

Resonant averaging for small-amplitude solutions of stochastic NLS equations

Sergei Kuksin,* Alberto Maiocchi §

Abstract

We consider the free linear Schrödinger equation on a torus \mathbb{T}^d , perturbed by a hamiltonian nonlinearity, driven by a random force and damped by a linear damping:

$$u_t - i\Delta u + i\nu\rho|u|^{2q_*}u = -\nu f(-\Delta)u + \sqrt{\nu} \frac{d}{dt} \sum_{\mathbf{k} \in \mathbb{Z}^d} b_{\mathbf{k}} \beta^{\mathbf{k}}(t) e^{i\mathbf{k} \cdot x} .$$

Here $u = u(t, x)$, $x \in \mathbb{T}^d$, $0 < \nu \ll 1$, $q_* \in \mathbb{N}$, f is a positive continuous function, ρ is a positive parameter and $\beta^{\mathbf{k}}(t)$ are standard independent complex Wiener processes. We are interested in limiting, as $\nu \rightarrow 0$, behaviour of distributions of solutions for this equation and of its stationary measure. Writing the equation in the slow time $\tau = \nu t$, we prove that the limiting behaviour of the both is described by the *effective equation*

$$u_\tau + f(-\Delta)u = -iF(u) + \frac{d}{d\tau} \sum b_{\mathbf{k}} \beta^{\mathbf{k}}(\tau) e^{i\mathbf{k} \cdot x} ,$$

where the nonlinearity $F(u)$ is made out of the resonant terms of the monomial $|u|^{2q_*}u$.

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*CNRS and I.M.J, Université Paris Diderot-Paris 7, Paris, France, e-mail: kuksin@math.jussieu.fr

§Laboratoire de Mathématiques, Université de Cergy-Pontoise, 2 avenue Adolphe Chauvin, Cergy-Pontoise, France, e-mail: alberto.maiocchi@unimi.it

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0 Introduction

0.1 Equations

The nonlinear Schrödinger equation on the torus with small nonlinearity

$$u_t(t, x) - i\Delta u(t, x) = -i\varepsilon^{2q_*} |u|^{2q_*} u, \quad u = u(t, x), \quad x \in \mathbb{T}_L^d = \mathbb{R}^d / (2\pi L\mathbb{Z}^d), \quad (0.1)$$

where $q_* \in \mathbb{N}$ and $0 < \varepsilon \leq 1$, is a popular model in various branches of science. The nonlinearity in (0.3) is hamiltonian and may be written as

$$-i\varepsilon^{2q_*} |u|^{2q_*} u = \varepsilon^{2q_*} i \nabla \mathcal{H}(u), \quad \mathcal{H}(u) = \mathcal{H}^{2q_*+2}(u) = -\frac{1}{2q_*+2} \int |u(x)|^{2q_*+2} dx, \quad (0.2)$$

so the equation describes a conservative system. To describe systems, interacting with the “environment”, physicists often add at some stage (to this and to other similar equations) terms, describing pumping the energy to the system and its dissipation (e.g. see [ZLF92], Section 2.2.3). A way to describe the pumping of energy is by adding to the equation a small random force, which usually is Gaussian, smooth in x , and often is white in time t , while to describe the dissipation a suitable function of the Laplacian usually is used. In this way we arrive at the equation (cf. [ZL75], eq. (5), and [CFG08], Section 1.2, eq. (1.2))

$$u_t - i\Delta u = -i\varepsilon^{2q_*} |u|^{2q_*} u - \nu f(-\Delta)u + \sqrt{\nu} \frac{d}{dt} \sum_{\mathbf{k} \in \mathbb{Z}_L^d} b_{\mathbf{k}} \beta^{\mathbf{k}}(t) e^{i\mathbf{k} \cdot x}. \quad x \in \mathbb{T}_L^d, \quad (0.3)$$

Here $0 < \nu \leq 1$ and \mathbb{Z}_L^d denotes the set of vectors of the form $\mathbf{k} = \mathbf{l}/L$ with $\mathbf{l} \in \mathbb{Z}^d$. The damping $-f(-\Delta)$ is the selfadjoint linear operator in $L_2(\mathbb{T}_L^d)$ which acts on the exponents $e^{i\mathbf{k} \cdot x}$, $\mathbf{k} \in \mathbb{Z}_L^d$, according to

$$f(-\Delta)e^{i\mathbf{k} \cdot x} = \gamma_{\mathbf{k}} e^{i\mathbf{k} \cdot x}, \quad \gamma_{\mathbf{k}} = f(\lambda_{\mathbf{k}}) \quad \text{where} \quad \lambda_{\mathbf{k}} = |\mathbf{k}|^2. \quad (0.4)$$

The real-valued smooth function $f(t), t \geq 0$, is positive and $f' > 0$. To avoid technicalities, not relevant for this work, we assume that $f(t) \geq C_1|t| + C_2$ for all t , for suitable positive constants C_1, C_2 (for example, $f(-\Delta)u = -\Delta u + u$). The processes $\beta^{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_L^d$, are standard independent complex Wiener processes, i.e., $\beta^{\mathbf{k}}(t) = \beta_+^{\mathbf{k}}(t) + i\beta_-^{\mathbf{k}}(t)$, where $\beta_{\pm}^{\mathbf{k}}(t)$ are standard

independent real Wiener processes. The real numbers $b_{\mathbf{k}}$ are all non-zero and decay fast when $|\mathbf{k}| \rightarrow \infty$. The factor in front of the random force is chosen to be $\sqrt{\nu}$ to guarantee that solutions of (0.3) stay of order one when $t \gg 1$ and $0 < \nu \ll 1$.

We *assume* that eq. (0.3) with sufficiently smooth initial data $u(0, x) = u_0(x)$ is well posed. It is well known that this assumption holds (at least) under some restriction on d, q_* and the growth of $f(t)$ at infinity, see in Section 1.1

The parameters ν and ε measure, respectively, the inverse time-scale of the forced oscillations, and their amplitude. Physicists consider different regimes, where the two parameters are tied in various ways.¹ To do this they assume some relations between ε and ν , explicitly or implicitly. In our work we choose

$$\varepsilon^{2q_*} = \rho\nu,$$

where $\rho > 0$ is a constant. This assumption is within the usually imposed bounds, see [Naz11]. Passing to the slow time $\tau = \nu t$, we get the rescaled equation

$$\dot{u} + i\nu^{-1}(-\Delta u) = -f(-\Delta)u - i\rho|u|^{2q_*}u + \sum_{\mathbf{k}} b_{\mathbf{k}}\dot{\beta}^{\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot x}, \quad (0.5)$$

where $u = u(\tau, x)$, $x \in \mathbb{T}_L^d$ and the upper dot $\dot{\cdot}$ stands for $\frac{d}{d\tau}$. If we write $u(\tau, x)$ as Fourier series, $u(\tau, x) = \sum_{\mathbf{k}} v_{\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot x}$, then in view of (0.2), eq. (0.5) may be written as the system

$$\dot{v}_{\mathbf{k}} + i\nu^{-1}\lambda_{\mathbf{k}}v_{\mathbf{k}} = -\gamma_{\mathbf{k}}v_{\mathbf{k}} + 2\rho i \frac{\partial \mathcal{H}(v)}{\partial \bar{v}_{\mathbf{k}}} + b_{\mathbf{k}}\dot{\beta}^{\mathbf{k}}(\tau), \quad \mathbf{k} \in \mathbb{Z}_L^d. \quad (0.6)$$

Here $\mathcal{H}(v)$ is the Hamiltonian \mathcal{H} , expressed in terms of the Fourier coefficients $v = (v_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_L^d)$:

$$\mathcal{H}(v) = -\frac{1}{2q_* + 2} \sum_{\mathbf{k}_1, \dots, \mathbf{k}_{2q_*+2} \in \mathbb{Z}_L^d} v_{\mathbf{k}_1} \dots v_{\mathbf{k}_{q_*+1}} \bar{v}_{\mathbf{k}_{q_*+2}} \dots \bar{v}_{\mathbf{k}_{2q_*+2}} \delta_{q_*+2, \dots, 2q_*+2}^{1, \dots, q_*+1}, \quad (0.7)$$

and we use the standard notation (see [Naz11]):

$$\delta_{q_*+2, \dots, 2q_*+2}^{1, \dots, q_*+1} = \begin{cases} 1 & \text{if } \mathbf{k}_1 + \dots + \mathbf{k}_{q_*+1} - \mathbf{k}_{q_*+2} - \dots - \mathbf{k}_{2q_*+2} = 0 \\ 0 & \text{otherwise} \end{cases}. \quad (0.8)$$

As before we are interested in the limit $\nu \rightarrow 0$.²

We note that the method of our work applies as well to equations (0.6) with the Hamiltonians \mathcal{H} of the form (0.2), where the density of the Hamiltonian is a real-valued polynomial of u and \bar{u} (not necessarily a polynomial of $|u|^2$). For instance, we could work with the cubic Hamiltonians $\mathcal{H}^3 = \int |u|^2(u + \bar{u}) dx$ or $\mathcal{H}^3 = \int (u^3 + \bar{u}^3) dx$.

0.2 Discrete Turbulence

In physics equations (0.1) and (0.5) with $\nu, \varepsilon \ll 1$ are treated by the theory of weak turbulence (WT); see the works, quoted above. That theory either deals with equation (0.5), where $L = \infty$ by formal replacing Fourier series for L -periodic functions with Fourier integrals and makes with

¹When forcing and dissipation are not present, a parameter T is introduced, which measures the time scale on which averaging is performed (see [Naz11]): to heuristically compare with the present case, we put $\nu = 1/T$.

²See [KN13] for a theory of equation (0.5) for the case when $f(t) = t + 1$ and $\nu = \infty$.

them bold transformations, or considers the limit $\nu, \varepsilon \rightarrow 0$ simultaneously with the limit $L \rightarrow \infty$. That is, considers the iterated limit

$$L \rightarrow \infty, \quad \varepsilon, \nu \rightarrow 0, \quad (0.9)$$

and treats it in an equally bold way. Concerning this limit the WT makes a number of remarkable predictions, based on tools and ideas, developed in the community, which can be traced back to the work [Pei97]. Relation between the parameters in (0.9) is not quite clear, and it may be better to talk about the WT limits (rather than about a single case).

