ*-a.a.* $\mathbf{b} \in B$ *the measure* $\theta_{\mathbf{b}}$ (*interpreted as a measure on Borel subsets of* $K_{\mathbf{b}}$) *is invariant for the dynamical system* (1.8).

Proof. 1. This group of assertions makes the statement of the disintegration theorem applied to a measurable map

$$\Psi\colon\ (K, \mathcal{A}, \theta) \to (B, \mathcal{B}, \lambda),$$

where $\mathcal{A}$ and $\mathcal{B}$ are the Borel sigma-algebras for the Polish spaces K and B. For the case when K and B are locally compact sets, the theorem is proved in [4]. The case of arbitrary Polish spaces reduces to the above case since the measure spaces $(K, \mathcal{A})$ and $(B, \mathcal{B})$ are both isomorphic to the unit segment $[0, 1]$ with the Borel sigma-algebra, by means of some measurable isomorphism (see [5]).

2. Since the measure θ is invariant for the dynamical system (1.7) and since Ψ commutes with the flow-maps S_t, for any Borel sets $A \subset K$ and $D \subset B$ we have $\int_D \theta_{\mathbf{b}}(A)\, d\lambda(\mathbf{b}) = \int_D \theta_{\mathbf{b}}(S_t A)\, d\lambda(\mathbf{b})$. That is,

$$\int\limits_D \theta_{\mathbf{b}}\, d\lambda(\mathbf{b}) = \int\limits_D (S_t \circ \theta_{\mathbf{b}})\, d\lambda(\mathbf{b}),$$

where the integrands are vector functions with values in the linear space of signed Borel measures on K. Hence, $\theta_{\mathbf{b}} = S_t \circ \theta_{\mathbf{b}}$ for λ-a.a. $\mathbf{b}$, as stated. □

2. THE KdV-MODEL

In this section we discuss the inviscid limit in the equation (3) as a model for the Eulerian limit. The results described below are obtained in [7].

The force η in (3) is assumed to have the same form as the force in Section 1:

$$\eta = \frac{\partial}{\partial t} \sum_{s \in \mathbb{Z} \setminus \{0\}} b_s \beta_s(t) e_s(x).$$

Here

$$e_s(x) = \begin{cases} \cos sx, & s > 0, \\ \sin sx, & s < 0, \end{cases}$$

$\{\beta_s(t)\}$ are standard independent Wiener processes, and the coefficients b_s satisfy

$$b_s = b_{-s} \neq 0 \quad \forall\, s, \qquad b_s \le C_m |s|^{-m} \quad \forall\, m, s$$

with suitable constants C_m.

For the same reasons as for the NSE, equation (3) has a unique stationary measure. Let $u_\nu(t,x)$, $t \ge 0$, be a corresponding stationary solution. It is also stationary in x, and its Sobolev norms satisfy the following estimates uniformly in $\nu > 0$:

$$\mathbf{E}\, e^{\sigma \|u_\nu(t)\|_0^2} \le C_\sigma < \infty, \qquad \mathbf{E} \|u_\nu\|_m^k \le C_{m,k} < \infty \tag{2.1}$$

for a suitable $\sigma > 0$ and for all m and k (see [7]). These estimates allow one to prove an analog of Theorem 1.1 for equation (3), i.e., to establish that any sequence $\widetilde{\nu}_j \to 0$ contains a subsequence $\nu_j \to 0$ such that

$$\mathcal{D} u_{\nu_j}(\cdot) \rightharpoonup \mathcal{D} U(\cdot) \qquad \text{in} \quad \mathcal{P}(C(0,\infty; H^m))$$

for each m. The limiting process $U(t) = U(t,x)$ is stationary in t and in x and satisfies estimates (2.1). Its marginal distribution

$$\theta = \mathcal{D} U(0)$$

is an invariant measure for KdV, and its every realisation $U(t,x)$ is a smooth solution for KdV. Solutions of that equation have infinitely many integrals of motion. The realisation of these integrals that was studied in [10, 6] (see also [7]) is the most convenient one for our purposes. These integrals are non-negative analytic functions $I_1, I_2, \dots$ on the space H^0, and in contrast to the integrals for the Euler equation, the structure of the iso-integral sets

