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# KdV Hamiltonian as a Function of Actions

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**Abstract** We prove that the non-linear part of the Hamiltonian of the KdV equation on the circle, written as a function of the actions, defines a continuous convex function on the $\ell^2$ space and derive for it lower and upper bounds in terms of some functions of the $\ell^2$-norm. The proof is based on a new representation of the Hamiltonian in terms of the quasimomentum, obtained via the conformal mapping theory.

**Keywords** KDV · Action variables

**Mathematics Subject Classification (2010)** 34A55 · 34B24 · 47E05

## 1 Introduction and Main Results

We consider the Korteweg de Vries (KdV) equation under zero mean-value periodic boundary conditions:

$$\begin{aligned} q_t &= -q_{xxx} + 6qq_x, \qquad\qquad x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}, \\ &\int_0^1 q(x,t)\,dx = 0. \end{aligned} \tag{1.1}$$

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For any $\alpha \in \mathbb{R}$ denote by $\mathscr{H}_\alpha$ the Sobolev space of real-valued 1-periodic functions with zero mean-value. In particular, we have

$$\mathscr{H}_\alpha = \mathscr{H}_\alpha(\mathbb{T}) = \left\{ q \in L^2(\mathbb{T}) : \; q^{(\alpha)} \in L^2(\mathbb{T}), \quad \int_0^1 q(x)\,dx = 0 \right\}, \qquad \alpha \geqslant 0.$$

We provide the spaces $\mathscr{H}_\alpha$ with the trigonometric base $\{e_{\pm1}, e_{\pm2}, e_{\pm3}, \ldots.\}$, where

$$e_j(x) = \sqrt{2}\cos 2\pi jx, \qquad e_{-j}(x) = -\sqrt{2}\sin 2\pi jx, \qquad j \geqslant 1.$$

We also introduce real spaces $\ell^p_\alpha$ of sequences $f = (f_n)_1^\infty$, equipped with the norms

$$\|f\|^p_{p,\alpha} = \sum_{n\geqslant1} (2\pi n)^{2\alpha} |f_n|^p, \qquad p \geqslant 1, \; \alpha \in \mathbb{R}, \tag{1.2}$$

and positive octants

$$\ell^p_{\alpha,+} = \left\{ f = (f_n)_1^\infty \in \ell^p_\alpha : f_n \geqslant 0, \; \forall\, n \geqslant 1 \right\}.$$

In the case $\alpha = 0$, we write $\ell^p = \ell^p_0$, $\ell^p_+ = \ell^p_{0,+}$ and $\|\cdot\|_p = \|\cdot\|_{p,0}$.

The operator $\frac{\partial}{\partial x}$ defines linear isomorphisms $\frac{\partial}{\partial x} : \mathscr{H}_\alpha \to \mathscr{H}_{\alpha-1}$. Denoting by $\left(\frac{\partial}{\partial x}\right)^{-1}$ the inverse operator, we provide the spaces $\mathscr{H}_\alpha, \alpha \geqslant 0$, with a symplectic structure by means of the 2-form $\omega_2$:

$$\omega_2(q_1, q_2) = -\left\langle (\partial/\partial x)^{-1} q_1, q_2 \right\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in $L^2(0,1)$. Then in any space $\mathscr{H}_\alpha, \alpha \geqslant 1$, the KdV equation (1.1) may be written as a Hamiltonian system with the Hamiltonian $H_2$, given by

$$H_2(q) = \frac{1}{2}\int_0^1 (q'(x)^2 + 2q^3(x))\,dx.$$

That is, as the system

$$q_t = \frac{\partial}{\partial x}\frac{\partial}{\partial q} H_2(q), \tag{1.3}$$

e.g., see [4, 7, 20] (note that $H_2$ is an analytic function on any space $\mathscr{H}_\alpha$, $\alpha \geqslant 1$).

It is well known after the celebrated work of Novikov, Lax, Its, and Matveev that the system (1.3) is integrable. It was shown by Kappeler and collaborators in a series of publications, starting with [6], that it admits global Birkhoff coordinates. Namely, for any $\alpha \in \mathbb{R}$ denote by $\mathfrak{h}_\alpha$ the Hilbert space, formed by real sequences $b = (b_n, b_{-n})_1^\infty$, equipped with the norm

$$\|b\|^2_{\mathfrak{h}_\alpha} = \sum_{n\geqslant1} (2\pi n)^{2\alpha} \left( b_n^2 + b_{-n}^2 \right).$$

We provide the spaces $\mathfrak{h}_\alpha$ with the usual symplectic form

$$\Omega_2 = \sum_{n\geqslant1} db_n \wedge db_{-n},$$

and define the actions $I = (I_n)_1^\infty$ and the angles $\phi = (\phi_n)_1^\infty$ by

$$I_n = \frac{1}{2}\left( b_n^2 + b_{-n}^2 \right), \qquad \phi_n = \arctan\frac{b_n}{b_{-n}}. \tag{1.4}$$

This is another set of symplectic coordinates on $\mathfrak{h}_\alpha$, since $\Omega_2 = dI \wedge d\phi$, at least formally. Then

1) There exists an analytic symplectomorphism $\Psi : \mathscr{H}_0 \to \mathfrak{h}_{\frac{1}{2}}$ which defines analytic diffeomorphisms $\Psi : \mathscr{H}_\alpha \to \mathfrak{h}_{\alpha+\frac{1}{2}}, \alpha \geqslant -1$, such that $d\Psi(0) = \Phi$, where

$$\Phi\left(\sum_{j\geqslant 1}\left(u_j e_j(x) + u_{-j}e_{-j}(x)\right)\right) = b, \qquad b_j = |2\pi j|^{\frac{1}{2}} u_j, \quad \forall j. \tag{1.5}$$

2) The transformed Hamiltonian $H_2\left(\Phi^{-1}(b)\right)$ (which is an anlytic function on the space $\mathfrak{h}_{\frac{3}{2}}$) depends solely on the actions $I$, i.e., $K(I(b)) = H_2\left(\Phi^{-1}(b)\right)$, where $K(I)$ is an analytic function on the octant $\ell^1_{\frac{3}{2},+}$. A curve $q(\cdot,t) \in C^1\left(\mathbb{R}, \mathscr{H}_0\right)$ is a solution of Eq. 1.1 if and only if $b(t) = \Phi(q(\cdot,t))$ satisfies the following system of equations

$$\frac{\partial b_n}{\partial t} = -b_{-n}\frac{\partial K}{\partial I_n}, \qquad \frac{\partial b_{-n}}{\partial t} = b_n \frac{\partial K}{\partial I_n}, \qquad n \geqslant 1, \tag{1.6}$$

where $I = (I(b))$.

For 1)-2) with $\alpha \geqslant 0$ see [7] and with $\alpha = -1$ see [8]. See [22] for the important quasilinearity property of the transformation $\Psi$.

Note that

$$\|b\|_{\mathfrak{h}} = 2|I|_{1,\alpha}.$$

Thus, if $I \in \ell^p_+$ for some $p < \infty$, then $I \in \ell^1_{-\frac{1}{2},+}$ and the corresponding potential $q \in \mathscr{H}_{-1}$.

By 2), in the action-angle variables $(I, \phi)$ the KdV equation takes the form

$$I_t = 0, \qquad \phi_t = \frac{\partial}{\partial I} K(I). \tag{1.7}$$

This reduction of KdV is due to McKean-Trubowitz [28] and was found before the Birkhoff form (1.6). The action maps $\psi \mapsto I_j$, $j \geqslant 1$, are given by explicit formulas due to Arnold and are defined in a unique way. So the Hamiltonian $K(I)$ also is uniquely defined, see [3]. But the symplectic angles are defined only up to rotations $\phi \mapsto \phi + (\partial/\partial I)g(I)$, where $g$ is any smooth function. So the transformation $\Psi$ is not unique.

The Birkhoff coordinates $b$ and the actions-angles $(I, \phi)$ make an effective tool to study properties of the KdV equation, see [8], and of its perturbations, see [4, 21]. For both these goals, it is important to understand properties of the Hamiltonian $K(I)$ which defines the dynamics (1.6) and (1.7). But the only information about the function $K(I)$ which follows from 1)-2) is that it is analytic on the spaces $\ell^1_{p,+}$, $p \geqslant 3/2$.[1]

Denote by $P_j$ moments of the actions $I$, given by

$$P_j = \sum_{n\geqslant 1}(2\pi n)^j I_n, \qquad j \in \mathbb{Z}. \tag{1.8}$$

Note that

$$P_1 = \frac{1}{2}\|q\|^2, \qquad \text{if} \quad I = I(b), \; b = \Phi(q), \tag{1.9}$$

[1] About the behavior of $K(I)$ of the finite-dimensional subspaces $\tilde{\ell}^N$, defined below in Eq. 1.11, we know more. See in [20] and below in Introduction.

this is the Parseval identity for the transformation $\Phi$, see [15, 27]. Due to Eq. 1.5, the linear part $dK(0)(I)$ of $K(I) = H_2\left(\Phi^{-1}(b)\right)$ at the origin equals

$$\frac{1}{2}\int_0^1 \left(\frac{\partial}{\partial x}\left(\Phi^{-1}b\right)\right)^2 dx = P_3.$$

Therefore

$$K(I) = P_3(I) + O\left(I^2\right).$$

The cubic part $\int_0^1 q^3(x)\,dx$ of the Hamiltonian $H_2(q)$ is more regular than its quadratic part $\frac{1}{2}\int_0^1 q'(x)^2\,dx$. Thus, it is natural to assume that the linear term $P_3$ is a singular part of $K(I)$ and to study smoothness of the more regular quadratic part $V$, given by

$$H_2(q) = K(I) = P_3(I) - V(I). \tag{1.10}$$

Here, the minus-sign is convenient, since below we will see that $V \geqslant 0$. For any $N \geqslant 1$ denote by $\widetilde{\ell}^N \subset \ell^2$ the N-dimensional subspace

$$\widetilde{\ell}^N = \left\{I = (I_n)_1^\infty,\ I_n = 0 \quad \forall\, n > N\right\}, \tag{1.11}$$

and set $\widetilde{\ell}^\infty = \cup \widetilde{\ell}^N$. Clearly, $V$ is analytic on each octant $\widetilde{\ell}^N_+$ (i.e., it analytically extends to a neighborhood of $\widetilde{\ell}^N_+$ in $\widetilde{\ell}^N$). So $V$ is Gato-analytic on $\widetilde{\ell}^\infty_+$. That is, it is analytic on each interval $\left\{(a+tc) \in \widetilde{\ell}^\infty_+ | t \in \mathbb{R}\right\}$, where $a, c \in \widetilde{\ell}^\infty_+$. It is known that

$$\frac{\partial^2 V(0)}{\partial I_i \partial I_j} = 6\delta_{i,j} \qquad \forall \quad i, j \geqslant 1, \tag{1.12}$$

see [1] and [4, 7, 20]. So $d^2V(0)(I) = 6\|I\|_2^2$. This suggests that the Hilbert space $\ell^2$ rather than the Banach space $\ell^1_{\frac{3}{2}}$ (which is contained in $\ell^2$) is a distinguished phase-space for the Hamiltonian $K(I)$. This guess is justified by the following theorem which is the main result of our work.