In order to understand the double limit above, it is natural to study first the limit $\nu \rightarrow 0$ (with L fixed). Its deterministic version recently got attention in physical literature as the “discrete turbulence (DT) limit”, see [Kar10] and [Naz11], Section 10. Similar limits were considered by mathematicians, interested in related problems (see [GG12]), and were used by them for intermediate arguments (e.g., see [FGH13]).

Our work is dedicated to rigorous justification of the DT limit for the damped-driven equation (0.3) $_{\nu \sim \varepsilon^{2q_*}} = (0.5)$. Namely, we show that *when $\nu \rightarrow 0$, statistical characteristics of actions of solution u^ν for (0.5) have limits of order one, described by actions of solutions for a certain effective equation which is a nonlinear stochastic equation with coefficients of order one and with a hamiltonian nonlinearity, made out of the resonant terms of the nonlinearity $|u|^{2q_*} u$.*

The effective equation above is a natural stochastic version of similar equations from the deterministic (physical) DT (see [Kar10, Naz11]). So, in a sense, our results justify the physical DT in the stochastic setting. But in the stochastic case we do more than that since we also treat the stationary regime for eq. (0.5) and show that it converges to that for the effective equation. So solutions of the latter approximate (in distribution) solutions of the former as $t \rightarrow \infty$ and $\nu \rightarrow 0$. Remarkably, in the stationary regime the effective equation approximates not only the actions of solutions with $\nu \ll 1$, but also their angles, see below.

As the title of the paper suggests, our argument is a form of averaging. The latter is a tool which is used by the WT community on a regular basis, either explicitly (e.g. see [Naz11]), or implicitly.

0.3 Inviscid limits for damped/driven hamiltonian PDE, effective equations and interaction representation

Equation (0.3) is the linear HPDE (=linear hamiltonian PDE) (0.1) $_{\varepsilon=0}$, driven by the random force, damped by the linear damping $-\nu f(-\Delta u)$ and perturbed by the hamiltonian nonlinearity $-\varepsilon^{2q_*} i\rho |u|^{2q_*} u$. Damped/driven HPDE and the inviscid limits in these equations when the random force and the damping go to zero, are very important for physics. In particular, since the d -dimensional Navier-Stokes equation (NSE) with a random force can be regarded as a damped/driven Euler equation (which is an HPDE), and the inviscid limit for the NSE describes the d -dimensional turbulence. The NSE with a random force, especially when $d = 2$, was intensively studied last years, but the corresponding inviscid limit turned out to be very complicated even for $d = 2$, see [KS12]. The problem of this limit becomes feasible when the underlying HPDE is integrable or linear. The most famous integrable PDE is the KdV equation. Its damped/driven perturbations and the corresponding inviscid limits were studied

in [KP08, Kuk10]. In [Kuk13] the method of those works was applied to the situation when the unperturbed HPDE is the Schrödinger equation

$$u_t + i(-\Delta u + V(x)u) = 0, \quad x \in \mathbb{T}_L^d, \quad (0.10)$$

where the potential $V(x)$ is in general position. Crucial for the just mentioned works is that there the unperturbed equation is free from strong resonances. For [KP08, Kuk10] it means that all solutions of KdV are almost-periodic functions of time, such that for a typical solution the corresponding frequency vector is free from resonances; while for [Kuk13] it means that for the typical potentials $V(x)$, considered in [Kuk13], the spectrum of the linear operator in (0.10) is non-resonant.

In contrast, now the linear operator in the unperturbed equation $(0.1)_{\varepsilon=0}$ has the eigenvalues $\lambda_{\mathbf{k}} \in \mathbf{k}^{-2}\mathbb{Z}$, $\mathbf{k} \in \mathbb{Z}_L^d$ (see (0.4)), which are highly resonant (accordingly, all solutions for eq. $(0.1)_{\varepsilon=0}$ are periodic with the same period $2\pi L^{-2}$). This gives rise to an additional difficulty. To explain it, we rewrite equation (0.5)=(0.6) as a fast-slow system, denoting $I_{\mathbf{k}} = \frac{1}{2}|v_{\mathbf{k}}|^2$, $\varphi_{\mathbf{k}} = \text{Arg } v_{\mathbf{k}}$ (these are the action-angles for the linear hamiltonian system $(0.1)_{\varepsilon=0}$). In the new variables eq. (0.5) reads

$$\dot{I}_{\mathbf{k}}(\tau) = v_{\mathbf{k}} \cdot P_{\mathbf{k}}(v) + b_{\mathbf{k}}^2 + b_{\mathbf{k}}(v_{\mathbf{k}} \cdot \dot{\beta}^{\mathbf{k}}), \quad (0.11)$$

$$\dot{\varphi}_{\mathbf{k}}(\tau) = -\nu^{-1}\lambda_{\mathbf{k}} + I_{\mathbf{k}}^{-1} \dots, \quad (0.12)$$

where $\mathbf{k} \in \mathbb{Z}_L^d$, the dot \cdot indicates the real scalar product in $\mathbb{C} \simeq \mathbb{R}^2$, $P(v)$ is the vector field in the r.h.s. of the v -equation (0.6) and \dots abbreviates a factor of order one (as $\nu \rightarrow 0$). If the frequencies $\{\lambda_{\mathbf{k}}\}$ are resonant, then equations for some linear combinations of the phases $\varphi_{\mathbf{k}}$ are slow, which make it more difficult to analyse the system. The method of resonant averaging treats this problem in finite dimension, see [AKN06] and Section 1.2 below. In the situation at hand, we have additional problem: the φ -equations (0.12) have singularities at the locus

$$\mathcal{D} = \{I : I_{\mathbf{k}} = 0 \text{ for some } \mathbf{k}\} \quad (0.13)$$

which is dense in the space of sequences $(I_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_L^d)$, and the averaged I -equations

$$\dot{I}_{\mathbf{k}}(\tau) = \langle v_{\mathbf{k}} \cdot P_{\mathbf{k}} \rangle(I) + b_{\mathbf{k}}^2 + b_{\mathbf{k}}\sqrt{2I_{\mathbf{k}}}\dot{\beta}^{\mathbf{k}}(\tau), \quad \mathbf{k} \in \mathbb{Z}_L^d, \quad (0.14)$$

where $\langle \cdot \rangle$ signifies the average in $\varphi \in \mathbb{T}^\infty$, have there weak singularities. A way to overcome these difficulties is to find for (0.11), (0.12) an *effective equation*, which is a system of regular equations

$$\dot{v}_{\mathbf{k}} = R_{\mathbf{k}}(v) + b_{\mathbf{k}}\dot{\beta}^{\mathbf{k}}(\tau), \quad \mathbf{k} \in \mathbb{Z}_L^d, \quad (0.15)$$

such that under the natural projection $v_{\mathbf{k}} \mapsto I_{\mathbf{k}} = \frac{1}{2}|v_{\mathbf{k}}|^2$, $\mathbf{k} \in \mathbb{Z}_L^d$, solutions of (0.15) transform to solutions of (0.14). In [Kuk10] this approach was used to study the perturbed KdV equation, written as a fast-slow system, similar to (0.11), (0.12). That system has strongly non-linear behaviour, and in [Kuk10] the effective equation was constructed as a kind of averaging of the corresponding I -equations. In [Kuk13] an effective equation for the damped/driven nonresonant equation (0.10) was derived in a similar way. If the introduced damping is linear and the nonlinearity is hamiltonian, like in eq. (0.3), then the effective equation in [Kuk13] is linear.

When the unperturbed hamiltonian system is linear, an alternative way to find an effective equation is to use the *interaction representation*. I.e., to pass from the complex variables $v_{\mathbf{k}}(\tau)$ (which diagonalise the linear system) to the fast rotating variables

$$a_{\mathbf{k}}(\tau) = e^{i\nu^{-1}\lambda_{\mathbf{k}}\tau} v_{\mathbf{k}}(\tau), \quad \mathbf{k} \in \mathbb{Z}_L^d. \quad (0.16)$$

Since $|a_{\mathbf{k}}| = |v_{\mathbf{k}}|$, then the limiting dynamics of the a -variables controls the limiting behaviour of the actions $I_{\mathbf{k}}$. So a regular system of equations, describing the limiting a -dynamics, is the effective equation. N. N. Bogolyubov used this approach for the finite-dimensional deterministic averaging, calling it *averaging in the quasilinear systems* (see in [AKN06]). The interaction representation is systematically used in the WT.