$$T_I = \{u \mid I_j(u) = I_j \ge 0\ \forall\, j\}$$

is now well understood. Namely, each T_I is an analytic torus in H^0 of dimension $|\mathcal{J}(I)| \le \infty$, where $\mathcal{J}(I) = \{j \mid I_j > 0\}$. Moreover, the torus T_I carries a cyclic coordinate $\varphi = \{\varphi_j,\, j \in \mathcal{J}(I)\}$ such that in the coordinates (I, φ) the KdV-dynamics takes the integrable form

$$\dot I = 0, \qquad \dot\varphi = W(I),$$

where the frequency vector $W(I)$ is analytic and non-degenerate in I. The Haar measure $d\varphi$ on any torus T_I is invariant for this dynamics. Since the map $I \mapsto W(I)$ is analytic and non-degenerate, this is a unique invariant Borel measure on T_I for a "typical" I. It turns out that the limiting measure θ is supported by the set $\{u \in H^\infty = \bigcap H^m \mid I_j(u) > 0\ \forall\, j\}$, which is measurably isomorphic to a Borel subset of $\mathbb{R}_+^\infty \times \mathbb{T}^\infty$. Under this isomorphism, θ can be written as

$$\theta = \lambda \times d\varphi, \tag{2.2}$$

where $\lambda = \mathcal{D}I(U(0))$ is a Borel measure on $\mathbb{R}_+^\infty$. This is a KdV-analog of the disintegration (1.9). It simplifies compared with the NSE-case since, firstly, the iso-integral sets T_I are isomorphic to tori, while the topology of the sets $K_{\mathbf{b}}$ is unknown, and, secondly, any set T_I carries the simple KdV-invariant measure $d\varphi$, which is typically a unique invariant measure, while the structure of the invariant measures $\theta_{\mathbf{b}}$ in (1.9) is unknown.

To describe the measure λ in (2.2), we apply Ito's formula to equation (3) and the map $u \mapsto I(u)$, and pass to the fast time $\tau = \nu^{-1}t$ in the obtained equation. We get a system

$$dI_k(\tau) = F_k(u)\, d\tau + \sum_j \sigma_{kj}(u)\, d\beta_j(\tau), \qquad k = 1, 2, \dots \tag{2.3}$$

(see [7, Section 3]). This is an infinite-dimensional SDE with the drift F_k and the diffusion matrix $a_{kl} = \sum_j \sigma_{kj}\sigma_{lj}$. Clearly, $I_\nu(t) = I(u_\nu(t))$ is a stationary solution for this system. Let us define the averaged drift and the averaged diffusion matrix as follows:

$$\langle F_k\rangle(I) = \int\limits_{T_I} F_k(u(\varphi))\, d\varphi, \qquad \langle a_{kl}\rangle(I) = \int\limits_{T_I} a_{kl}(u(\varphi))\, d\varphi \tag{2.4}$$

($u(\varphi)$ is a point in T_I with the coordinate φ).

Theorem 2.1. *Every sequence $\widetilde\nu_j \to 0$ contains a subsequence $\nu_j \to 0$ such that the process $I(u_{\nu_j}(\tau))$ (where $\tau = \nu^{-1}t$) $*$-weakly converges in distribution in the space $\mathcal{P}(C(0,\infty; H^0))$ to a stationary process $I(\tau)$, which is a solution for the martingale problem with the drift and diffusion as in (2.4). The law $\mathcal{D}I(0)$ equals $\lambda = \mathcal{D}I(U(0))$. Its finite-dimensional projections*[3] *are absolutely continuous with respect to the Lebesgue measure, and all moments of all norms $|I|_m = \sum |I_j| j^m$ with respect to the measure λ are finite.*

Thus, the averaged martingale problem, which can be formally written as

$$dI(\tau) = \langle F\rangle(I)\, d\tau + \big(\langle a\rangle\langle a\rangle^t\big)^{1/2}(I)\, d\beta(\tau), \tag{2.5}$$

describes the limiting behaviour of the stationary measures $\mathcal{D}u_{\nu_j}(t)$. Indeed, the limiting (as $\nu_j \to 0$) measure may be written in the form (2.2), where λ is a stationary measure for (2.5). It is plausible that (2.5) has a unique stationary measure λ. If so, then $\mathcal{D}u_\nu(t) \rightharpoonup \lambda \times d\varphi$ as $\nu \to 0$.