**Theorem 1.1** *The function $V : \widetilde{\ell}^\infty_+ \to \mathbb{R}$ extends to a non-negative continuous function on the $\ell^2$-octant $\ell^2_+$, such that $V(I) = 0$ for some $I \in \ell^2_+$ iff $I = 0$. Moreover,*

$$0 \leqslant V(I) \leqslant 8P_1P_{-1}, \qquad \forall\, I \in \ell^1_{1,+}, \tag{1.13}$$

*and*

$$\frac{\pi}{10}\frac{\|I\|_2^2}{\left(1 + P_{-1}^{\frac{1}{2}}\right)} \leqslant V \leqslant \left(4^{\frac{11}{2}}\left(1 + P_{-1}^{\frac{1}{2}}\right)^{\frac{1}{2}} P_{-1}^2 + 6\pi e^{\sqrt{P_{-1}}}\|I\|_2\right)\|I\|_2, \qquad \forall\, I \in \ell^2. \tag{1.14}$$

Let $X$ be a Banach space which contains $\widetilde{\ell}^\infty$ as a dense subsets. We say that the function $V(I)$ *agrees with the norm* $\|I\|_X$ if $V$ extends to a continuous function on $X_+$ (= the closure of $\widetilde{\ell}^\infty_+$ in $X$) and

$$F_1(\|I\|_X) \leqslant V(I) \leqslant F_2(\|I\|_X), \qquad \forall\, I \in X_+,$$

where $F_1, F_2$ are monotonous continuous functions from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that $F_j(0) = 0$ and $F_j(t) \to \infty$ as $t \to \infty$, $j = 1, 2$. It is easy to see that there exists at most one Banach space $X$ as above (i.e., if $X'$ is another space, then $X = X'$ and the two norms are equivalent).

Estimates (1.14) imply that the function $V(I)$ agrees with the norm $\|I\|_2$. So $\ell^2$ is the natural phase space for the non-linear part $V$ of the Hamiltonian $K(I)$. In Section 3, we show that $\ell^2$-sequences $I$ correspond to potentials $q \in \mathscr{H}_{-1}$ and in general these potentials do not belong to $\mathscr{H}_{1/2}$ (see there Remark 2).

A proof of the theorem is based on a new identity (see Theorem 4.2), representing $V(I)$ in terms of the quasimomentum of the Hill operator with a potential $q$. It uses properties of the conformal mapping, associated with this quasimomentum, developed in [10–16].

*Remark* 1) Equation 1.13 improves the known estimate $|H_2(q)| \leqslant 4^5 P_3 \left(1 + P_3^{\frac{4}{3}}\right)$ from [13].

2) We claim that the function $V$ is real analytic on $\ell^2_+$. This will be proven elsewhere.
3) The complete Hamiltonian $K(I)$ is analytic on the space $\ell^1_{3/2,+}$. Our results show that the function $K(I) - dK(0)(I) = -V(I)$ is smoother and continuously extends to a larger space $\ell^2_+$. A natural question is if the function

$$K(I) - dK(0)(I) - \tfrac{1}{2}d^2K(0)(I,I) = K(I) - P_3 + 3\|I\|_2^2$$

is even smoother and continuously extends to a larger space, etc. We do not know the answer.
4) By Theorem 1.1, $V(I)$ admits a quadratic upper bound in terms of $P_1$. The estimate (1.14) implies the exponential upper bound for $V$ in terms of $\|I\|_2$. The bottle neck of our proof which yields the unpleasant exponential factor in Eq. 1.14 is the Bernstein inequality, used in Section 3 to prove Lemma 3.1. We conjecture that, in fact, $V(I)$ is bounded by a polynomials of $\|I\|_2$.

Consider the restriction of the function $V(I)$ to $\widetilde{\ell}^N_+$ with any $N \geqslant 1$. It is known that the corresponding Hessian is non-degenerate:

$$\det \left\{ \frac{\partial^2 V(I)}{\partial I_i \partial I_j} \right\}_{1 \leqslant i,j \leqslant N} \neq 0, \qquad \forall\, I \in \ell^+_N. \tag{1.15}$$

This result was proven in [19] with serious omissions, fixed in [2] (see also Appendix 3.6 in [20] and [4]). Since $V$ is analytic on $\widetilde{\ell}^N_+$, then Eqs. 1.12 and 1.15 yield that the Hessian of $V|_{\widetilde{\ell}^N_+}$ is a positive $N \times N$ matrix. Thus $V$ is convex on $\ell^+_N$. Since $\widetilde{\ell}^\infty = \cup \widetilde{\ell}^N$ is dense in $\ell^2$, where $V$ is continuous, we get

**Corollary 1.2** *The function $V(I)$ is convex on $\ell^2_+$.*

*Remark* 5) By Eq. 1.12 and Remark 2, the function $V$ is strictly convex in some vicinity of the origin in $\ell^2_+$ (note that $\ell^2_+$ is the only phase-space where $V$ is strictly convex). We conjecture that it is strictly convex everywhere in $\ell^2_+$.

In difference with $V(I)$, the total Hamiltonian $K(I)$ is not continuous on $\ell^2_+$ since its linear part $P_3(I)$ is there an unbounded linear functional. But $P_3(I)$ contributes to Eq. 1.6 the linear rotations

$$\frac{\partial b_n}{\partial t} = -(2\pi n)^3 b_{-n}, \qquad \frac{\partial b_{-n}}{\partial t} = (2\pi n)^3 b_n, \qquad n \geqslant 1.$$

So the properties of Eq. 1.6 essentially are determined by the component $-V(I)$ of the Hamiltonian $K(I)$. We also note that since $P_3(I)$ is a bounded linear functional on the space $\ell^1_{3/2,+} \subset \ell^2$, then the complete Hamiltonian $K(I) = P_3 - V$ is concave on $\ell^1_{3/2,+}$. The flow of the KdV equation in the action-angle variables (1.7) is

$$(I, \phi) \to \big(I, \phi(t) = \phi + tK'(I)\big), \quad t \in \mathbb{R}, \qquad K'(I) = \frac{\partial K(I)}{\partial I}.$$

Since the function $K$ is concave and analytic on $\ell^1_{3/2,+}$, then the flow-maps are twisting:

$$\langle \phi(t; I_{(2)}, \phi_{(1)}) - \phi(t; I_{(1)}, \phi_{(1)}), I_{(2)} - I_{(1)} \rangle = t\langle K'(I_{(2)}) - K'(I_{(1)}), I_{(2)} - I_{(1)} \rangle \leqslant 0 \qquad \forall t \geqslant 0.$$

If the assertion of Remark 5 above holds true, then L.H.S. is $\leqslant -Ct\|I_{(2)} - I_{(1)}\|_2^2$, where the positive constant $C$ depends on $I_{(2)}, I_{(1)}$.

In the finite-dimensional case convexity (and strict convexity) of an integrable Hamiltonian significantly simplifies the study of long time behavior of actions of solutions for perturbed equations. Similarly, we are certain that results of this work will help to study perturbations of the KdV equation (1.1), especially those which are Hamiltonian . It is important that as a phase space, our results suggest the Hilbert space $\ell^2$, rather than a weighted $\ell^1$-space .

## 2 Momentum, Quasimomentum, and KdV Equation

### 2.1 Spectrum of the Hill Operator

We consider the Hill operator $T$ acting in $L^2(\mathbb{R})$ and given by

$$T = -\frac{d^2}{dx^2} + q_0 + q(x),$$

where a 1-periodic potential $q$ (with zero mean-value) belongs to the Sobolev space $\mathscr{H}_\alpha, \alpha \geqslant -1$ and $q_0$ is a constant (so the potential $q_0 + q$ may be a distribution). Below, we recall the results from [12] on the Hill operator with potentials $q \in \mathscr{H}_{-1}$. The spectrum of $T$ is absolutely continuous and consists of intervals (*spectral bands*) $\mathfrak{S}_n$, separated by *gaps* $\gamma_n$ and is given by (see also Fig. 1)

$$\mathfrak{S}_n = [\lambda^+_{n-1}, \lambda^-_n], \qquad \gamma_n = (\lambda^-_n, \lambda^+_n), \qquad \text{where} \qquad \lambda^-_{n-1} \leqslant \lambda^-_n \leqslant \lambda^+_n, \quad n \geqslant 1.$$

We choose the constant $q_0 = q_0(q)$ in such a way that $\lambda^+_0 = 0$; in view of (2.19) $q_0 \geq 0$. Note that a gap-length $|\gamma_n| \geqslant 0$ may be zero. If the $n$th gap degenerates, that is, $\gamma_n = \emptyset$, then the corresponding spectral bands $\mathfrak{S}_n$ and $\mathfrak{S}_{n+1}$ merge. The sequence $0 = \lambda^+_0 < \lambda^-_1 \leqslant \lambda^+_1 < \ldots$ form the *energy spectrum* of $T$ and is the spectrum of the equation $-y'' + (q_0 + q)y = \lambda y$ with the 2-periodic boundary conditions, i.e., $y(x + 2) = y(x), x \in \mathbb{R}$. Here, the equality means that $\lambda^-_n = \lambda^+_n$ is a double eigenvalue. The eigenfunctions, corresponding to $\lambda^\pm_n$, have period 1 when $n$ is even, and they are antiperiodic, i.e., $y(x + 1) = -y(x), \quad x \in \mathbb{R}$, when $n$ is odd.