Now consider the fast-slow equations (0.11), (0.12) which come from eq. (0.6), where the fast motion (0.12) is highly resonant. Repeating the construction of the effective equation from [Kuk13], but replacing there the usual averaging by the resonant averaging, we find an effective equation, corresponding to (0.6). It turned out to be another damped/driven hamiltonian system with a Hamiltonian \mathcal{H}^{res} , obtained by the resonant averaging of $\mathcal{H}(v)$, see Section 2.2. As we said above, an alternative way to derive the effective equation is through the interaction representation, i.e., by transition from the v -variables to the a -variables (0.16). In view of (0.6), the a -variables satisfy the system of equations

$$\begin{aligned} \dot{a}_{\mathbf{k}} = & -\gamma_{\mathbf{k}} a_{\mathbf{k}} + e^{i\nu^{-1}\lambda_{\mathbf{k}}\tau} b_{\mathbf{k}} \dot{\beta}^{\mathbf{k}}(\tau) \\ & - \rho i \sum_{\mathbf{k}_1, \dots, \mathbf{k}_{2q_*+1} \in \mathbb{Z}_L^d} a_{\mathbf{k}_1} \dots a_{\mathbf{k}_{q_*+1}} \bar{a}_{\mathbf{k}_{q_*+2}} \dots \bar{a}_{\mathbf{k}_{2q_*+1}} \delta_{q_*+2 \dots 2q_*+1}^{1 \dots q_*+1} \\ & \times \exp(-i\nu^{-1}\tau(\lambda_{\mathbf{k}_1} + \dots + \lambda_{\mathbf{k}_{q_*+1}} - \lambda_{\mathbf{k}_{q_*+2}} - \dots - \lambda_{\mathbf{k}_{2q_*+1}} - \lambda_{\mathbf{k}})), \quad \mathbf{k} \in \mathbb{Z}_L^d. \end{aligned} \quad (0.17)$$

The terms, constituting the nonlinearity, oscillate fast as ν goes to zero, unless the sum of the eigenvalues in the exponent in the third line vanishes. The processes $\{e^{i\nu^{-1}\lambda_{\mathbf{k}}\tau} \dot{\beta}^{\mathbf{k}}(\tau), \mathbf{k} \in \mathbb{Z}_L^d\}$, make another set of standard independent complex white noises. This leads to the right guess that only the terms for which this sum equals zero (i.e., the resonant terms), contribute to the limiting dynamics, and that the effective equation is the following damped/driven hamiltonian system

$$\dot{v}_{\mathbf{k}} = -\gamma_{\mathbf{k}} v_{\mathbf{k}} + 2\rho i \frac{\partial \mathcal{H}^{\text{res}}(v)}{\partial \bar{v}_{\mathbf{k}}} + b_{\mathbf{k}} \dot{\beta}^{\mathbf{k}}(\tau), \quad \mathbf{k} \in \mathbb{Z}_L^d. \quad (0.18)$$

Here the Hamiltonian $\mathcal{H}^{\text{res}}(v)$ is given by the sum

$$-\frac{1}{2q_*+2} \sum_{\mathbf{k}_1, \dots, \mathbf{k}_{2q_*+2} \in \mathbb{Z}_L^d} v_{\mathbf{k}_1} \dots v_{\mathbf{k}_{q_*+1}} \bar{v}_{\mathbf{k}_{q_*+2}} \dots \bar{v}_{\mathbf{k}_{2q_*+2}} \delta_{q_*+2 \dots 2q_*+2}^{1 \dots q_*+1} \delta(\lambda_{q_*+2 \dots 2q_*+2}^{1 \dots q_*+1}), \quad (0.19)$$

so that $2\rho i \frac{\partial \mathcal{H}^{\text{res}}}{\partial \bar{v}_{\mathbf{k}}}(v)$ is

$$-\rho i \sum_{\mathbf{k}_1, \dots, \mathbf{k}_{2q_*+2} \in \mathbb{Z}_L^d} v_{\mathbf{k}_1} \dots v_{\mathbf{k}_{q_*+1}} \bar{v}_{\mathbf{k}_{q_*+2}} \dots \bar{v}_{\mathbf{k}_{2q_*+2}} \delta_{q_*+2 \dots 2q_*+1}^{1 \dots q_*+1} \delta(\lambda_{q_*+2 \dots 2q_*+1}^{1 \dots q_*+1}), \quad (0.20)$$

where we use another physical abbreviation:

$$\delta(\lambda_{q_*+2 \dots 2q_*+2}^{1 \dots q_*+1}) = \begin{cases} 1 & \text{if } \lambda_{\mathbf{k}_1} + \dots + \lambda_{\mathbf{k}_{q_*+1}} - \lambda_{\mathbf{k}_{q_*+2}} - \dots - \lambda_{\mathbf{k}_{2q_*+2}} = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (0.21)$$

This representation for \mathcal{H}^{res} is different from that given by the resonant averaging. Its advantage is the natural relation with the a -variables, which is convenient to study the limit $\nu \rightarrow 0$. The representation for \mathcal{H}^{res} by means of the resonant averaging turned out to be more useful to study properties of \mathcal{H}^{res} and of the corresponding hamiltonian vector field.

We saw that the effective equation can be obtained from the system (0.6) by a simple procedure: drop the fast rotations and replace the Hamiltonian \mathcal{H} by its resonant average \mathcal{H}^{res} . In difference with the non-resonant case, this is a nonlinear system. The corresponding hamiltonian system

$$\dot{v}_{\mathbf{k}} = 2\rho i \frac{\partial \mathcal{H}^{\text{res}}(v)}{\partial \bar{v}_{\mathbf{k}}}, \quad \mathbf{k} \in \mathbb{Z}_L^d, \quad (0.22)$$

has a vector field, locally Lipschitz in sufficiently smooth spaces, so equation (0.18) is well posed locally in time. In fact, it is globally well posed. We get this result in Section 4.1 as a simple consequence of our main theorems.

The Hamiltonian \mathcal{H}^{res} has two convex quadratic integrals,

$$H_0(v) = \frac{1}{2} \sum |v_{\mathbf{k}}|^2, \quad H_1 = \frac{1}{2} \sum \lambda_{\mathbf{k}} |v_{\mathbf{k}}|^2,$$

which are similar to the energy and the enstrophy integrals for the 2d Euler equation on \mathbb{T}^2 (see (2.32)), and the vector-integral of moments $M(u) = \frac{1}{2} \sum \mathbf{k} |u_{\mathbf{k}}|^2 \in \mathbb{R}^d$, which can be compared with the extra integrals of the 2d Euler. Besides, the vector-field (0.20) is non-linear homogeneous and hamiltonian, as that of the Euler equation. This makes the effective equation (0.18) similar to the 2d Navier-Stokes system on \mathbb{T}^2 . Fortunately the former is significantly simpler than the latter.

0.4 Results

Main results of our work are stated and proved in Section 4, based on properties of the effective equation, established earlier. They imply that the long-time behaviour of solutions for equations (0.5), when $\nu \rightarrow 0$, is controlled in distribution by solutions for the effective equation. We start with the results on the Cauchy problem. So, let $v^\nu(\tau)$ be a solution of (0.6) such that $v^\nu(0) = v_0$, where $v_0 = (v_{0\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_L^d)$ corresponds to a sufficiently smooth function $u_0(x)$. Let us fix any $T > 0$.

Consider the vector of actions $I(v^\nu(\tau)) = \{I_{\mathbf{k}}(v^\nu(\tau)), \mathbf{k} \in \mathbb{Z}_L^d\}$.

Theorem 1. When $\nu \rightarrow 0$, we have the weak convergence of measures

$$\mathcal{D}(I(v^\nu(\tau))) \rightharpoonup \mathcal{D}(I(v^0(\tau))), \quad (0.23)$$

where $v^0(\tau)$, $0 \leq \tau \leq T$, is a unique solution of equation (0.18) such that $v^0(0) = v_0$.

For any $\xi \in \mathbb{Z}_0^\infty$, where \mathbb{Z}_0^∞ is the set of integer vectors $(\xi_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_L^d)$ of finite length, we denote $\Phi^\xi(v^\nu(\tau)) := \sum \xi_{\mathbf{k}} \varphi_{\mathbf{k}}(v^\nu(\tau)) \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Then, in addition to (0.23), for a resonant vector $\xi \in \mathbb{Z}_0^\infty$ ³ the distribution of $\Phi^\xi(v^\nu(\tau))$, mollified in τ , converges as $\nu \rightarrow 0$ to that of $\Phi^\xi(v^0(\tau))$. On the contrary, if ξ is non-resonant, then the measure $\mathcal{D}(\Phi^\xi(v^\nu(\tau)))$, mollified in τ , converges to the Lebesgue measure on S^1 . All this is proved in Section 4.1, using the interaction representation (0.17) for equation (0.3).

³i.e. for a vector ξ such that $\sum_{\mathbf{k}} \xi_{\mathbf{k}} \lambda_{\mathbf{k}} = 0$.

The limiting behaviour of solutions $v^\nu(\tau)$ can be described without evoking the effective equation. See Proposition 4.5.

Now consider a stationary measure μ^ν for equation (0.5) (it always exist). We have **Theorem 2**. Every sequence $\nu'_j \rightarrow 0$ has a subsequence $\nu_j \rightarrow 0$ such that

$$I \circ \mu^{\nu_j} \rightharpoonup I \circ m^0, \quad \Phi^{(\xi)} \circ \mu^{\nu_j} \rightharpoonup \Phi^{(\xi)} \circ m^0$$

for any resonant vector $\xi \in \mathbb{Z}_0^\infty$, where m^0 is a stationary measure for equation (0.18). If a vector ξ is non-resonant, then the measure $\Phi^{(\xi)} \circ \mu^\nu$ converges, as $\nu \rightarrow 0$, to the Lebesgue measure on S^1 .

If the effective equation has a unique stationary measure m^0 , then the limits in Theorem 2 do not depend on the sequence $\nu_j \rightarrow 0$, so the convergences hold as $\nu \rightarrow 0$. Remarkably, in this case the measure m^0 controls not only the fast, but also the slow components of the measures μ^ν :

Theorem 3. If the effective equation has a unique stationary measure m^0 , then $\mu^\nu \rightharpoonup m^0$ as $\nu \rightarrow 0$.

In particular, if the effective equation has a unique stationary measure m^0 and the equation (0.3) is mixing,⁴ then m^0 describes asymptotical behaviour of distributions of solutions $u(t)$ for (0.3) as $t \rightarrow \infty$ and $\nu \rightarrow 0$:

$$\lim_{\nu \rightarrow 0} \lim_{t \rightarrow \infty} \mathcal{D}(u(t)) = m^0.$$

In view of the last theorem, it is important to understand when the effective equation has a unique stationary measure and is mixing. This is discussed in Section 4.3. In particular, the mixing holds if $q_* = 1$, $f(t) = t + 1$ and $d \leq 3$.