[3]That is, images of λ under the maps $I \mapsto (I_1, \dots, I_M) \subset \mathbb{R}_+^M$, $M \in \mathbb{N}$.

3. ERGODIC HYPOTHESIS FOR THE EULERIAN LIMIT

We believe that the damped/driven KdV equation (3) is a right model for the Eulerian limit (that is, a right model for the space-periodic stationary 2D turbulence). Guided by this belief, below we make statements 1–3 which specify the results of Section 1 concerning the Eulerian limit. Crucial among them is the first one, which is an ergodic hypothesis for the Euler equation on a typical iso-integral set $K_{\mathbf{b}}$. We cannot prove the three statements, but, firstly, we believe that the validity of their analogs for the KdV-model justifies the assumptions to some extent and, secondly, we can prove some fragments of the picture given by the assumptions (the corresponding results will be published elsewhere).

1. Every non-empty set $K_{\mathbf{b}}$ carries a measure $m_{\mathbf{b}}$, invariant for the Euler flow (1.8), such that for a.a. $u \in \mathcal{H}$ with respect to any stationary measure μ_ν we have

$$\lim_{T\to\infty} T^{-1}\int_0^T f(S_t u)\,dt = \langle f, m_{\mathbf{b}}\rangle, \qquad \mathbf{b} = \Psi(u). \tag{3.1}$$

For λ-a.a. $\mathbf{b}$ the measure $m_{\mathbf{b}}$ coincides with $\theta_{\mathbf{b}}$ in (1.9).

Let $u(t,x)$ be a solution of (1). Then the vector $\mathbf{b}(t) = \Psi(u(t)) \in B$ satisfies an SDE. To describe it, we introduce the space $\mathcal{C} = C_b(\mathbb{R}) \times \mathbb{R}$ and denote by $\langle\cdot,\cdot\rangle$ its natural pairing with $\mathcal{P}(\mathbb{R}) \times \mathbb{R} \supset B$. For any $\mathbf{f} \in \mathcal{C}$ we set

$$\Psi_{\mathbf{f}}(u) = \langle \Psi(u), \mathbf{f}\rangle.$$

Applying Ito's formula to $u_{\mathbf{f}} = \Psi_{\mathbf{f}}(u)$, where $u(t)$ satisfies (1), and passing to the fast time $\tau = \nu t$, we get

$$du_{\mathbf{f}}(\tau) = F_{\mathbf{f}}(u(\tau))\,d\tau + \sum_s \sigma_{\mathbf{f}s}(u(\tau))\,d\beta_s(\tau). \tag{3.2}$$

Here $F_{\mathbf{f}}$ and $\{\sigma_{\mathbf{f}s},\ s \in \mathbb{Z}^2 \setminus 0\}$ are smooth functions on $\mathcal{H}$ (depending on the coefficients b_s) and $\{\beta_s(\tau)\}$ are new standard Wiener processes.

2. Let $u_\nu(t)$ be a stationary solution of (1). Then, along a subsequence $\nu_j \to 0$, the process $\Psi(u_\nu(\tau))$ converges in distribution to a limiting process $\mathbf{b}(\tau)$. This is a stationary Ito process in B such that for any $\mathbf{f} \in \mathcal{C}$ the process $b_{\mathbf{f}}(\tau) = \langle \mathbf{b}(\tau), \mathbf{f}\rangle$ has the drift