In order to study the actions $I_n, n \geq 1$, we introduce the *quasimomentum function*. We cannot introduce the standard fundamental solutions for the operator $T$, since the perturbation $q$ is too singular if $\alpha < 0$. Instead, we use another representation of $T$. Define a function $\rho(x)$ by

$$\rho(x) = e^{\int_0^x q_*(t)dt}, \qquad \text{where} \qquad q_* \in \mathscr{H}_0 \qquad q'_* = q.$$

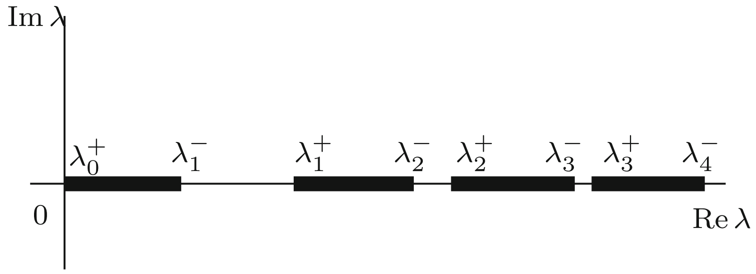


**Fig. 1** The spectral domain $\mathbb{C} \setminus \cup \mathfrak{S}_n$ and the bands $\mathfrak{S}_n = \left[\lambda_{n-1}^+, \lambda_n^-\right], n \geqslant 1$

Consider the unitary transformation $U : L^2\left(\mathbb{R}, \rho^2\, dx\right) \to L^2(\mathbb{R}, dx)$ given by the multiplication by $\rho$. Then $T$ is unitarily equivalent to

$$T_1 y = U^{-1} T U y = -\frac{1}{\rho^2}\left(\rho^2 y'\right)' + \left(q_0 - q_*^2\right) y = -y'' - 2q_* y' + \left(q_0 - q_*^2\right) y,$$

acting in $L^2\left(\mathbb{R}, \rho^2\, dx\right)$. Note that the norm in this space is equivalent to the original $L^2-$norm. This representation clearly is more convenient, since $q_*$ and $q_*^2$ are regular functions. It is convenient to write the the spectral parameter $\lambda$ as

$$\lambda = z^2.$$

Let $\varphi(x, z)$ and $\vartheta(x, z)$ be solutions of the equation

$$-y'' - 2q_* y' + \left(q_0 - q_*^2\right) y = z^2 y, \quad z \in \mathbb{C}, \tag{2.1}$$

satisfying $\vartheta'(0, z) = \vartheta(0, z) = 1$ and $\varphi(0, z) = \vartheta'(0, z) = 0$. The *Lyapunov function* is defined by

$$\Delta(z) = \frac{1}{2}(\varphi'(1, z) + \vartheta(1, z)). \tag{2.2}$$

This function is entire and even, i.e., $\Delta(-z) = \Delta(z)$ for all $z \in \mathbb{C}$. It is known that $\Delta\left(\sqrt{\lambda_n^\pm}\right) = (-1)^n, n \geqslant 0$ and the function $\Delta'(z)$ has a unique zero $z_n$ in each gap $\left[\sqrt{\lambda_n^-}, \sqrt{\lambda_n^+}\right] \subset \mathbb{R}_+$ (see e.g., [17, 18]).

## 2.2 Momentum and Quasimomentum

Below we consider the conformal mappings from the spectral domain (see Fig. 2) onto the quasimomentum domain (see Fig. 3). Consider a strongly increasing odd sequence $u_n, n \in \mathbb{Z}$, of real numbers, $u_n = -u_{-n}$, such that $u_n \to \pm\infty$ as $n \to \pm\infty$, and a non-negative sequence $h = (h_n)_1^\infty \in \ell_+^\infty$. We define the following domains (see also Fig. 3)

$$\mathcal{K}(h) = \mathbb{C} \setminus \cup_{n\in\mathbb{Z}} \overline{\Gamma}_n, \qquad \mathcal{K}_+(h) = \mathbb{C}_+ \cap \mathcal{K}(h),$$

where

$$\Gamma_0 = \emptyset, \quad \Gamma_n = (u_n - ih_n, u_n + ih_n) = -\Gamma_{-n}, \qquad \text{and} \qquad \mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}.$$

We call $\mathcal{K}_+(h)$ the "*comb*" and denote its points by $k = u + iv$. Then there exists a unique conformal mapping $z = z(k)$:

$$z : \mathcal{K}_+(h) \to \mathbb{C}_+,$$

normalized by the condition $z(0) = 0$ and the asymptotics:

$$z(iv) = iv + o(v) \quad \text{as} \quad v \to +\infty, \quad \text{where} \quad z = x + iy, \quad k = u + iv. \tag{2.3}$$

We call $z(k)$ "*the comb mapping*." Define the inverse mapping

$$k = z^{-1} : \mathbb{C}_+ \to \mathcal{K}_+(h), \qquad k(z) = u(z) + iv(z). \tag{2.4}$$

This function is continuous in $\mathbb{C}_+$ up to the boundary, i.e., on the closure $\overline{\mathbb{C}}_+$. It is convenient to introduce "gap" $g_n$, "bands" $\sigma_n$, and the "spectrum" $\sigma$ of the comb mapping by:

$$g_n = \left(z_n^-, z_n^+\right) = (z(u_n - 0), z(u_n + 0)), \quad \sigma_n = \left[z_{n-1}^+, z_n^-\right], \quad \sigma = \cup \sigma_n, \quad g_0 = \emptyset, \quad z_0^\pm = 0.$$

Note that the identities $\lambda_n^\pm = z_n^{\pm 2}$ yields

$$|\gamma_n| = z_n^{+2} - z_n^{-2} = |g_n| \left(z_n^+ + z_n^-\right), \quad \forall\, n \geqslant 1. \tag{2.5}$$

Define the momentum domain (see also Fig. 2)

$$\mathcal{Z} = \mathbb{C} \setminus \cup_{n \in \mathbb{Z}} \overline{g}_n.$$

The function $k(z)$ may be continued from $\mathbb{C}_+$ to the domain $\mathcal{Z}$ by the symmetry, using the formula $k(z) = \overline{k}(\overline{z})$, $\operatorname{Im} z < 0$. Thus, we obtain a conformal mapping $k : \mathcal{Z} \to \mathcal{K}(h)$, called *the quasimomentum mapping* (or shortly the quasimomentum), which generalizes the classical quasimomentum ( see, e.g., [30]). A point $z \in \mathcal{Z}$ is called *momentum* and a point $k \in \mathcal{K}(h)$ is called *quasimomentum*. It is odd, i.e., $k(-z) = -k(z)$, since the domains $\mathcal{K}(h)$ and $\mathcal{Z}$ both are invariant under the inversion $z \to -z$.

If the spectrum of the comb mapping $k(z)$ has only finite number of open gaps, then $k(z)$ is called a *finite-gap quasimomentum*. Different properties of the finite-gap quasimomentum (and of more general conformal mappings) were studied by Hilbert one hundred years ago, see in [5].

The abstract quasimomentum, which we have just defined, is related to the spectral theory of the Hill operator $T$ by the following construction invented in [26]. Namely, let $\{z_n^\pm, n \in \mathbb{Z}\}$ be an odd sequence as above. For $n \geqslant 0$ denote $\lambda_n^\pm = \left(z_n^\pm\right)^2$. Then $\left\{\lambda_n^\pm, n \geqslant 0\right\}$ is the energy spectrum of the Hill operator $T$ with a potential $q_0 + q$, where $q \in \mathscr{H}_\alpha$, $\alpha \geqslant 0$, if and only if the corresponding comb domain $\mathcal{K}(h)$ is such that $u_n = \pi n, n \in \mathbb{Z}$ and $h = (h_n)_1^\infty \in \ell^2_{\alpha+1}$. Moreover, in this case, $\cos k(z) = \Delta(z)$ is the Lyapunov function for $T$.

In [12], the construction was generalized for potentials from $\mathscr{H}_{-1}$, see below Theorem 2.1.

Despite the objects, treated by Theorem 1.1, are defined in terms of Hill operators with periodic potentials, for the proofs in Sections 3–4 below, we need the quasimomentum mapping $k(z)$, $k = u + iv$, $z = x + iy$, corresponding to general odd sequences $\{u_n\}$. Now we summarize their basic properties, referring for a proof to [9–17, 23–27].

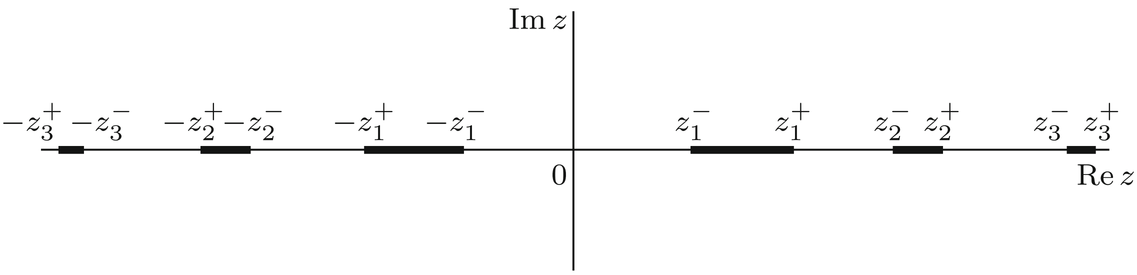


**Fig. 2** $z$-domain $\mathcal{Z} = \mathbb{C} \setminus \cup g_n$, where $z = \sqrt{\lambda}$ and momentum gaps $g_n = \left(z_n^-, z_n^+\right)$

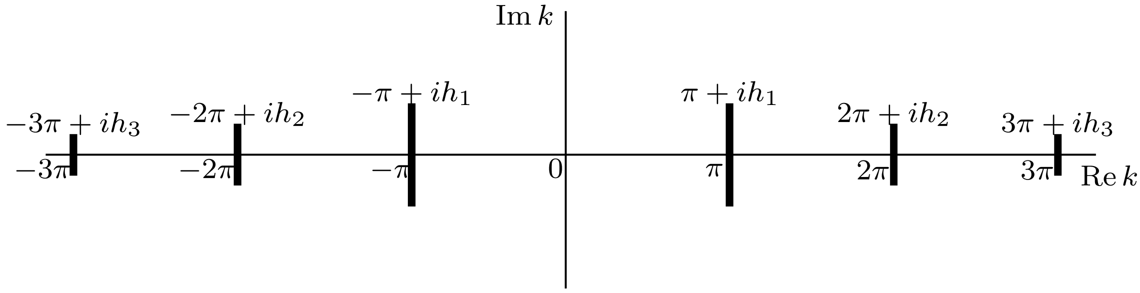


**Fig. 3** $k$-plane and cuts $\Gamma_n = (\pi n - ih_n, \pi n + ih_n)$, $n \in \mathbb{Z}$

1) $v(z) \geqslant Im\, z > 0$ *and* $v(z) = -v(\overline{z})$ *for all* $z \in \mathbb{C}_+$ *and*

$$k(-z) = -k(z), \qquad k(z) = \overline{k}(\overline{z}), \quad \alpha\, z \in \mathcal{Z}, \tag{2.6}$$

2) $v(z) = 0$ for all $z \in \sigma_n = \left[z_{n-1}^+, z_n^-\right]$, $n \in \mathcal{Z}$.

3) If some $g_n \neq \emptyset$, $n \in \mathcal{Z}$, then

$$h_n \geqslant v(z + i0) = -v(z - i0) > 0, \qquad v''(z + i0) < 0 \qquad \forall\, z \in g_n, \tag{2.7}$$

see Fig. 4. The function $v(z + i0)|_{g_n} > 0$ attains its maximum at a point $z_n \in g_n$, where

$$h_n = v(z_n + i0), \quad v'(z_n) = 0. \tag{2.8}$$

Moreover,

$$v = 0 \qquad \text{on} \quad \mathbb{R} \setminus \cup_{n \in \mathbb{Z}} g_n, \tag{2.9}$$

$$v(z + i0) > v_n(z) = \left|(z - z_n^-)\left(z - z_n^+\right)\right|^{\frac{1}{2}} > 0, \qquad \forall\, z \in g_n, \tag{2.10}$$

$$|g_n| \leqslant 2h_n, \qquad |\sigma_n| \leqslant u_n - u_{n-1}, \qquad \forall\, n \in \mathbb{Z}. \tag{2.11}$$

4) $u'(z) > 0$ on each $\sigma_n$, and

$$u(z) = \pi n, \qquad \forall\, z \in g_n \neq \emptyset, n \in \mathbb{Z}. \tag{2.12}$$

5) The function $k(z)$ maps a horizontal cut (a gap ) $\overline{g}_n$ onto the vertical cut $\overline{\Gamma}_n$ and maps a spectral band $\sigma_n$ onto the segment $[\pi(n-1), \pi n]$, for all $n \in \mathbb{Z}$.