Other equations. Our approach applies to other equations, usually considered in the WT. In particular, in [KM14] we apply it to the 2d quasigeostrophic equation on the β -plane with random force:

$$(-\Delta + K)\psi - \rho J(\psi, \Delta\psi) - \beta\psi_x = \langle \text{random force} \rangle - \kappa\Delta^2\psi + \Delta\psi. \quad (0.24)$$

Here ψ is the stream function, $\psi = \psi(t, x, y)$, where $x \in \mathbb{R}/L\mathbb{Z}$ and $y \in \mathbb{R}/\mathbb{Z}$; $\Delta\psi$ is the Ekman damping, $-\kappa\Delta^2\psi$ is the kinematic viscosity and the random force is similar to that in eq. (0.3)=(0.5). The equation has the same structure as (0.5), and our approach applies to prove that for typical values of the horizontal period L (when the structure of resonances is relatively simple) the limiting, as $\beta \rightarrow \infty$, behaviour of solutions for (0.24) exists, is uniform in $\kappa \in (0, 1]$, and is described by an effective equation which is an infinite system of stochastic equations. This system splits to invariant systems of complex dimension ≤ 3 ; each of them is an integrable hamiltonian system, coupled with a Langevin thermostat. Under the iterated limits $\lim_{L=\rho \rightarrow \infty} \lim_{\beta \rightarrow \infty}$ and $\lim_{\kappa \rightarrow 0} \lim_{\beta \rightarrow \infty}$ we get similar systems. In particular, none of the three limiting systems exhibits the energy cascade to high frequencies.

⁴both these conditions hold, e.g. if $q_* = 1$ and $f(\lambda) = c_1 + \lambda^{c_d}$, where c_d is sufficiently big in terms of d .

0.5 Weak Turbulence

The most famous prediction of the WT (see Section 0.2) deals with the distribution of the energy of solutions for (0.1) and (0.3) between the frequencies. To describe the corresponding claims, consider the quantity $\mathbf{E}|v_{\mathbf{k}}(\tau)|^2$, average it in time⁵ τ and in wave-vectors $\mathbf{k} \in \mathbb{Z}_L^d$ such that $|\mathbf{k}| \approx r > 0$; next properly scale this and denote the result E_r . The function $r \rightarrow E_r$ is called the *energy spectrum*. It is predicted by the WT that, in a certain *inertial range* $r \in [r_1, r_2]$, which is contained in the spectral zone where the random force is negligible (i.e., where $|b_{\mathbf{k}}| \ll (\mathbf{E}|v_{\mathbf{k}}|^2)^{1/2}$), the energy spectrum has an algebraic behaviour:

$$E_r \sim r^{-\alpha} \quad \text{for } r \in [r_1, r_2], \quad (0.25)$$

for a suitable $\alpha > 0$. If the WT is stated in terms of the iterated limits (0.9), then only the limits which lead to the algebraic energy spectra (0.25) are relevant.

In our sequel work [KM13a] we study the effective equation for (0.5) under the limit $L \rightarrow \infty$, evoking the heuristic tools from WT, as presented in [ZLF92], and mimicking the logic of that book. There we show (heuristically) that a suitable choice of the function $\rho(L)$ leads, in the limit of $L \rightarrow \infty$, to a wave kinetic equation for the averaged actions $n_{\mathbf{k}}(t) = \frac{1}{2}\mathbf{E}|v_{\mathbf{k}}(t)|^2$. That equation is different from the heuristic kinetic equations, obtained by the WT methods (see [CFG08], sec. 1.2, and [ZLF92], sec. 2.2.3), but is closely related to them, so that the Zakharov ansatz applies and allows to obtain stationary solutions of the equation, algebraic in $r = |\mathbf{k}|$ and corresponding to energy spectra of the desired form (0.25).⁶

The rigorous and heuristic results, obtained in this work and in [KM13a], encourage us to pursue our program to study the WT in the model, given by eq. (0.3), which brings to the WT the advantage of a rigorous foundation, based on the recent results of stochastic calculus. We believe that some predictions of the WT (including the fact that the limiting behaviour of the averaged actions $n_{\mathbf{k}}(t)$ is described by a certain wave kinetic equation which admits stationary solutions of the algebraical form (0.25)), may be obtained if not under the iterated limit “first $\nu \rightarrow 0$, next $L \rightarrow 0$ ”, then under its suitable modification (e.g., “ $\nu \rightarrow 0$ and $L \rightarrow \infty$ in such a way that $\nu L \rightarrow 0$ ”).

Notation and Agreement. The *stochastic terminology* we use agrees with [KS91]. All filtered probability spaces we work with satisfy the *usual condition* (see [KS91]).⁷ Sometime we forget to mention that a certain relation holds a.s.

Spaces of integer vectors. We denote by \mathbb{Z}_0^∞ the set of vectors in \mathbb{Z}^∞ of finite length, and denote $\mathbb{Z}_{+0}^\infty = \{s \in \mathbb{Z}_0^\infty : s_k \geq 0 \forall k\}$. Also see (1.15) and (2.9).

⁵Certainly this is not needed if we consider stationary solutions of the equation.

⁶The work [CFG08] mainly treats the systems, where the sources and sinks of energy are well separated, so that there is a large spectral region where the forcing and the dissipation are practically absent and where the kinetic equation does not depend explicitly on the form of the forcing and dissipation. For this reason, the stationary spectra, found in [CFG08] for (0.3), are equal to those obtained by the WT methods for the HPDE (0.1). This explains the difference between the kinetic equation in [CFG08] and that in [KM13a] (which deals with the case when the dissipation is present on the whole spectral range). Contrary to [CFG08], the method of the work [ZL75] applies, at least formally, to all kinds of forcing and damping, so the kinetic equation of that work may depend on the forcing and dissipation.

⁷I.e., the corresponding filtrations $\{\mathcal{F}_t\}$ are continuous from the right, and each \mathcal{F}_t contains all negligible sets.

Infinite vectors. For an infinite vector $\xi = (\xi_1, \xi_2, \dots)$ (integer, real or complex) and $N \in \mathbb{N}$ we denote by ξ^N the vector (ξ_1, \dots, ξ_N) , or the vector $(\xi_1, \dots, \xi_N, 0, \dots)$, depending on the context. For a complex vector ξ and $s \in \mathbb{Z}_{+0}^\infty$ we denote $\xi^s = \prod_j \xi_j^{s_j}$.

Norms. We use $|\cdot|$ to denote the Euclidean norm in \mathbb{R}^d and in $\mathbb{C} \simeq \mathbb{R}^2$, as well as the ℓ_1 -norm in \mathbb{Z}_0^∞ . For the norms $|\cdot|_{h^m}$ and $|\cdot|_{h_T^p}$ see (1.13) and below that.

Scalar products. The notation “ \cdot ” stands for the scalar product in \mathbb{Z}_0^∞ , the pairing of \mathbb{Z}_0^∞ with \mathbb{Z}^∞ , the Euclidean scalar product in \mathbb{R}^d and in \mathbb{C} . The latter means that if $u, v \in \mathbb{C}$, then $u \cdot v = \operatorname{Re}(\bar{u}v)$. The L_2 -product is denoted $\langle \cdot, \cdot \rangle$, and we also denote by $\langle f, \mu \rangle = \langle \mu, f \rangle$ the integral of a function f against a measure μ .

Max/Min. We denote $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$.

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1 Preliminaries

Since in this work we are not interested in the dependence of the results on L , from now on it will be kept fixed and equal to 1, apart from Section 3. There we make explicit calculations, controlling how their results depend on L .

1.1 A priori estimates.

In this section we discuss preliminary properties of solutions for (0.5). We found it convenient to parametrise the vectors from the trigonometric basis $\{e^{i\mathbf{k}\cdot x}\}$ by natural numbers and to normalise them. That is, to use the basis $\{e^j(x), j \geq 1\}$, where

$$e^j(x) = (2\pi)^{-d/2} e^{i\mathbf{k}\cdot x}, \quad \mathbf{k} = \mathbf{k}(j). \quad (1.1)$$

The functions $e^j(x)$ are eigen-vectors of the Laplacian, $-\Delta e^j = \lambda_j e^j$, so ordered that $0 = \lambda_1 < \lambda_2 \leq \dots$. Accordingly eq. (0.5) reads

$$\dot{u} + i\nu^{-1}(-\Delta u) = -f(-\Delta)u - i\rho|u|^{2q^*}u + \frac{d}{d\tau} \sum_{j=1}^{\infty} b_j \beta^j(\tau) e^j(x), \quad (1.2)$$

$u = u(\tau, x)$, where $f(-\Delta)e^j = \gamma_j e^j$ with $\gamma_j = f(\lambda_j)$. The processes $\beta^j = \beta^j + i\beta^{-j}$, $j \geq 1$, are standard independent complex Wiener processes. The real numbers b_j are such that for a suitable sufficiently large even integer r (defined below in (2.13)) we have

$$B_r := 2 \sum_{j=1}^{\infty} \lambda_j^r b_j^2 < \infty.$$

By \mathcal{H}^p , $p \in \mathbb{R}$, we denote the Sobolev space $\mathcal{H}^p = H^p(\mathbb{T}^d, \mathbb{C})$, regarded as a real Hilbert space, and denote by $\langle \cdot, \cdot \rangle$ the real L^2 -scalar product on \mathbb{T}^d . We provide \mathcal{H}^p with the norm $\|\cdot\|_p$,

$$\|u\|_p^2 = \sum_{j=1}^{\infty} |u_j|^2 (\lambda_j \vee 1)^p \quad \text{for } u(x) = \sum_{j=1}^{\infty} u_j e^j(x).$$

Let $u(t, x)$ be a solution of (1.2) such that $u(0, x) = u_0$. It satisfies standard a-priori estimates which we now discuss, following [Kuk13]. Firstly, for a suitable $\varepsilon_0 > 0$, uniformly in $\nu > 0$ one has

$$\mathbf{E} e^{\varepsilon_0 \|u(\tau)\|_0^2} \leq C(B_0, \|u_0\|_0) \quad \forall \tau \geq 0. \quad (1.3)$$

Assume that

$$q_* < \infty \quad \text{if } d = 1, 2, \quad q_* < \frac{2}{d-2} \quad \text{if } d \geq 3. \quad (1.4)$$

Then, the following bounds on the Sobolev norms of the solution hold for each $2m \leq r$ and every n :

$$\begin{aligned} \mathbf{E} \left(\sup_{0 \leq \tau \leq T} \|u(\tau)\|_{2m}^{2n} + \int_0^T \|u(s)\|_{2m+1}^2 \|u(s)\|_{2m}^{2n-2} ds \right) \\ \leq \|u_0\|_{2m}^{2n} + C(m, n, T)(1 + \|u_0\|_0^{c_{m,n}}), \end{aligned} \quad (1.5)$$

$$\mathbf{E} \|u(\tau)\|_{2m}^{2n} \leq C(m, n) \quad \forall \tau \geq 0, \quad (1.6)$$

where $C(m, n, T)$ and $C(m, n)$ also depend on B_{2m} .