$$\langle F\rangle_{\mathbf{f}}(\mathbf{b}) = \langle F_{\mathbf{f}}(u), m_{\mathbf{b}}\rangle = \int_{K_{\mathbf{b}}} F_{\mathbf{f}}(u)\, m_{\mathbf{b}}(du), \tag{3.3}$$

and for any $\mathbf{f}_1, \mathbf{f}_2 \in \mathcal{C}$ the processes $b_{\mathbf{f}_1}$ and $b_{\mathbf{f}_2}$ have the covariation

$$\langle a\rangle_{\mathbf{f}_1\mathbf{f}_2}(\mathbf{b}) = \Bigl\langle \sum_s \sigma_{\mathbf{f}_1 s}(u)\sigma_{\mathbf{f}_2 s}(u), m_{\mathbf{b}}\Bigr\rangle; \tag{3.4}$$

i.e., the processes $\mathbf{b}_{\mathbf{f}_1}(\tau) - \int_0^\tau \langle F\rangle_{\mathbf{f}_1}(\mathbf{b}(s))\,ds$ and $\mathbf{b}_{\mathbf{f}_2}(\tau) - \int_0^\tau \langle F\rangle_{\mathbf{f}_2}(\mathbf{b}(s))\,ds$ are martingales and their bracket equals $\langle a\rangle_{\mathbf{f}_1\mathbf{f}_2}(\mathbf{b}(\tau))$.

That is to say, the limiting process $\mathbf{b}(\tau)$ is a martingale solution for a stochastic differential equation in the space B with the drift $\langle F\rangle(\mathbf{b})$ and the covariance $\langle a\rangle(\mathbf{b})$. This is the *Whitham equation for the* 2D *NSE* (1).

In contrast to the KdV case, the space B does not have a natural basis, and we cannot *naturally* write this equation as a system of infinitely many SDEs. Still we can find a countable system of vectors $f_j \in \mathcal{C}$ such that their linear combinations are dense in $\mathcal{C}$, and use these vectors in (3.3) and (3.4). In this way we write the Whitham equation above as an over-determined system of SDEs.

We can write down the Whitham equation using in (3.3) and (3.4) the measures $\theta_{\mathbf{b}}$ instead of the measures $m_{\mathbf{b}}$, i.e., without invoking the ergodic hypothesis 1. But if the hypothesis fails, then the measures $\theta_{\mathbf{b}}$ may depend on the sequence $\nu_j \to 0$ in Theorem 1.1. In this case the Whitham equation seems to be a useless object.

3. The distribution of the limiting process $\mathbf{b}(\tau)$ is independent of the sequence $\{\nu_j \to 0\}$. Accordingly, the measure $\theta = \mathcal{D}\mathbf{b}(0)$ is also independent of the sequence, and

$$\Psi \circ \mu_\nu \rightharpoonup \theta, \qquad \mu_\nu \rightharpoonup \mu_0 = \int m_{\mathbf{b}}\, \theta(d\mathbf{b})$$

as $\nu \to 0$.

The crucial assumption for the scenario above is the ergodic hypothesis 1. We assume there that (3.1) holds for μ_ν-a.a. initial data (rather than for all of them) since the analogy with the damped/driven KdV suggests that the dynamics $\{S_t\}$ may be non-ergodic (i.e., "resonant") on some atypical sets $K_{\mathbf{b}}$.

The task of proving hypothesis 1 and finding out the properties of the measures $m_{\mathbf{b}}$ seems to be very difficult. In particular, it is not clear for us which role in this problem plays the fact that the group of area-preserving diffeomorphisms of $\mathbb{T}^2$ acts transitively on the set of equidistributed functions $K_m = \bigcup_{b \in \mathbb{R}_+} K_{(m,b)}$, $m \in \mathcal{P}(\mathbb{R})$.

Although the measures $m_{\mathbf{b}}$ are unknown, the averaged drift and covariance (3.3) and (3.4), which characterise the limiting equation, can be calculated by replacing the ensemble-average by the time-average (see (3.1)). So the validity of the suggested scenario 1–3 may be verified numerically by comparing $\Psi(u_\nu(\tau))$ with solutions for the limiting equation.

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