6) The following asymptotics hold true:

$$z_n^{\pm} = \pi n + o(1) \qquad \text{as} \quad n \to \infty. \tag{2.13}$$

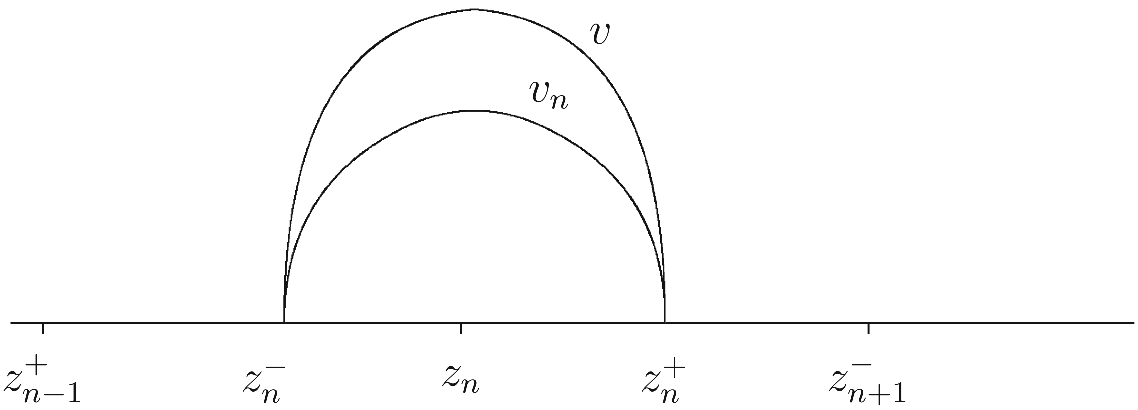


**Fig. 4** The graph of $v(z + i0)$, $z \in g_n \cup \sigma_n \cup \sigma_{n+1}$ and $|h_n| = v(z_n + i0) > 0$

7) If $h \in \ell^2$ and $\inf_{n\geqslant 1}(u_{n+1}-u_n)>0$, then $v(z+i0)$, $z\in\mathbb{R}$ belongs to $L^1(\mathbb{R})$ and the following identity holds true:

$$k(z)=z+\frac{1}{\pi}\int_{\bigcup_{n\in\mathbb{Z}} g_n}\frac{v(t)}{t-z}dt, \qquad \forall z\in\mathcal{Z}. \tag{2.14}$$

For additional properties of the comb mapping $z(k)$, see [9–17, 23–26].

### 2.3 Quasimomentum and the KdV Hamiltonian

Recall that we choose the constant $q_0\geq 0$ in such a way that $\lambda_0^+=0$. If $q\in\mathscr{H}_0(\mathbb{T})$, then the quasimomentum $k(\cdot)$ has asymptotics

$$k(z)=z-\frac{Q_0}{z}-\frac{Q_2+o(1)}{z^3} \qquad as \quad \operatorname{Im} z\to\infty, \tag{2.15}$$

see [15]. If $q,q'\in\mathscr{H}_0$, then the asymptotics 2.15 may be improved:

$$k(z)=z-\frac{Q_0}{z}-\frac{Q_2}{z^3}-\frac{Q_4+o(1)}{z^5} \qquad as \quad z\to+i\infty, \tag{2.16}$$

and

$$k^2(z)=\lambda-S_{-1}-\frac{S_0}{\lambda}-\frac{S_1+o(1)}{\lambda^2} \qquad \text{as} \quad \lambda=z^2, \quad z\to+i\infty, \tag{2.17}$$

where

$$Q_j=\frac{1}{\pi}\int_{\mathbb{R}} z^j v(z+i0)\,dz\geqslant 0, \quad j\geqslant 0, \qquad S_j=\frac{4}{\pi}\int_0^\infty z^{2j+1}u(z)v(z+i0)\,dz, \quad j\geqslant -1. \tag{2.18}$$

Note that $Q_{2j+1}=0$ for all $j\geq 0$ by the symmetry. The involved quantities $Q_j$ and $S_j$ are defined by converging integrals (see [10, 12, 15]), and satisfy the following identities

$$q_0(q)=S_{-1}=2Q_0 \qquad \text{if} \qquad q\in\mathscr{H}_{-1}, \tag{2.19}$$

$$H_1(q)=\int_0^1 q^2(x)\,dx=2P_1=4S_0=8Q_2-4Q_0^2 \qquad \text{if} \qquad q\in\mathscr{H}_0, \tag{2.20}$$

$$H_2(q)=8(S_1-S_{-1}S_0), \qquad S_1+2Q_0Q_2=2Q_4 \qquad \text{if} \qquad q\in\mathscr{H}_1, \tag{2.21}$$

$$8Q_2=\|q\|^2+q_0^2, \qquad 2^4Q_4=H_2(q+q_0). \tag{2.22}$$

See [12, 15, 29].

### 2.4 The KdV Actions

The components $I_n$ of the action vector $I=(I_n)_1^\infty$ (see Eq. 1.4) may be calculated with the help of a general formula due to Arnold, which in the KdV-case takes the form

$$I_n=\frac{(-1)^{n+1}2}{\pi}\int_{g_n}\frac{z^2\Delta'(z)dz}{\left|\Delta^2(z)-1\right|^{\frac12}}\geqslant 0, \qquad n\geqslant 1, \tag{2.23}$$

see [3]. These integrals may be re-written using the quasimomentum. Indeed, since $\sin k(z)=\sqrt{1-\Delta^2(z)}$, then

$$I_n=-\frac{1}{\pi i}\int_{c_n} z^2\frac{\Delta'(z)}{\sin k(z)}\,dz,$$

where $c_n$ is a contour around $g_n$. It is convenient to introduce contours $\chi_n$ around $\Gamma_n$ by

$$\chi_n = \left\{ k \in \mathcal{K}(h) : \text{dist } (k, \Gamma_n) = \frac{\pi}{4} \right\} \subset \mathcal{K}(h), \qquad n \geqslant 1, \tag{2.24}$$

and define the contours $c_n$ as

$$c_n = z(\chi_n) \subset \mathcal{Z}, \qquad \forall\, n \geqslant 1.$$

The differentiation of $\Delta(z) = \cos k(z)$ gives $k'(z) = -\Delta'(z)/\sin k(z)$. This yields

$$I_n = \frac{1}{i\pi} \int_{c_n} z^2 k'(z)\, dz = -\frac{2}{i\pi} \int_{c_n} z k(z)\, dz = \frac{4}{/\pi} \int_{g_n} z v(z + i0)\, dz \geqslant 0, \tag{2.25}$$

since on $g_n$ the function $k = \pi n + iv$ and $v$ satisfies (2.7). This representation for $I_n$ is convenient and is crucial for our work. In particular, below in Lemma 3.1, we derive from Eq. 2.25 the following two-sided estimates:

$$\frac{2}{3\pi} h_n |\gamma_n| < I_n \leqslant \frac{2h_n|\gamma_n|}{\pi}, \qquad \text{if} \quad |\gamma_n| > 0.$$

Using Eq. 2.25 jointly with Eqs. 2.12 and 2.9, we easily see that

$$P_3 = \sum_{n \geqslant 1} (2\pi n)^3 I_n = \frac{32}{\pi} \int_0^\infty z u^3(z) v(z + i0)\, dz. \tag{2.26}$$

Recall that $P_3(I)$ is the linear in $I$ part of the Hamiltonian $H_2$, see Eq. 1.10.

## 2.5 Marchenko-Ostrovski Construction for Potentials $q \in \mathscr{H}_{-1}$

The Marchenko-Ostrovski construction, described in Section 2.1, defines the mapping $q \to h$, acting from $\mathscr{H}_{j-1}$ into $\ell^2_j$, $j = 0, 1, \ldots$.. The results below are proven in [26] for $j \geqslant 1$ and in [12] for $j = 0$.

**Theorem 2.1** *The mapping $q \to h$ acting from $\mathscr{H}_{j-1}$ into $\ell^2_j$, $j = 0, 1$ is a surjection. It satisfies the following estimates*

$$\|q\|_{-1} \leqslant 2\|h\|_2(1 + 4\|h\|_2), \qquad \|h\|_2 \leqslant 3\|q\|_{-1}(1 + 2\|q\|_{-1})^2, \quad \forall\, q \in \mathscr{H}_{-1}, \tag{2.27}$$

*where $\|q\|_{-1} = \|q\|_{\mathscr{H}_{-1}}$. For each $h \in \ell^2$, there exists a function $q \in \mathscr{H}_{-1}$ such that $h = h(q)$, and a unique conformal mapping $k(\cdot, h) : \mathcal{Z} \to K(h)$ defined in* Eq. 2.4. *Moreover,*

$$\cos k(z, h) = \Delta(z, h), \qquad z \in \mathcal{Z}, \tag{2.28}$$

*where $\Delta(z, h)$ is the Lyapunov function for $q$, and $k(z)$ satisfy*

$$k(z, h) = z - \frac{Q_0 + o(1)}{z} \qquad \text{as} \quad z \to i\infty, \tag{2.29}$$

$$k\left(z_n^{\pm}, h\right) = \pi n \pm i0, \qquad k(z_n \pm i0, h) = \pi n \pm i h_n, \quad n \geqslant 1. \tag{2.30}$$

*Moreover, the real numbers $z_n^{\pm 2}$, satisfying* (2.30)*, form the energy spectrum of the operator $T$. Furthermore, if a sequence $h^\nu$, $\nu \geqslant 1$ converges strongly in $\ell^2$ to $h$ as $\nu \to \infty$, then $\Delta(z, h^\nu) \to \Delta(z, h)$ uniformly on bounded subsets of $\mathbb{C}$.*

In order to prove our main result, Theorem 1.1, we use Theorem 2.1 to reformulate it as questions from the conformal mapping theory in terms of quasimomentum of the Hill operator. To proceed, we need auxiliary results from previous work of the first author:

**Lemma 2.2** *Let $h \in \ell^2$ and let each $u_n = \pi n, n \geqslant 1$. Then following estimates hold true:*

$$\frac{\pi}{4} Q_0 \leqslant \|h\|_2^2 \leqslant \frac{\pi^2}{2}\left(1 + \frac{\sqrt{2}}{\pi} Q_0^{\frac{1}{2}}\right) Q_0, \tag{2.31}$$

$$\|\rho\|_2^2 \leqslant (16)^2 Q_0, \tag{2.32}$$

$$\frac{\|h\|_\infty^2}{2} \leqslant Q_0 \leqslant \frac{2}{\pi}\int_0^\infty \frac{zv(z)\,dz}{u(z)} = \sum_{j\geqslant 1} \frac{I_j}{2\pi j} = P_{-1}, \tag{2.33}$$

$$\|h\|_\infty \leqslant \frac{4}{\pi}\sum_{j=1}^\infty \frac{|\gamma_j|}{2\pi j}. \tag{2.34}$$

*Proof* Estimates (2.31), (2.32) were proved in Theorem 2.1 from [11]. The first estimate in Eq. 2.33, $\|h\|_\infty^2 \leqslant 2Q_0$ was established in [16]. The second estimate in Eq. 2.33 $Q_0 \leqslant \frac{2}{\pi}\int_0^\infty zv(z)\,dz/u(z)$ was proved in [15] (see p. 398). The identities (2.25), (2.9), and Eq. 2.12 imply

$$\frac{2}{\pi}\int_0^\infty \frac{zv(z)\,dz}{u(z)} = \sum_{j\geqslant 1} \frac{I_j}{2\pi j},$$

and we get (2.33). Finally, using Eqs. 2.33, 2.8, and 2.12, we obtain

$$\|h\|_\infty^2 \leqslant \frac{4}{\pi}\int_0^\infty \frac{zv(z)\,dz}{u(z)} \leqslant \frac{4}{\pi}\sum_{j\geqslant 1} h_j \int_{g_j} \frac{z\,dz}{u(z)} = \frac{2}{\pi}\sum_{j\geqslant 1} h_j \frac{|\gamma_j|}{\pi j} \leqslant \frac{2\|h\|_\infty}{\pi}\sum_{j\geqslant 1}\frac{|\gamma_j|}{\pi j}, \tag{2.35}$$

which gives (2.34). □

## 3 Local Estimates

In this section, we derive estimates for $h_n, I_n$, and $|\gamma_n|$ with a fixed $n \geqslant 1$. We use the following constants

$$C_- = e^{\sqrt{P_{-1}}}, \qquad C_I = 1 + \sqrt{P_{-1}} \qquad C_0 = \chi\|h\|_\infty \leqslant e^{\sqrt{2P_{-1}}}, \tag{3.1}$$

where the inequality follows from lemma below.

**Lemma 3.1** *Let $h \in \ell^2$ and let each $u_n = \pi n, n \geqslant 1$. Then for each $n \geqslant 1$ the following estimates hold true:*

$$\frac{2}{3\pi} h_n|\gamma_n| < \frac{2}{3\pi} h_n|g_n|\left(z_n + z_n^- + z_n^+\right) < I_n \leqslant \frac{2h_n|\gamma_n|}{\pi}, \qquad \text{if} \quad |\gamma_n| > 0, \tag{3.2}$$

$$z_n^\pm \leqslant \pi n + \sum_{j=1}^n |g_j|, \tag{3.3}$$

$$\pi n \leqslant 2z_n^\pm + \frac{\|\rho\|_2^2}{\pi}, \qquad \rho = (\rho_n)_1^\infty, \quad \rho_n = \pi - |\sigma_n|, \tag{3.4}$$

$$2n \leqslant C_0 z_n^\pm, \qquad h_n \leqslant \frac{\sqrt{C_0}}{2}|g_n|, \tag{3.5}$$

$$2\pi n h_n^2 \leqslant \sqrt{C_0}\frac{3\pi}{2} I_n + 2\frac{\|\rho\|^2}{\pi} h_n^2, \tag{3.6}$$

$$\frac{1}{C_I}\frac{|\gamma_n|}{4\pi n} \leqslant h_n \leqslant \frac{\pi C_0^{\frac{3}{2}}|\gamma_n|}{8\pi n}. \tag{3.7}$$

$$\frac{1}{3\pi C_I}\frac{|\gamma_n|^2}{(2\pi n)} \leqslant I_n \leqslant \frac{C_0^{\frac{3}{2}}}{2}\frac{|\gamma_n|^2}{(2\pi n)}, \tag{3.8}$$

$$\frac{8C_0^{-\frac{3}{2}}}{3\pi^2}(2\pi n)|h_n|^2 \leqslant I_n \leqslant 8nC_I h_n^2. \tag{3.9}$$

*Proof* We show (3.2). Using Eqs. 2.7, 2.8, and standard convexity arguments (see Fig. 5), we have

$$v\left(z_n^- + t + i0\right) \geqslant f_-(t) = t\frac{h_n}{\varepsilon_-}, \qquad t \in (0, \varepsilon_-), \quad \varepsilon_- = z_n - z_n^- > 0. \tag{3.10}$$

This yields

$$\begin{aligned} I_n^- &= \frac{4}{\pi}\int_{z_n^-}^{z_n} zv(z)\,dz \geqslant \frac{4}{\pi}\int_0^{\varepsilon_-}(z_n^- + t)f_-(t)dt = \frac{4}{\pi}\frac{h_n}{\varepsilon_-}\left(z_n^-\frac{\varepsilon_-^2}{2} + \frac{\varepsilon_-^3}{3}\right) \\ &= \frac{2}{\pi}h_n\varepsilon_-\left(z_n - \frac{\varepsilon_-}{3}\right). \end{aligned}$$

Let $f_+(t) = (\varepsilon_+ - t)\frac{h_n}{\varepsilon_+}$, $t \in (0, \varepsilon_+)$, $\varepsilon_+ = z_n^+ - z_n > 0$. A similar argument gives

$$\begin{aligned} I_n^+ &= \frac{4}{\pi}\int_{z_n}^{z_n^+} zv(z)\,dz \geqslant \frac{4}{\pi}\int_0^{\varepsilon_+}\left(z_n^+ + t - \varepsilon_+\right)f_+(t)dt = \frac{4}{\pi}\frac{h_n}{\varepsilon_+}\left(z_n^+\frac{\varepsilon_+^2}{2} - \frac{\varepsilon_+^3}{3}\right) \\ &= \frac{2}{\pi}h_n\varepsilon_+\left(z_n + \frac{\varepsilon_+}{3}\right). \end{aligned}$$

Denoting $z_n^0 = \frac{1}{2}\left(z_n^+ + z_n^-\right)$, we obtain

$$I_n = I_n^- + I_n^+ > \frac{2}{\pi}h_n\left(\varepsilon_-\left(z_n - \frac{\varepsilon_-}{3}\right) + \varepsilon_+\left(z_n + \frac{\varepsilon_+}{3}\right)\right) = \frac{2}{\pi}h_n\left(z_n|g_n| + \frac{\varepsilon_+^2 - \varepsilon_-^2}{3}\right)$$

$$= \frac{2}{\pi}h_n|g_n|\left(z_n + 2\frac{z_n^0 - z_n}{3}\right) = \frac{2}{3\pi}h_n|g_n|\left(z_n + 2z_n^0\right) \geqslant \frac{2}{3\pi}h_n|g_n|2z_n^0 = \frac{2}{3\pi}h_n|\gamma_n|.$$

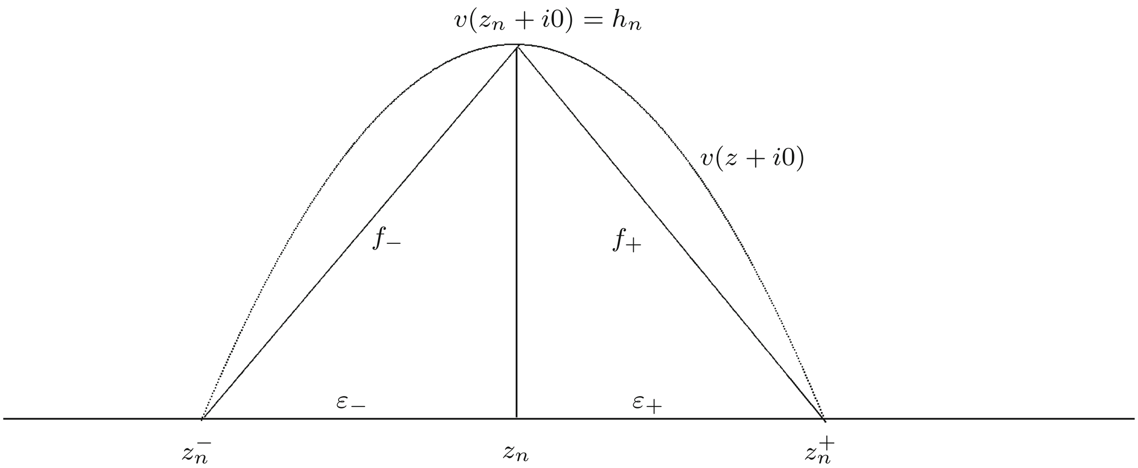


**Fig. 5** The graphs of $v(z + i0)$ and $f_\pm$

This implies the first two estimates in Eq. 3.2. Using Eq. 2.7 for all $z \in g_n$, we get $I_n < \frac{4}{\pi} h_n \int_{g_n} z\,dz = \frac{2}{\pi} h_n |\gamma_n|$, which gives us the last estimate in Eq. 3.2.

We show (3.3). It is clear that

$$z_n^+ = \pi n + \sum_1^n \left(|g_j| - (\pi - |\sigma_j|)\right), \qquad \text{all } n \geqslant 1. \tag{3.11}$$

Since by Eq. 2.11, $\rho_j = \pi - |\sigma_j| \geqslant 0$ and $|g_n| = z_n^+ - z_n^-$, we get (3.3).

We show (3.4). Using identities (3.11), we obtain

$$\pi n \leqslant z_n^\pm + \sum_1^n \rho_j \leqslant z_n^\pm + n^{\frac{1}{2}} \|\rho\|_2 \leqslant z_n^\pm + \frac{\pi n}{2} + \frac{\|\rho\|_2^2}{2\pi},$$

which yields (3.4).

In order to prove (3.5), we use an argument from [26]. This is a weak point in our proof, which gives the exponential factor in Eq. 3.5 and later in Eq. 1.14. The Taylor formula implies

$$2 = \left|\Delta\left(z_n^-\right) - \Delta\left(z_{n-1}^+\right)\right| \leqslant \left|\Delta'(\widetilde{z}_n)||\sigma_n\right|, \tag{3.12}$$

for some $\widetilde{z}_n \in \sigma_n = \left[z_{n-1}^+, z_n^-\right]$ and all $n \geqslant 1$. Using the Bernstein inequality for the bounded exponential type functions (see [10, 12, 15]) we obtain

$$\sup_{z\in\mathbb{R}} |\Delta'(z)| \leqslant \sup_{z\in\mathbb{R}} |\Delta(z)| = C_0 = \operatorname{ch} \|h\|_\infty \leqslant e^{\|h\|_\infty}. \tag{3.13}$$

Combining (3.12) and (3.13), we get $2n \leqslant C_0 z_n^\pm$ for all $n \geqslant 1$, which yields the first estimate in Eq. 3.4.