Estimates (1.5), (1.6) are assumed everywhere in our work. As we have explained, they are fulfilled under the assumption (1.4), but if the function $f(t)$ grows super-linearly, then the restriction (1.4) may be weakened.

Relations (1.5) in the usual way (cf. [Hai02, KS04, Oda06, Shi06]) imply that eq. (1.2) is *regular in the space \mathcal{H}^r* in the sense that for any $u_0 \in \mathcal{H}^r$ it has a unique strong solution $u(t, x)$, equal to u_0 at $t = 0$, and satisfying estimates (1.3), (1.5) for any n . By the Bogolyubov-Krylov argument, applied to a solution of (1.2), starting from the origin at $t = 0$, this equation has a stationary measure μ^ν , supported by the space \mathcal{H}^r , and a corresponding stationary solution $u^\nu(\tau)$, $\mathcal{D}u^\nu(\tau) \equiv \mu^\nu$, also satisfies (1.3) and (1.6).

1.2 Resonant averaging

Let $W \in \mathbb{Z}^n$, $n \geq 1$, be a non-zero integer vector such that its components are relatively prime (so if $W = mV$, where $m \in \mathbb{Z}$ and $V \in \mathbb{Z}^n$, then $m = \pm 1$). We call the set

$$\mathcal{A} = \mathcal{A}(W) := \{s \in \mathbb{Z}^n : W \cdot s = 0\} \quad (1.7)$$

the *set of resonances for W* . This is a \mathbb{Z} -module. Denote its rank by r . Here and everywhere below the finite-dimensional vectors are regarded as column-vectors.

Lemma 1.1. *The rank r equals $n - 1$. There exists a system ζ^1, \dots, ζ^n of integer vectors in \mathbb{Z}^n such that $\text{span}_{\mathbb{Z}}\{\zeta^1, \dots, \zeta^{n-1}\} = \mathcal{A}$, and the $n \times n$ matrix $R = (\zeta^1 \zeta^2 \dots \zeta^n)$ is unimodular (i.e., $\det R = \pm 1$).*

That is, the vectors $(\zeta^1, \dots, \zeta^{n-1})$ make an integer basis of the hyperspace $W^\perp \subset \mathbb{R}^n$.

Proof. We restrict ourselves to the case when some component of the vector W equals one since this is the result we need below. For the general case and for a more general statement see, for example, [Bou71], Section 7.

Without loss of generality we assume that $W_n = 1$. Consider the matrix such that its n -th column is W and for $j < n$ the j -th column is the vector $e^j = (e_1^j, \dots, e_n^j)^T$, where $e_l^j = \delta_{j,l}$. It is unimodular and transforms the basis vector e^n to W . Its inverse is an unimodular matrix B such that $BW = e^n$. Let s be any vector in \mathcal{A} . Since

$$W \cdot s = 0 \Leftrightarrow BW \cdot (B^T)^{-1}s = 0 \Leftrightarrow e^n \cdot (B^T)^{-1}s = 0,$$

then $(B^T)^{-1}s = \sum_{j=1}^{n-1} m_j e^j$, where m_j 's are some integers. This proves the lemma if we choose $\zeta^j = B^T e^j$, $j = 1, \dots, n$. Note that the matrix R equals B^T . \square

Since $R^T W = BW = e^n$, then the automorphism of the torus $\mathbb{T}^n \rightarrow \mathbb{T}^n$, $\varphi \rightarrow y = R^T \varphi$, “resolves the resonances” in the differential equation $\dot{\varphi} = W$ in the sense that it transforms it to the equation

$$\dot{y} = R^T W = (0, \dots, 0, 1)^T. \quad (1.8)$$

Let us consider a mapping $L = L_{\mathcal{A}} : \mathbb{T}^n \rightarrow \mathbb{T}^{n-1}$, “dual to the module \mathcal{A} ”:

$$L : \mathbb{T}^n \ni \varphi \rightarrow (\varphi \cdot Re^1, \dots, \varphi \cdot Re^{n-1})^T \in \mathbb{T}^{n-1}. \quad (1.9)$$

The basis $\{\eta^j = (R^T)^{-1}e^j, 1 \leq j \leq n\}$, is dual to the basis $\{\zeta^j = Re^j, 1 \leq j \leq n\}$, since

$$\eta^j \cdot \zeta^l = (R^T)^{-1}e^j \cdot Re^l = \delta_{j,l}.$$

Therefore if we decompose $\varphi \in \mathbb{T}^n$ in the η -basis, $\varphi = \sum_k y_k \eta^k = (R^T)^{-1}y$, then $L\varphi = (y_1, \dots, y_{n-1})^T$. That is,

$$L \circ (R^T)^{-1}(y_1, \dots, y_n)^T = (y_1, \dots, y_{n-1})^T. \quad (1.10)$$

In particular, the fibers of the mapping L are the circles $R(\{\mathbf{y}\} \times S^1)$, where $\mathbf{y} = (y_1, \dots, y_{n-1})^T \in \mathbb{T}^{n-1}$.

For a continuous function f on \mathbb{T}^n we define its *resonant average with respect to the integer vector W* as the function

$$\langle f \rangle_W(\varphi) := \int_0^{2\pi} f(\varphi + tW) dt, \quad (1.11)$$

where we have set $dt := \frac{1}{2\pi} dt$.

Lemma 1.2. *Let f be a C^∞ -function on \mathbb{T}^n , $f(\varphi) = \sum f_s e^{is \cdot \varphi}$. Then*

$$\langle f \rangle_W(\varphi) = \sum f_s \delta_{0, s \cdot W} e^{is \cdot \varphi} = \sum_{s \in \mathcal{A}(W)} f_s e^{is \cdot \varphi}. \quad (1.12)$$

Proof. It is immediate that (1.12) holds for trigonometrical polynomial. Since for C^∞ -functions the series in (1.12) converges well, then by continuity the result holds for smooth functions f . \square

1.3 Resonant averaging in a Hilbert space

Consider the Fourier transform for complex functions on \mathbb{T}^d which we write as the mapping

$$\mathcal{F} : \mathcal{H} \ni u(x) \mapsto v = (v_1, v_2, \dots) \in \mathbb{C}^\infty ,$$

defined by the relation $u(x) = \sum v_k e^{kx}$. In the space of complex sequences we introduce the norms

$$|v|_{h^p}^2 = \sum_{k \geq 1} |v_k|^2 (\lambda_k \vee 1)^p, \quad p \in \mathbb{R}, \quad (1.13)$$

and set $h^p = \{v \mid |v|_{h^p} < \infty\}$. Then

$$|\mathcal{F}u|_{h^p} = \|u\|_p \quad \forall p.$$

For $k \geq 1$ let us denote $I_k = I(v_k) = \frac{1}{2}|v_k|^2$ and $\varphi_k = \varphi(v_k)$, where for $v \in \mathbb{C}$ $\varphi(v) = \text{Arg } v \in S^1$ if $v \neq 0$, and $\varphi(0) = 0 \in S^1$. For any $r \geq 0$ consider the mappings

$$\Pi_I : h^r \ni v \mapsto I = (I_1, I_2, \dots) \in h_{I+}^r, \quad \Pi_\varphi : h^r \ni v \mapsto \varphi = (\varphi_1, \varphi_2, \dots) \in \mathbb{T}^\infty. \quad (1.14)$$

Here h_{I+}^r is the positive octant $\{I : I_k \geq 0 \forall k\}$ in the space h_I^r , where

$$h_I^r = \{I \mid |I|_{h_I^r} = 2 \sum_k (\lambda_k \vee 1)^r |I_k| < \infty\}.$$

Abusing a bit notation we will write $\Pi_I(\mathcal{F}(u)) = I(u)$, $\Pi_\varphi(\mathcal{F}(u)) = \varphi(u)$. The mapping $I : \mathcal{H}^r \rightarrow h_I^r$ is 2-homogeneous continuous, while the mapping $\varphi : \mathcal{H}^r \rightarrow \mathbb{T}^\infty$ is Borel-measurable (the torus \mathbb{T}^∞ is given the Tikhonov topology and the corresponding Borel sigma-algebra).

For infinite integer vectors $s = (s_1, s_2, \dots)$ (and only for them) we will write the l_1 -norm of s as $|s|$,

$$|s| = \sum_j |s_j|.$$

We denote $\mathbb{Z}_0^\infty = \{s \in \mathbb{Z}^\infty : |s| < \infty\}$, and for a vector $s = (s_1, s_2, \dots) \in \mathbb{Z}_0^\infty$ write

$$\Lambda \cdot s = \sum_k \lambda_k s_k, \quad \text{supp } s = \{k : s_k \neq 0\}, \quad [s] = \max\{k : s_k \neq 0\}. \quad (1.15)$$

Similar for $\varphi \in \mathbb{T}^\infty$ and $s \in \mathbb{Z}_0^\infty$ we write $\varphi \cdot s = s \cdot \varphi = \sum_k \varphi_k s_k \in S^1$.

Let us fix some $m \in \mathbb{N} \cup \infty$ and define the set of resonances of order m for the (integer) frequency-vector $\Lambda = (\lambda_1, \lambda_2, \dots)$ as

$$\mathcal{A}(\Lambda, m) = \{s \in \mathbb{Z}_0^\infty : |s| \leq m, \Lambda \cdot s = 0\}. \quad (1.16)$$

We will abbreviate $\mathcal{A}(\Lambda) = \mathcal{A}(\Lambda, \infty) = \{s \in \mathbb{Z}_0^\infty : \Lambda \cdot s = 0\}$.

Let us denote $\mathbb{Z}_{+0}^\infty = \{s \in \mathbb{Z}_0^\infty : s_k \geq 0 \forall k\}$, and consider a series on some space h^r , $r \geq 0$:

$$F(v) = \sum_{p, q, l \in \mathbb{Z}_{+0}^\infty} C_{pql} (2I)^p v^q \bar{v}^l, \quad (1.17)$$

where $I = I(v)$, $C_{pql} = 0$ if $\text{supp } q \cap \text{supp } l \neq \emptyset$ and for $v \in h^r$, $q \in \mathbb{Z}_{+0}^\infty$ we write $v^q = \prod v_j^{q_j}$. We assume that the series converges normally in h^r in the sense that for each $R > 0$ we have

$$\sum_{p,q,l \in \mathbb{Z}_{+0}^\infty} |C_{pql}| \sup_{|v|_{h^r}, |w|_{h^r} \leq R} |v^p w^p v^q w^l| < \infty. \quad (1.18)$$

Clearly $F(v) = \mathbf{F}(v, \bar{v})$, where \mathbf{F} is a (complex) analytic function on $h^r \times h^r$. Abusing language and following a physical tradition we will say that F is analytic in v and \bar{v} . In particular, $F(v)$ is a real-analytic (so continuous) function of v , and the series (1.17) converges absolutely.

The resonant averaging of F can be conveniently defined by introducing, for any $\theta \in \mathbb{T}^\infty$, the rotation operator Ψ_θ , which is a linear operator in h^0 :

$$\Psi_\theta(v) = v', \quad v'_k = e^{i\theta_k} v_k.$$

Clearly this is an unitary isomorphism of every space h^r . Note that $(I \times \varphi)(\Psi_\theta v) \equiv (I(v), \varphi(v) + \theta)$. Using that Λ is an integer vector and based on definition (1.11), we give the following

Definition. If a function $F \in C(h^r)$ is given by a normally converging series (1.17), then its resonant average with respect to Λ is the function

$$\langle F \rangle_\Lambda(v) := \int_0^{2\pi} F(\Psi_{t\Lambda}(v)) dt, \quad dt = dt/2\pi. \quad (1.19)$$

Defining a function $\tilde{F}(I, \varphi)$ by the relation $F(v) = \tilde{F}(I(v), \varphi(v))$, we see that $\langle F \rangle_\Lambda(v) = \int_0^{2\pi} \tilde{F}(I, \varphi + t\Lambda) dt$. So this definition well agrees with (1.11).

Consider a monomial $F = (2I)^p v^q \bar{v}^l$. By Lemma 1.2 we have

$$\langle (2I)^p v^q \bar{v}^l \rangle_\Lambda = (2I)^p v^q \bar{v}^l \delta_{0, (q-l) \cdot \Lambda}.$$

Now assume that F is given by a normally convergent series (1.17) and has degree $\leq m \leq \infty$ in sense that $C_{pql} = 0$ unless $|q| + |l| \leq m$. Then

$$\langle F \rangle_\Lambda(v) = \sum_{q-l \in \mathcal{A}(\Lambda, m)} C_{pql} (2I)^p v^q \bar{v}^l = \sum_{(q-l) \cdot \Lambda = 0} C_{pql} (2I)^p v^q \bar{v}^l. \quad (1.20)$$

If the series (1.17) converges normally, then the series in the r.h.s. above also does. It defines an analytic in (v, \bar{v}) function. Note that in view of (1.20)

$$\langle F \rangle_\Lambda \text{ is a function of } I_1, I_2 \dots \text{ and the variables } \{s \cdot \varphi, s \in \mathcal{A}(\Lambda, m)\}. \quad (1.21)$$

2 Averaging for equation (1.2).

Everywhere below T is a fixed positive number.

2.1 Equation (1.2) in v -variables, resonant monomials and combinations of phases.

Let us pass in eq. (1.2) with $u \in \mathcal{H}^r$, $r > d/2$, to the v -variables, $v = \mathcal{F}(u) \in h^r$:

$$dv_k + i\nu^{-1} \lambda_k v_k d\tau = P_k(v) d\tau + b_k d\beta^k(\tau), \quad k \geq 1; \quad v(0) = \mathcal{F}(u_0) =: v_0. \quad (2.1)$$

Here

$$P_k = P_k^1 + P_k^0, \quad (2.2)$$

where P^1 and P^0 are, correspondingly, the linear and nonlinear hamiltonian parts of the perturbation. So P_k^1 is the Fourier-image of $-f(-\Delta)$, i.e. $P_k^1 = \text{diag}\{-\gamma_k, k \geq 1\}$, while the operator P^0 is the mapping $u \mapsto -i\rho|u|^{2q_*}u$, written in the v -variables. I.e.,

$$P^0(v) = -i\rho\mathcal{F}(|u|^{2q_*}u), \quad u = \mathcal{F}^{-1}(v).$$

Every its component P_k^0 is a sum of monomials:

$$P_k^0(v) = \sum_{p,q,l \in \mathbb{Z}_{+0}^\infty} C_k^{pql} (2I)^p v^q \bar{v}^l = \sum_{p,q,l \in \mathbb{Z}_{+0}^\infty} P_k^{0pql}(v), \quad k \geq 1, \quad (2.3)$$

where $C_k^{pql} = 0$ unless $2|p| + |q| + |l| = 2q_* + 1$ and $|q| = |l| + 1$. It is straightforward that $P_k^0(I, \varphi)$ (see (1.14)) is a function of $\varphi = (\varphi_j, j \geq 1)$ of order $2q_* + 1$, and that the mapping P^0 is analytic of polynomial growth:

Lemma 2.1. *The nonlinearity P^0 defines a real-analytic transformation of h^r if $r > \frac{d}{2}$. The mapping $P^0(v)$ and its differential $dP^0(v)$ both have polynomial growth in $|v|_{hr}$.*

We will refer to equations (2.1) as to the v -equations.

For any $s \in \mathbb{Z}_0^\infty$ consider the linear combination of phases

$$\Phi^s : h^0 \rightarrow S^1, \quad v \mapsto s \cdot \varphi(v).$$

We fix

$$m = 2q_* + 2,$$

and find the corresponding set $\mathcal{A} = \mathcal{A}(\Lambda, m)$ of resonances or order m (see (1.16)). We order vectors in the set \mathcal{A} , that is write it as $\mathcal{A} = \{s^{(1)}, s^{(2)}, \dots\}$, in such a way that $[s^{(j_1)}] \leq [s^{(j_2)}]$ if $j_1 \leq j_2$, and for $N \geq 1$ denote

$$J(N) = \max\{j : [s^{(j)}] \leq N\}. \quad (2.4)$$

For any $s^{(j)} \in \mathcal{A}$ consider the corresponding resonant combination of phases $\varphi(v)$, $\Phi_j(v) = \Phi^{s^{(j)}}(v)$, and introduce the Borel-measurable mappings

$$\begin{aligned} h^r \ni v &\mapsto \Phi = (\Phi_1, \Phi_2, \dots) \in S^1 \times S^1 \times \dots =: \mathcal{T}^\infty, \\ h^r \ni v &\mapsto (I \times \Phi) \in h_{I^+}^r \times \mathcal{T}^\infty. \end{aligned}$$

Note that the system Φ of resonant combinations is highly over-determined: there are many linear relations between its components Φ_j .

Let us pass in eq. (2.1) from the complex variables v_k to the action-angle variables I, φ :

$$dI_k(\tau) = (v_k \cdot P_k)(v) d\tau + b_k^2 d\tau + b_k(v_k \cdot d\beta^k) \quad (2.5)$$

(here \cdot indicates the real scalar product in $\mathbb{C} \simeq \mathbb{R}^2$), and

$$d\varphi_k(\tau) = \left(-\nu^{-1}\lambda_k + |v_k|^{-2}(iv_k \cdot P_k(v)) \right) d\tau + |v_k|^{-2}b_k(iv_k \cdot d\beta^k). \quad (2.6)$$

The equations for the actions are slow, while equations for the angles are fast since $d\varphi_k \sim \nu^{-1}$. But the resonant combinations Φ_j of angles satisfy slow equations:

$$d\Phi_j(\tau) = \sum_{k \geq 1} s_k^{(j)} \left(|v_k|^{-2} (iv_k \cdot P_k) d\tau + |v_k|^{-2} b_k (iv_k \cdot d\beta^k) \right), \quad j \geq 1. \quad (2.7)$$

Repeating for equations (2.1) and (2.5) the argument from Section 7 in [KP08] (also see Section 6.2 in [Kuk10]), we get low bounds for the norms of the components $v_k(\tau)$ of $v(\tau)$:

Lemma 2.2. *Let $v^\nu(\tau)$ be a solution of (2.1) and $I^\nu(\tau) = I(v^\nu(\tau))$. Then for any $k \geq 1$ the following convergence holds uniformly in $\nu > 0$:*

$$\int_0^T \mathbf{P}\{I_k^\nu(\tau) \leq \delta\} d\tau \rightarrow 0 \quad \text{as } \delta \rightarrow 0 \quad (2.8)$$

(the rate of the convergence depends on k).