Let $n$ be even and let $|z_n^- - z_n| \leqslant |g_n|/2$ (for other cases the proof is similar). The identity $\chi h_n = \Delta(z_n)$ (which follows from Eq. 2.30) and the Taylor formula imply

$$\frac{h_n^2}{2} \leqslant \operatorname{ch} h_n - 1 = \Delta(z_n) - 1 = \frac{1}{2}\Delta''\left(\widetilde{z}_n^-\right)\left(z_n^- - z_n\right)^2 \tag{3.14}$$

for some $\widetilde{z}_n^- \in (z_n^-, z_n)$. Using again the Bernstein inequality, we obtain

$$\sup_{z\in\mathbb{R}} |\Delta''(z)| \leqslant \sup_{z\in\mathbb{R}} |\Delta(z)| = C_0. \tag{3.15}$$

Then combining (3.14) and (3.15), we get $h_n^2 \leqslant \frac{C_0}{4}|g_n|^2$, which gives the second estimate in Eq. 3.5.

We show (3.6). Using Eqs. 3.4, 3.5, 3.2, and 2.5, we obtain

$$\begin{aligned}\pi n h_n^2 &\leqslant \left(z_n^- + z_n^+\right) h_n^2 + \frac{\|\rho\|_2^2}{\pi} h_n^2 \leqslant \left(z_n^- + z_n^+\right) \frac{\sqrt{C_0}}{2} |g_n| h_n + \frac{\|\rho\|_2^2}{\pi} h_n^2 \\ &= \frac{\sqrt{C_0}}{2} |\gamma_n| h_n + \frac{\|\rho\|_2^2}{\pi} h_n^2 \leqslant \frac{\sqrt{C_0}}{2} \frac{3\pi}{2} I_n + \frac{\|\rho\|_2^2}{\pi} h_n^2,\end{aligned}$$

which yields (3.6).

We show (3.7). Using Eqs. 2.5, 3.3, 2.33, and 2.11, we obtain

$$\frac{|\gamma_n|}{4\pi n} = \frac{\left(z_n^- + z_n^+\right)|g_n|}{4\pi n} \leqslant \left(1 + \frac{\|g\|_\infty}{\pi}\right) h_n \leqslant \left(1 + \frac{\sqrt{8P_{-1}}}{\pi}\right) h_n \leqslant C_I h_n.$$

Recalling that $C_0 = \operatorname{ch} \|h\|_\infty$ and using Eqs. 3.5 and 2.5, we obtain

$$h_n \leqslant \frac{C_0^{\frac{1}{2}} |g_n|}{2} = \frac{C_0^{\frac{1}{2}} |\gamma_n|}{2\left(z_n^- + z_n^+\right)} \leqslant \frac{\pi C_0^{\frac{3}{2}} |\gamma_n|}{8\pi n},$$

and Eq. 3.7 is proven.

Estimates (3.7) and (3.2) imply the first estimate in Eq. 3.8:

$$\frac{|\gamma_n|^2}{(2\pi n)} \leqslant 2C_I h_n |\gamma_n| \leqslant 3\pi C_I I_n.$$

Combining the last estimate in Eqs. 3.7 and 3.2, we obtain the second estimate in Eq. 3.8.

We show 3.9. Using Eqs. 3.2 and 3.7, we obtain

$$I_n \leqslant \frac{2h_n|\gamma_n|}{\pi} \leqslant 8nC_I h_n^2,$$

which yields the second estimate in Eq. 3.9. Using Eqs. 3.7 and 3.2, we obtain

$$2\pi n h_n^2 \leqslant \frac{\pi}{4} C_0^{\frac{3}{2}} h_n |\gamma_n| \leqslant \frac{3\pi^2}{8} C_0^{\frac{3}{2}} I_n,$$

and get the first. □

For any $h \in \ell^\infty$ we define integrals $V_n$ as

$$V_n = \frac{8}{\pi} \int_{g_n} z v^3(z)\, dz \geqslant 0, \qquad n \geqslant 1. \tag{3.16}$$

These quantities are important for our argument since, as we show below, $V = \sum_{n\geqslant 1} (4\pi n) V_n$ for $I \in \ell_+^2$.

**Lemma 3.2** *Let $h \in \ell^\infty$ and let each $u_n = \pi n, n \geqslant 1$. Then for $n \geqslant 1$ the following relations hold true:*

$$\frac{1}{5} h_n^2 I_n \leqslant \frac{2}{5\pi} h_n^3 |\gamma_n| \leqslant \frac{2}{5\pi} h_n^3 |g_n| \left(3z_n + 2z_n^0\right) \leqslant V_n \leqslant 2h_n^2 I_n, \tag{3.17}$$

$$\frac{4}{\pi i} \int_{c_n} z k^4(z)\, dz = (4\pi n) V_n - (2\pi n)^3 I_n. \tag{3.18}$$

*Proof* We show 3.17. Let $g_n \neq \emptyset$. Using Eq. 3.10, we get $v\left(z_n^- + t + i0\right) \geqslant f_-(t) = t h_n/\varepsilon_-, t \in (0, \varepsilon_-)$, where $\varepsilon_- = z_n - z_n^- > 0$. Therefore

$$V_n^- := \frac{8}{\pi} \int_{z_n^-}^{z_n} z v^3(z)\, dz \geqslant \frac{8}{\pi} \int_0^{\varepsilon_-} (z_n^- + t) f_-^3(t) dt = \frac{8}{\pi} \frac{h_n^3}{\varepsilon_-^3} \left(z_n^- \frac{\varepsilon_-^4}{4} + \frac{\varepsilon_-^5}{5}\right)$$

$$= \frac{2}{\pi} h_n^3 \varepsilon_- \left(z_n^- + \frac{4\varepsilon_-}{5}\right) = \frac{2}{\pi} h_n^3 \varepsilon_- \left(z_n - \frac{\varepsilon_-}{5}\right).$$

Similar argument yields $v(z_n + t + i0) \geqslant f_+(t) = (\varepsilon_+ - t) h_n/\varepsilon_+, t \in (0, \varepsilon_+)$, where $\varepsilon_+ = z_n^+ - z_n > 0$. Thus

$$V_n^+ = \frac{8}{\pi} \int_{z_n}^{z_n^+} z v^3(z)\, dz \geqslant \frac{8}{\pi} \int_0^{\varepsilon_+} \left(z_n^+ + t - \varepsilon_+\right) f_+^3(t) dt = \frac{8}{\pi} \frac{h_n^3}{\varepsilon_+^3} \left(z_n^+ \frac{\varepsilon_+^4}{4} - \frac{\varepsilon_+^5}{5}\right)$$

$$= \frac{2}{\pi} h_n^3 \varepsilon_+ \left(z_n^+ - \frac{4\varepsilon_+}{5}\right) = \frac{2}{\pi} h_n^3 \varepsilon_+ \left(z_n + \frac{\varepsilon_+}{5}\right).$$

Summing these relations, we obtain

$$V_n = V_n^- + V_n^+ > \frac{2}{\pi}h_n^3\left(\varepsilon_-\left(z_n - \frac{\varepsilon_-}{5}\right) + \varepsilon_+\left(z_n + \frac{\varepsilon_+}{5}\right)\right) = \frac{2}{\pi}h_n^3\left(z_n|g_n| + \frac{\varepsilon_+^2 - \varepsilon_-^2}{5}\right)$$

$$= \frac{2}{\pi}h_n^3|g_n|\left(z_n + 2\frac{z_n^0 - z_n}{5}\right) = \frac{2}{5\pi}h_n^3|g_n|\left(3z_n + 2z_n^0\right).$$

Using Eq. 2.7, we get $V_n \leqslant \left(8h_n^2/\pi\right)\int_{g_n} zv(z,h)\,dz = 2h_n^2 I_n$, which yields the last two estimates in Eq. 3.17. The second follows from Eq. 2.5 and the first follows from the last estimate in Eq. 3.2.

Using Eqs. 2.7, 2.12, and 2.25, we obtain

$$\frac{4}{\pi i}\int_{c_n} zk^4(z)\,dz = \frac{4}{\pi i}\int_{c_n} z\left(u^2 - v^2 + 2iuv\right)^2 dz = -\frac{8}{\pi i}\int_{g_n} 2z\left(u^2 - v^2\right)(2iuv)\,dz$$

$$= \frac{32}{\pi}\int_{g_n} z\left(v^2 - u^2\right)uv\,dz = (4\pi n)V_n - (2\pi n)^3 I_n.$$

This proves (3.18). □

*Remark* 1) In particular, Eq. 3.8 yields that $I \in \ell_+^2$ iff $\gamma \in \ell_{-1}^4$.

2) Due to [12], for any $N_0 > 1$ and $\varepsilon \in \left(0, \frac{1}{4}\right)$, there exists a potential $q \in \mathscr{H}_{-1}$ such that $|\gamma_n| = n^\varepsilon$, for all $n > N_0$. Then $\gamma = (|\gamma_n|)_1^\infty \in \ell_{-1}^4$ and Eq. 3.8 gives that $I \in \ell_+^2$. It is clear that $q \notin L^2(\mathbb{T})$, since the gap length $|\gamma_n|$ is increasing.

   Note that if $I \in \ell_+^2$, then 3.7 gives $\sum_{n\geqslant 1}|\gamma_n|^4/n^2 < \infty$, which yields $(|\gamma_n|)_1^\infty \in \ell_{-\frac{1}{2}}^2$. This and the standard relationship between the gap lengths and the Fourier coefficients of potential imply $q \in \mathscr{H}_{-\frac{1}{2}}$ (e.g., see [12], where an analogy of this relation is established for $(|\gamma_n|)_1^\infty \in \ell_{-1}^2$ and for $(|\gamma_n|)_1^\infty \in \ell^2$ ).

3) Relations 3.8 show that asymptotically the actions $I_n$ are equivalent to the weighted squared gap-length $|\gamma_n|^2/2\pi n$. It is known that the gap-length $|\gamma_n|$ is asymptotically equivalent to the module of the Fourier coefficients $|\hat{q}_n|$ of the potential $q$ (see [26, 27] for the equivalence and see [14] for the corresponding estimates). The asymptotical equivalences $nI_n \sim |\gamma_n|^2 \sim |\hat{q}_n|^2$ and the corresponding estimates are important for the spectral theory of the Hill operator $T$ and the theory of KdV.

## 4 Proof of Theorem 1.1

We remind that $V(I) = P_3 - H_2$ is the non-linear part of the KdV Hamiltonian, written as a function of actions, see Eqs. 1.8 and 1.10. Identities (2.21) and (2.25) represent $H_2$ and $I_j$ as integrals in terms of the quasimomentum $k = u + iv$. They allow to write $V$ in a similar form. We start with an integral representation for $V$ for the case of smooth potentials.