Now we define and study corresponding resonant monomials of v . For any $s \in \mathbb{Z}_0^\infty$, vectors $s^+, s^- \in \mathbb{Z}_{+0}^\infty$ such that $s = s^+ - s^-$ and $\text{supp } s = \text{supp } s^+ \cup \text{supp } s^-$, $\text{supp } s^+ \cap \text{supp } s^- = \emptyset$ are uniquely defined. Denote by V^s the monomial

$$V^s(v) = v^{s^+} \bar{v}^{s^-} = \prod_l v_l^{s_l^+} \prod_l \bar{v}_l^{s_l^-}. \quad (2.9)$$

This is a real-analytic function on every space h^l , and $\varphi(V^s(v)) = \Phi^s(v)$. *Resonant monomials* are the functions ⁸

$$V_j(v) = V^{s^{(j)}}(v), \quad j = 1, 2, \dots$$

Clearly they satisfy

$$I(V_j(v)) = (2I)^{\frac{1}{2}|s^{(j)}|} := \prod_l (2I_l)^{\frac{1}{2}|s_l^{(j)}|}, \quad \varphi(V_j(v)) = \Phi_j(v). \quad (2.10)$$

Now consider the mapping

$$V : h^l \ni v \mapsto (V_1, V_2, \dots) \in \mathbb{C}^\infty, \quad (2.11)$$

where \mathbb{C}^∞ is given the Tikhonov topology. It is continuous for any l . For $N \geq 1$ denote

$$V^{(N)}(v) = (V_1, \dots, V_J(v)) \in \mathbb{C}^J,$$

where $J = J(N)$, see (2.4).

For any $s \in \mathbb{Z}_0^\infty$, applying the Ito formula to the process $V^s(v(\tau))$, we get that

$$dV^s = V^s \left(-i\nu^{-1} (\Lambda \cdot s) d\tau + \sum_{j \in \text{supp } s^+} s_j^+ v_j^{-1} (P_j(v) d\tau + b_j d\beta_j) + \sum_{j \in \text{supp } s^-} s_j^- \bar{v}_j^{-1} (\bar{P}_j(v) d\tau + b_j d\bar{\beta}_j) \right). \quad (2.12)$$

⁸It may be better to call $V_j(v)$ a minimal resonant monomial since for any $l \in \mathbb{Z}_{+0}^\infty$ the monomial $I^l V_j(v)$ is also resonant and corresponds to the same resonance.

If $s = \tilde{s} \in \mathbb{Z}_0^\infty$ is perpendicular to Λ , then the first term in the r.h.s. vanishes. So $V^{\tilde{s}}(\tau)$ is a slow process, $dV^{\tilde{s}} \sim 1$. In particular, the processes $dV_j, j \geq 1$, are slow.

Estimates (1.5) and equation (2.12) readily imply

Lemma 2.3. *For any $j \geq 1$ we have $\mathbf{E}|V_j(v(\cdot))|_{C^{1/3}[0,T]} \leq C_j(T) < \infty$, uniformly in $0 < \nu \leq 1$.*

Let us provide the space $C([0, T]; \mathbb{C}^\infty)$ with the Tikhonov topology, identifying it with the space $C([0, T]; \mathbb{C})^\infty$. This topology is metrisable by the Tikhonov distance. From now on we fix an even integer r ,

$$r \geq \frac{d}{2} + 1, \quad (2.13)$$

and abbreviate

$$h^r = h, \quad h_I^r = h_I, \quad C([0, T], h_{I+}) \times C([0, T], \mathbb{C}^\infty) =: \mathcal{H}_{I,V}.$$

We provide $\mathcal{H}_{I,V}$ with Tikhonov's distance, the corresponding Borel σ -algebra and the natural filtration of the sigma-algebras $\{\mathcal{F}_t, 0 \leq t \leq T\}$.

Let us consider a solution $u^\nu(\tau)$ of eq. (1.2), satisfying $u(0) = u_0$, denote $v^\nu(\tau) = \mathcal{F}(u^\nu(\tau))$ and abbreviate

$$I(v^\nu(\tau)) = I^\nu(\tau), \quad V(v^\nu(\tau)) = V^\nu(\tau) \in \mathbb{C}^\infty.$$

Lemma 2.4. *1) Assume that $u_0 \in \mathcal{H}^r$. Then the set of laws $\mathcal{D}(I^\nu(\cdot), V^\nu(\cdot))$, $0 < \nu \leq 1$, is tight in $\mathcal{H}_{I,V}$.*

2) Any limiting measure \mathcal{Q} for the set of laws in 1) satisfies

$$\begin{aligned} \mathbf{E}^{\mathcal{Q}}|I|_{C([0,T],h_I^r)}^n &\leq C_n \quad \forall n \in \mathbb{N}, & \mathbf{E}^{\mathcal{Q}} \int_0^T |I(\tau)|_{h_I^{r+1}} d\tau &\leq C', \\ \mathbf{E}^{\mathcal{Q}} e^{\varepsilon_0 |I(\tau)|_{h_I^0}} &\leq C'' \quad \forall \tau \in [0, T]. \end{aligned} \quad (2.14)$$

Proof. 1) Due to Lemma 2.3 and the Arzelà Theorem, the laws of processes $V_j(v^\nu(\cdot))$, $0 < \nu \leq 1$, are tight in $C([0, T], \mathbb{C})$, for any j . Due to estimates (1.5) with $n = 1$ and since the actions I_k^ν satisfy slow equations (2.5), the laws of processes $I^\nu(\tau)$ are tight in $C([0, T], h_{I+})$ (e.g. see in [VF88]). Therefore, for every N , any sequence $\nu_\ell \rightarrow 0$ contains a subsequence such that the laws $\mathcal{D}(I^\nu(\cdot), V^{(N)}(v^\nu(\cdot)))$ converges along it to a limit. Applying the diagonal process we get another subsequence ν'_ℓ such that the convergence holds for each N . The corresponding limit is a measure m^N on the space $C([0, T], h_{I+}) \times C([0, T], \mathbb{C})^{J(N)}$. Different measures m^N agree, so by Kolmogorov's theorem they correspond to some measure m on the sigma-algebra, generated by cylindric subsets of the space $C([0, T], h_{I+}) \times C([0, T], \mathbb{C})^\infty$, which coincides with the Borel sigma-algebra for that space. It is not hard to check that $\mathcal{D}(I^\nu(\cdot), V^\nu(\cdot)) \rightharpoonup m$ as $\nu = \nu'_\ell \rightarrow 0$. This proves the first assertion.

2) Estimates (2.14) follow from (1.3), (1.5), the weak convergence to \mathcal{Q} and the Fatou lemma; cf. Lemma 1.2.17 in [KS12]. \square

2.2 Averaged equations, effective equation, interaction representation

Fix $u_0 \in \mathcal{H}^r$ and consider any limiting measure \mathcal{Q}^0 for the laws

$$\mathcal{D}(I^{\nu_\ell}(\cdot), V^{\nu_\ell}(\cdot)) \rightarrow \mathcal{Q}^0 \quad \text{as } \nu_\ell \rightarrow 0, \quad (2.15)$$

existing by Lemma 2.4. Our goal is to show that the limit \mathcal{Q}^0 does not depend on the sequence $\nu_\ell \rightarrow 0$ and develop tools for its study. We begin with writing down averaged equations for the slow components I and Φ of the process $v(\tau)$, using the rules of the stochastic calculus (see [Kha68, FW03]), and formally replacing there the usual averaging in φ by the resonant averaging $\langle \cdot \rangle_\Lambda$. Let us first consider the I -equations (2.5). The drift in the k -th equation is

$$b_k^2 + v_k \cdot P_k = b_k^2 + v_k \cdot P_k^1 + v_k \cdot P_k^0,$$

where $v_k \cdot P_k^1 = -2\gamma_k I_k$ and $v_k \cdot P_k^0(v) = \sum_{p,q,l \in \mathbb{Z}_{+0}^\infty} v_k \cdot P_k^{0pql}(v)$, see (2.3). By Section 3 the sum converges normally, so the resonant averaging of the drift is well defined. The dispersion matrix for eq. (2.5) with respect to the real Wiener processes $(\beta^1, \beta^{-1}, \beta^2, \dots)$ is $\text{diag} \{b_k(\text{Re } v_k \text{ Im } v_k), k \geq 1\}$ (it is formed by 1×2 -blocks). The diffusion matrix equals the dispersion matrix times its conjugated and equals $\text{diag} \{b_k^2 |v_k|^2, k \geq 1\}$. It is independent from the angles, so the averaging does not change it. For its square-root we take $\text{diag} \{b_k \sqrt{2I_k}\}$, and accordingly write the Λ -averaged I -equations as

$$dI_k(\tau) = \langle v_k \cdot P_k \rangle_\Lambda(I, V) d\tau + b_k^2 d\tau + b_k \sqrt{2I_k} d\beta^k(\tau), \quad k \geq 1 \quad (2.16)$$

(see (1.21)).

Now consider equations (2.7) for resonant combinations Φ_j of the angles. The corresponding dispersion matrix $D = (D_{jk})$ is formed by 1×2 -blocks

$$D_{jk} = -s_k^{(j)} b_k (2I_k)^{-1} (\text{Im } v_k - \text{Re } v_k).$$

Again the diffusion matrix does not depend on the angles and equals $M = (M_{j_1 j_2})$, $M_{j_1 j_2} = \sum_k s_k^{(j_1)} s_k^{(j_2)} b_k^2 (2I_k)^{-1}$. The matrix D^{new} with the entries $D_{jk}^{new} = s_k^{(j)} b_k (2I_k)^{-1/2}$ satisfies $|D^{new}|^2 = M$, and we write the averaged equations for Φ_j 's as

$$d\Phi_j(\tau) = \sum_{k \geq 1} s_k^{(j)} \left(\frac{\langle i v_k \cdot P_k \rangle_\Lambda(I, V)}{2I_k} d\tau + \frac{b_k}{\sqrt{2I_k}} d\beta^{-k}(\tau) \right), \quad j \geq 1 \quad (2.17)$$

(we use here Wiener processes, independent from those in eq. (2.16) since the differentials $v_k \cdot d\beta^k$ and $i v_k \cdot d\beta^k$, corresponding to the noises in equations (2.5) and (2.6), are independent).

Equations (2.16), (2.17) is a system of stochastic differential equations for the process $(I, V)(\tau)$ since each Φ_j is a function of I and V_j . It is over-determined as there are linear relations between various Φ_j 's. Besides, eq. (2.16) has a weak singularity at the locus $\partial(h) = \cup_k \{v \in h : v_k = 0\}$, while eq. (2.17) has there a strong singularity.