**Lemma 4.1** *Let $q \in \mathscr{H}_1$. Then $V$ is finite, nonnegative and satisfies*

$$V = \frac{32}{\pi}\int_0^\infty zu(z)v^3(z)\,dz. \tag{4.1}$$

*Proof* Since $q \in \mathscr{H}_1$, then $H_2$ is finite and $I \in \ell^1_{3/2,+}$. So $P_3$ is finite, as well as $V$. To prove Eq. 4.1 we start with a finite-gap aproximation for the momentum spectrum $\sigma_M = \mathbb{R} \setminus \cup g_n$ of the potential $q_0 + q$, obtained by closing the gaps $g_m$ with large $m$. Namely, we fix $r > 0$ and consider a new momentum spectrum $\sigma_M^r = \sigma_M \cup (-\infty, -r) \cup (r, \infty)$, where the new gaps $g_n^r$ are given by

$$g_n^r = \begin{cases} g_n & if \quad g_n \subset (-r, r) \\ \emptyset & if \quad g_n \not\subset (-r, r). \end{cases}$$

The variables corresponding to $\sigma_M^r$ will be indicated the upper index $r$. Due to the general construction, presented in Section 2.2, for the finite-gap momentum spectrum $\sigma_M^r$ there exists a unique conformal mapping

$$k^r : \mathbb{C} \setminus g^r \to \mathbb{C} \setminus \Gamma_n^r, \qquad \Gamma_n^r = \left(u_n^r + ih_n^r, u_n^r - ih_n^r\right), \quad h_n^r \geqslant 0,$$

satisfying the asymptotics $k^r(z) = z - O(1)z^{-1}$ as $|z| \to \infty$. By Eq. 2.14, each function $k^r(z) - z$ is analytic at $\infty$. The sequence of real numbers $u_n^r, n \in \mathbb{Z}$ is odd, strongly increasing and $u_n^r \to \pm\infty$ as $n \to \pm\infty$. In general, $k^r$ is not a quasimomentum for some periodic potential, since not necessarily $u_n = \pi n$ for all $n$.

For each $r$, we introduce $Q_m^r, S_m^r, P_3^r$, and $V^r$ by relations (2.18), (2.26), and (4.1) respectively, where $k = k^r, u = u^r$ and $v = v^r$. Since $v^r(x) = 0$ for large real $x$ and $v^r(x + i0), u^r(x) \geqslant 0$ for real $x \geqslant 0$, then all these quantities are finite and non-negative. It is known (see [23–25]) that

$$v^r(x) \nearrow v(x), \quad |u^r(x)| \nearrow |u(x)| \qquad x \in \mathbb{R}, \quad \text{as} \quad r \to \infty,$$

and that $k^r$ converges to $k$ uniformly on compact sets from $\mathbb{C} \setminus \sigma_M$. From these convergence and the Beppo Levi theorem, it follows that

$$Q_m^r \nearrow Q_m, \qquad S_m^r \nearrow S_m, \qquad P_m^r \nearrow P_m, \qquad V^r \nearrow V \quad \text{as} \quad r \to \infty, \tag{4.2}$$

for $m = -1, 0, 1, 2, \ldots$ (some limits may be infinite).

Assume that for each $r$ sufficiently large, we have proved that

$$8\left(S_1^r - S_{-1}^r S_0^r\right) = P_3^r - V^r. \tag{4.3}$$

Then sending $r \to \infty$ using Eq. 4.2 and evoking (2.21), we get that $H_2 = P_3 -$ (r.h.s. of Eq. 4.1). Since $H_2 = P_3 - V$, we recover (4.1).

So it remains to show (4.3). Fix $r > 1$ large enough and consider the integral $\int_{|z|=t} zk^4(z)\,dz$. The function $z(k^r(z))^4$ is analytic in $\{|z| > r\}$. For any $m \geq 1$ we write its Tailor series at infinity, omitting the index $r$ for brevity:

$$k(z) = z - \frac{Q_0}{z} - \frac{Q_2}{z^3} \ldots - \frac{Q_{2m}}{z^{2m+1}} + \frac{O(1)}{z^{2m+2}} \qquad \text{as} \qquad |z| \to \infty. \tag{4.4}$$

Due to Eq. 4.4, we get

$$zk^4 = z\left(z^2 - S_{-1} - \frac{S_0}{z^2} - \frac{S_1 + o(1)}{z^4}\right)^2 = z^5\left(1 - \frac{S_{-1}}{z^2} - \frac{S_0}{z^4} - \frac{S_1 + o(1)}{z^6}\right)^2$$

$$= z^5\left(1 - 2\left(\frac{S_{-1}}{z^2} + \frac{S_0}{z^4} + \frac{S_1 + o(1)}{z^6}\right) + \left(\frac{S_{-1}}{z^2} + \frac{S_0}{z^4}\right)^2 + ..\right) = z^5 + .. - 2\frac{S_1 - S_0 S_{-1}}{z} + \frac{O(1)}{z^2}.$$

If $t > r$, then

$$\frac{1}{2\pi i}\int_{|z|=t} zk^4(z)\,dz = -2(S_1 - S_0 S_{-1}), \tag{4.5}$$

since $v(z) = 0$ for real $z$ such that $|z| > r$. Thus

$$\frac{1}{2\pi i}\int_{|z|=t} zk^4(z)\,dz = \frac{1}{2\pi i}\int_{|z|=t} z\left(u^2 - v^2 + 2iuv\right)^2 dz = \frac{-1}{\pi i}\int_{\mathbb{R}} 2z\left(u^2 - v^2\right)(2iuv)dz$$

$$= -\frac{8}{\pi}\int_{\cup_{n\geq 1} g_n} z\left(u^2 - v^2\right)uv\,dz. \tag{4.6}$$

By Eqs. 2.21, 4.5, 2.26, and 4.6, we get that

$$8\left(S_1 - S_0 S_{-1}\right) = \frac{32}{\pi}\int_0^\infty z\left(u^2 - v^2\right)uv\,dz = \frac{32}{\pi}\sum_{n\geqslant 1}\int_{g_n} z\left(u^2 - v^2\right)uv\,dz = P_3 - V,$$

which yields 4.3. □

Let $0 < a < \frac{1}{4}$. We note that since $|h\|^2_{2,a} = \sum_{n\geqslant 1}(2\pi n)^{2a-1}\left(2\pi n h_n^2\right)$, then

$$\|h\|^2_{2,a} \leqslant C_{2-4a}\|h\|_{4,1}, \qquad where \quad C_t^2 = \sum_{n\geqslant 1}\frac{1}{(2\pi n)^t} < \infty \quad \text{if } t > 1. \tag{4.7}$$

**Theorem 4.2** *A sequence $h = h(I)$ belongs to $\ell_1^4$ if and only if $I \in \ell_+^2$. The series*

$$W = \sum_{n\geqslant 1}(4\pi n)V_n, \tag{4.8}$$

*where $V_n = V_n(I) \geqslant 0$ is defined by* Eq. 3.16*, converges for $I \in \ell_+^2$ and defines there a finite non-negative function $W(I)$. Moreover,*

i) *this function equals to*

$$\frac{32}{\pi}\int_0^\infty zu(z)v^3(z)\,dz,$$

ii) *satisfies the following estimates*

$$\frac{1}{5}\sum_{n\geqslant 1}(4\pi n)h_n^2 I_n \leqslant W \leqslant 2\sum_{n\geqslant 1}(4\pi n)h_n^2 I_n, \tag{4.9}$$

$$W \leqslant 4\|h\|^2_\infty P_1; \tag{4.10}$$

iii) *is continuous on $\ell_+^2$;*

iv) *on the octant $\tilde{\ell}^\infty_{3/2,+}$ it coincides with $V(I)$.*

*Proof* Estimates (3.9) imply that $h \in \ell_1^4$ iff $I \in \ell_+^2$.

Due to the last inequality in Eq. 3.17,

$$W \leq \sum_{n\geqslant 1}(4\pi n)2h_n^2 I_n \leqslant 4\|I\|_2^{1/2}\|h\|^2_{4,1},$$

So $W$ is defined by a converging series and satisfies the second estimate in Eq. 4.9. Using the lower bound for $V_n$ in Eq. 3.17, we recover the first estimate in Eq. 4.9. Estimate (4.10) follows from Eq. 4.9.

Since $u = \pi n$ on $g_n$ and $u$ vanishes outside $\cup g_n$, then $W(I)$ has the integral representation, required by (i).

Let a sequence $I^s = \left(I_n^s\right)_1^\infty \to I$ strongly in $\ell^2$ as $s \to \infty$. To prove (iii), we need to show that

$$W(I^s) \to W(I) \qquad \text{as} \qquad s \to \infty. \tag{4.11}$$

Using Eqs. 3.9 and 3.1, we have

$$\|h^s\|_{4,1}^4 \leqslant \frac{3\pi^2}{8} C_0^{\frac{3}{2}} \|I^s\|_2^2,$$

where $C_0 \leqslant \exp\sqrt{2P_{-1}(I^s)}$ and $P_{-1}(I) = \sum_{n\geqslant 1} I_n/(2\pi n) \leqslant C_2\|I\|_2$. Together with Eq. 4.7 this yields the estimates

$$\sup_{s\geqslant 1}\|h^s\|_{4,1} < \infty, \qquad \sup_{s\geqslant 1}\|h^s\|_{2,a} < \infty \;\text{ if }\; a < \frac{1}{4}. \tag{4.12}$$

We claim that

$$h^s \to h \qquad \text{weakly in} \quad \ell_a^2 \quad as \quad s \to \infty, \tag{4.13}$$

for some $h \in \ell_a^2$. Indeed, assume that this is not the case. Then by Eq. 4.12, there are two different vectors $h', h'' \in \ell_a^2$ and two subsequence $\left\{s_j'\right\}$ and $\left\{s_j''\right\}$ such that

$$h^{s_j'} \to h', \quad h^{s_j''} \to h'' \qquad \text{weakly in} \quad \ell_a^2. \tag{4.14}$$

Then

$$h^{s_j'} \to h', \quad h^{s_j''} \to h'' \qquad \text{strongly in} \quad \ell_\nu^2,$$

for each $\nu < a$. Using Theorem 2.1 and the identity

$$k(z,h) = \int_0^z \frac{\Delta'(t,h)}{\sqrt{1-\Delta^2(t,h)}}dt, \qquad z \in \mathcal{Z},$$

which easily follows from Eq. 2.28, we deduce that the corresponding conformal mappings $k$ converge to limits:

$$k\left(z, h^{s_j'}\right) \to k\left(z, h'\right), \qquad k\left(z, h^{s_j'}\right) \to k\left(z, h'\right) \qquad as \quad j \to \infty, \tag{4.15}$$

uniformly on bounded subsets in $\mathbb{C}$. These convergences and Eq. 2.25 imply that for each $n$ the actions $I_n\left(h^{s_j'}\right)$ and $I_n\left(h^{s_j''}\right)$ converge to limits $I_n(h')$ and $I_n(h'')$, which must equal $I_n$. That is, $h'$ and $h''$ belong to the same iso-spectral class. Since $h', h'' \in \ell^2$, then by Theorem 2.1 we have $h' = h''$. This proves (4.13).

Due to Eq. 3.18

$$(4\pi n)V_n - (2\pi n)^3 I_n = \frac{4}{\pi i}\int_{c_n} zk^4(z)\,dz = -\frac{8}{\pi i}\int_{\chi_n} z^2(k,h)k^3 dk.$$

By this relation, Eqs. 2.25 and 4.15, we have

$$V_n(I^{s_j}) \to V_n(I) \quad \text{ as } \quad j \to \infty \quad \text{for each} \quad n = 1, 2, 3..... \tag{4.16}$$

For any $N$ denote

$$W^{(N)}(I^s) = \sum_{n\geqslant N}(4\pi n)V_n(I^s).$$

Using Eq. 3.17, we get

$$W^{(N)}(I^s) \leq \sum_{n\geqslant N}(8\pi n)(h_n^s)^2 I_n^s = A_N + B_N,$$

where

$$A_N = \sum_{n\geqslant N}(8\pi n)\left(h_n^s\right)^2 I_n, \qquad B_N = \sum_{n\geqslant N}(8\pi n)\left(h_n^s\right)^2\left(I_n^s - I_n\right).$$

Since

$$A_N \leqslant 4\|h^s\|_{4,1}\left(\sum_{n>N} I_n^2\right)^{\frac{1}{2}}, \quad B_N \leqslant 4\|h^s\|_{4,1}\|I^s - I\|_2,$$

then Eqs. 4.12 and 4.16 yield the required convergence (4.11).

The last assertion follows from the integral representation for $W(I)$ and Eq. 4.1 since $q \in \mathscr{H}_1$ means that $I \in \ell^1_{3/2,+}$. □

*Proof of Theorem 1.1.* Theorem 4.2 gives that the function $V : \widetilde{\ell}^\infty \to \mathbb{R}$ extends to a non-negative continuous function on the octant $\ell^2_+$. Using estimates (4.10) and (2.33), we obtain $V \leqslant 4\|h\|^2_\infty P_1 \leqslant 8P_{-1}P_1$, which yields (1.13). If $I = 0$, then Eq. 1.13 implies $V(I) = 0$. Finally, let $V = 0$ for some $I$. Since the terms $V_n$ are non-negative, then each $V_n = 0$ and Eq. 3.17 implies that $I = 0$.

It remains to prove (1.14). Estimates (4.9) and (3.9) give

$$V \geqslant \frac{2}{5}\sum_{n\geqslant 1}(2\pi n)h_n^2 I_n \geqslant \frac{\pi}{10C_I}\|I\|_2^2, \tag{4.17}$$

which yields the first inequality in Eq. 1.14. Now we show the second. Using Eqs. 4.9 and 3.6, we find that

$$V \leqslant \sum_{n\geqslant 1}\left(8\pi n h_n^2\right) I_n \leqslant \sum_{n\geqslant 1}\left(C_0^{\frac{1}{2}} 6\pi I_n^2 + 8\frac{\|\rho\|_2^2}{\pi}h_n^2 I_n\right) \leqslant 6\pi C_0^{\frac{1}{2}}\|I\|_2^2 + 8\frac{\|\rho\|_2^2}{\pi}\|h\|_2\|h\|_\infty\|I\|_2.$$

Using Eqs. 2.31, 2.32, and 2.33, we obtain

$$\|\rho\|_2^2\|h\|_2\|h\|_\infty \leqslant \pi 4^4\left(1 + Q_0^{\frac{1}{2}}\right)^{\frac{1}{2}} Q_0^2.$$

Combining these estimates, we get that

$$V \leqslant 6\pi\sqrt{C_0}\|I\|_2^2 + 4^{\frac{11}{2}}\left(1 + Q_0^{\frac{1}{2}}\right)^{\frac{1}{2}} Q_0^2\|I\|_2.$$

Together with Eq. 2.33, this yields the required estimate, since $C_0 \leqslant C_- = \exp\sqrt{2P_{-1}}$. □

Finally, as a by-product of some relations, derived above in this work, we get two-sided algebraical bounds on the norm $\|q'\|$ in terms of $P_3 = \|I\|_{1,\frac{3}{2}}$ (see [14] for two-sided algebraical estimates of $\left\|q^{(m)}\right\|$ in terms of $P_{m+\frac{1}{2}}$ for all $m \geqslant 0$).

**Proposition 4.3** *The following estimates hold true:*

$$\|q'\|^2 \leqslant 4\left(P_3 + 2P_1^2\right), \tag{4.18}$$

$$P_3 \leqslant \frac{\|q'\|^2}{2} + \frac{\|q'\|}{\sqrt{2}}\|q\|^2 + 2\pi\|q\|^3\left(1 + \|q\|^{\frac{1}{3}}\right). \tag{4.19}$$

*Proof* Since $H_2 = P_3 - V$ and $\|q\|_\infty = \sup_{x\in[0,1]} |q(x)| \leqslant \frac{\|q'\|}{\sqrt{2}}$, then

$$\frac{\|q'\|^2}{2} = H_2(q) - \int_0^1 q^3(x)\,dx \leqslant P_3 + \|q\|_\infty \|q\|^2 \leqslant P_3 + \frac{\|q'\|}{\sqrt{2}}\|q\|^2 \leqslant P_3 + \frac{\|q\|^4}{2} + \frac{\|q'\|^2}{4},$$

which together with Eq. 1.9 yields (4.18). Using Eq. 4.10 and relations $\|q\|_\infty \leqslant \frac{\|q'\|}{\sqrt{2}}$, $\|q\|^2 = 2P_1$ (see Eq. 1.9), we obtain

$$\begin{aligned} P_3 &= \frac{\|q'\|^2}{2} + \int_0^1 q^3(x)\,dx - V \leqslant \frac{\|q'\|^2}{2} + \frac{\|q'\|}{\sqrt{2}}\|q\|^2 + 4\|h\|_\infty^2 P_1 \\ &= \frac{\|q'\|^2}{2} + \frac{\|q'\|}{\sqrt{2}}\|q\|^2 + 2\|h\|_\infty^2 \|q\|^2. \end{aligned}$$

As $\|h\|_\infty^2 \leqslant \pi \|q\| \left(1 + \|q\|^{\frac{1}{3}}\right)$ (see Theorem 2.3 in [13]), then we get (4.19). □

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## References

1. Bobenko A, Kuksin S. Finite-gap periodic solutions of the KdV equation are nondegenerate. Phys Lett A. 1991;161(3):274–276.
2. Bikbaev R, Kuksin S. On the parametrization of finite-gap solutions by frequency and wavenumber vectors and a theorem of I. Krichever. Lett Math Phys. 1993;28(2):115–122.
3. Flaschka H, McLaughlin D. Canonically conjugate variables for the Korteveg- de Vries equation and the Toda lattice with periodic boundary conditions. Prog Theor Phys. 1976;55:438–456.
4. Huang G, Kuksin SB. KdV equation under periodic boundary conditions and its perturbations. Nonlinearity. 2014;27:1–28.
5. Jenkins A. Univalent functions and conformal mapping. Berlin: Springer; 1958.
6. Kappeler T. Fibration of the phase space for the Korteveg-de-Vries equation. Ann Inst Fourier (Grenoble). 1991;41(1):539-575.
7. Kappeler T, Pöschel J. KdV & KAM: Springer; 2003.
8. Kappeler T, Topalov P. Global wellposedness of KdV in $H^{-1}(\mathbb{T}, \mathbb{R})$, Duke. Math J. 2006;135(2):327–360.
9. Kargaev P, Korotyaev E. The inverse problem for the Hill operator, direct approach. Invent Math. 1997;129(3):567-593.
10. Kargaev P, Korotyaev E. Effective masses conformal mappings. Commun Math Phys. 1995;169:597–625.
11. Korotyaev E. Metric properties of conformal mappings on the complex plane with parallel slits. Inter Math Reseach Notices. 1996;10:493-503.
12. Korotyaev E. Characterization of the spectrum of Schrödinger operators with periodic distributions. Int Math Res Not. 2003;37:2019-2031.
13. Korotyaev E. Estimates for the Hill operator.I. J Diff Eq. 2000;162:1–26.
14. Korotyaev E. Estimate for the Hill operator.II. J Differential Equations. 2006;223(2):229–260.
15. Korotyaev E. The estimates of periodic potentials in terms of effective masses. Commun Math Phys. 1997;183:383–400.
16. Korotyaev E. Estimate of periodic potentials in terms of gap lengths. Commun Math Phys. 1998;197(3):521–526.
17. Korotyaev E. Periodic "weighted" operators. J Differential Equations. 2003;189(2):461–486.
18. Krein M. On the characteristic function $A(\lambda)$ of a linear canonical system differential equation of the second order with periodic coefficients (Russian). Prikl Mat Meh. 1957;21:320–329.

19. Krichever IM. Perturbation theory in periodic problems for two-dimensional integrable systems. Sov., Sci Rev C Math Phys. 1991;9:1–101.
20. Kuksin SB. Analysis of Hamiltonian PDEs, Oxford lecture series in mathematics and its applications, 19. Oxford: Oxford University Press; 2000.
21. Kuksin SB. Damped-driven KdV and effective equations for long-time behavior of its solutions. Geom Funct Anal. 2010;20(6):1431-1463.
22. Kuksin SB, Perelman G. Vey theorem in infinite dimensions and its application to KdV. DCDS-A. 2010;27:1–24.
23. Levin BYa. Majorants in classes of subharmonic functions. (Russian) Teor. Funktsii Funktsional. Anal. i Prilozhen, 51(1989), 3–17; translation in J. Soviet Math. 52 (1990), no. 6, 3441–3451.
24. Levin BYa. The connection of a majorant with a conformal mapping. II. (Russian) Teor. Funktsii Funktsional. Anal. i Prilozhen No. 52 (1989), 3-21; translation in J. Soviet Math. 52 (1990), no. 5, 3351–3364.
25. Levin BYa. Classification of closed sets on $R$ and representation of a majorant. III. (Russian) Teor. Funktsii Funktsional. Anal. i Prilozhen No. 52 (1989), 21–33; translation in J. Soviet Math., 52 (1990), no. 5, 3364–3372.
26. Marchenko V, Ostrovski I. A characterization of the spectrum of the Hill operator. Math. USSR Sbornik. 1975;26:493–554.
27. Marchenko V. Sturm-Liouville operators and applications, Revised edition. Providence, RI: AMS Chelsea Publishing; 2011.
28. McKean H., Trubowitz E. Hill's operator and hyperelliptic function theory in the presence of infinitely many branching points. Comm Pure Appl Math. 1976;29:143–226.
29. McKean H, Trubowitz E. Hill's surfaces and their theta functions, Bull. Am Math Soc. 1978;84:1042–1085.
30. Reed M, Simon B. Methods of modern mathematical physics. IV. Analysis of operators. New York-London: Academic Press; 1978.
31. Stein EM. Functions of exponential type. Ann Math. 1957;65(2):582–592.