Consider a component $\langle v_k \cdot P_k^0 \rangle_\Lambda(v)$ of the averaged drift in the equation for I_k . It may be written as

$$\langle v_k \cdot P_k^0 \rangle_\Lambda(v) = \int_0^{2\pi} v_k \cdot \left(e^{-it\lambda_k} P_k^0(\Psi_{t\Lambda}(v)) \right) dt = v_k \cdot R_k^0(v), \quad (2.18)$$

where we set $R_k^0(v) = \int_0^{2\pi} e^{-it\lambda_k} P_k^0(\Psi_{t\Lambda}(v)) dt$. That is,

$$R^0(v) = \int_0^{2\pi} \Psi_{-t\Lambda} P^0(\Psi_{t\Lambda} v) dt. \quad (2.19)$$

Repeating the derivation of (1.20) and using that $|q| + |l| \leq m - 1$, we see that

$$R_k^0(v) = \sum_{\substack{p,q,l \in \mathbb{Z}_{+0}^{\infty} \\ q-l \in \mathcal{A}(\Lambda, m) + e^k \\ |q| + |l| + 1 \leq m}} C_k^{pq} (2I)^p v^q \bar{v}^l. \quad (2.20)$$

The relation (2.20) interprets $R^0(v)$ as a sum of resonant terms of the mapping $P^0(v)$, very much in the spirit of the WT, while (2.19) interpret it a result of the resonant averaging of P^0 .

The vector field R^0 defines locally-Lipschitz operators in the spaces h^p , $p > d/2$:

$$|R^0(v) - R^0(w)|_{h^p} \leq C_p (|v|_{h^p} \vee |w|_{h^p})^{2q^*} |v - w|_{h^p}. \quad (2.21)$$

Indeed, in view of (2.19), for any v, w such that $|v|_{h^p}, |w|_{h^p} \leq R$ we have

$$|(R^0(v) - R^0(w))|_{h^p} \leq \int_0^{2\pi} \left| \Psi_{-t\Lambda} (P^0(\Psi_{t\Lambda} v) - P^0(\Psi_{t\Lambda} w)) \right|_{h^p} dt. \quad (2.22)$$

Since $P^0(v) = -i\rho\mathcal{F}(|\hat{v}|^{2q^*} \hat{v})$, where $\hat{v} = \mathcal{F}^{-1}v$, then denoting $\Psi_{t\Lambda} v = v_t$, defining w_t similarly and using that the operators Ψ_θ define isometries of h^p , we bound the r.h.s. of (2.22) by

$$\begin{aligned} \int_0^{2\pi} |P^0(v_t) - P^0(w_t)|_{h^p} dt &= \rho \int_0^{2\pi} \left\| |\hat{v}_t|^{2q^*} \hat{v}_t - |\hat{w}_t|^{2q^*} \hat{w}_t \right\|_p dt \\ &\leq \rho C_p R^{2q^*} \int_0^{2\pi} \|\hat{v}_t - \hat{w}_t\|_p dt \leq \rho C_p R^{2q^*} |v - w|_{h^p}. \end{aligned}$$

Finally we set

$$R = R^0 + R^1, \quad \text{where} \quad R_k^1(v) = P_k^1(v) = -\gamma_k v_k.$$

Since $\langle v_k \cdot P_k^1 \rangle_\Lambda = \langle -\sum 2\gamma_k I_k \rangle_\Lambda = v_k \cdot P_k^1 = v_k \cdot R_k^1$, then in view of (2.18) we have

$$\langle v_k \cdot P_k \rangle_\Lambda(v) = v_k \cdot R_k(v). \quad (2.23)$$

For further usage we note that by the same argument, $\langle iv_k \cdot P_k^0 \rangle_\Lambda = iv_k \cdot R_k^0$ and $\langle iv_k \cdot P_k^1 \rangle_\Lambda = 0 = iv_k \cdot R_k^1$. So also

$$\langle iv_k \cdot P_k \rangle_\Lambda(v) = iv_k \cdot R_k(v). \quad (2.24)$$

Motivated by the averaging theory for equations without resonances in [Kuk10, Kuk13], we now consider the following *effective equation* for the slow dynamics in eq. (2.5):

$$dv_k = R_k(v) d\tau + b_k d\beta^k, \quad k \geq 1. \quad (2.25)$$

In difference with the averaged equations (2.16) and (2.17), the effective equation is regular, i.e. it does not have singularities at the locus $\mathfrak{D}(h)$. Since $R^0 : h \rightarrow h$ is locally Lipschitz, then strong solutions for (2.25) exist locally in time and are unique:

Lemma 2.5. *A strong solution of eq. (2.25) with a specified initial data $v(0) = v_0 \in h$ is unique, a.s.*

The relevance of the effective equation for the study of the long-time dynamics in equations (1.2)=(2.1) is clear from the next lemma:

Lemma 2.6. *Let a continuous process $v(\tau) \in h$ be a weak solution of (2.25) such that all moments of the random variable $\max_{0 \leq \tau \leq T} |v(\tau)|_h$ are finite. Then $I(v(\tau))$ is a weak solution of (2.16). Let stopping times $0 \leq \tau_1 < \tau_2 \leq T$ and numbers $\delta_* > 0, N \in \mathbb{N}$ be such that*

$$I_k(v(\tau)) \geq \delta_* \quad \text{for } \tau_1 \leq \tau \leq \tau_2 \text{ and } k \leq N. \quad (2.26)$$

Then the process $(I(v(\tau)), \Phi_j(v(\tau)), j \leq J(N))$ is a weak solution of the system of averaged equations⁹ (2.16), (2.17) _{$j \leq J$} .

Proof. Let $v(\tau)$ satisfies (2.25). Applying Ito's formula to $I_k(v(\tau))$ and $\Phi_j(v(\tau)), j \leq J$, we get that

$$dI_k = v_k \cdot R_k d\tau + b_k^2 d\tau + b_k v_k \cdot d\beta^k \quad (2.27)$$

and

$$d\Phi_j = \sum_{k \in \text{supp } s^{(j)}} s_k^{(j)} \left(\frac{iv_k \cdot R_k}{|v_k|^2} d\tau + \frac{b_k}{|v_k|^2} iv_k \cdot d\beta^k \right).$$

Using (2.23) and (2.24) we see that (2.27) has the same drift and diffusion as (2.16). So $I(v(\tau))$ is a weak solution of (2.16) (see [Yor74, MR99]). Similar, for $\tau \in [\tau_1, \tau_2]$, in view of (2.24), the process $(I, \Phi_j, j \leq J)$, is a weak solution of the system (2.16), (2.17) _{$j \leq J$} . \square

Now we show that the effective equation describes the limiting (as $\nu \rightarrow 0$) dynamics for the equations of motions, written in the a -variables of the interaction representation (0.16). Indeed, let $u^\nu(\tau)$ be a solution of eq. (1.2), satisfying $u(0) = u_0$. Denote $v^\nu(\tau) = \mathcal{F}(u^\nu(\tau))$ and consider the vector of a -variables $a^\nu(\tau) = (a_k^\nu(\tau) = e^{i\nu^{-1}\lambda_k\tau} v_k^\nu(\tau), k \geq 1)$ (cf. (0.16)). Notice that we obviously have

$$|v^\nu(\tau)|_{h^m} \equiv |a^\nu(\tau)|_{h^m} \quad \forall m, \quad I(v^\nu(\tau)) \equiv I(a^\nu(\tau)), \quad V(v^\nu(\tau)) \equiv V(a^\nu(\tau)) \quad (2.28)$$

(see (2.11)). From (2.1) we obtain the following system of equations for the vector $a^\nu(\tau)$:

$$da_k^\nu = (R_k(a^\nu) + \mathcal{R}_k(a^\nu, \nu^{-1}\tau)) d\tau + b_k e^{i\nu^{-1}\lambda_k\tau} d\beta^k(\tau), \quad k \geq 1,$$

where we have denoted

$$\mathcal{R}_k(a, \nu^{-1}\tau) = \sum_{\substack{p, q, l \in \mathbb{Z}_{+0}^\infty \\ q-l-e^k \notin \mathcal{A}(\Lambda, m) \\ |q|+|l|+1 \leq m}} P_k^{0pql}(a) \exp\left(-i\nu^{-1}\tau (\Lambda \cdot (q-l-e^k))\right). \quad (2.29)$$

This is the nonresonant, fast oscillating part of the nonlinearity (because $|\Lambda \cdot (q-l-e^k)| \geq 1$). Since $\{\beta^k(\tau) := \int e^{i\nu^{-1}\lambda_k\tau} d\beta^k(\tau), k \geq 1\}$ is another set of standard independent complex Wiener processes, then the process $a^\nu(\tau)$ is a weak solution of the system of equations

$$da_k^\nu = (R_k(a^\nu) + \mathcal{R}_k(a^\nu, \nu^{-1}\tau)) d\tau + b_k d\beta^k(\tau), \quad k \geq 1. \quad (2.30)$$

⁹This system is heavily under-determined.

We will refer to equations (2.30) as to the *a-equations*. It is crucial that they are identical to the effective equation (2.25), apart from terms which oscillate fast as $\nu \rightarrow 0$.

2.3 Properties of resonant Hamiltonian \mathcal{H}^{res} and effective equation

Lemma 2.7. *The vector field R^0 is hamiltonian:*

$$R^0 = i\rho\nabla\mathcal{H}^{\text{res}}(v), \quad \forall v \in h^p, \quad p > d/2, \quad (2.31)$$

where $\mathcal{H}^{\text{res}}(v) = \langle \mathcal{H} \rangle_{\Lambda}(v)$ and \mathcal{H} is the Hamiltonian (0.2).

Proof. Indeed, since $P^0(v) = i\rho\nabla\mathcal{H}(v)$, then

$$R^0(v) = \int_0^{2\pi} \Psi_{-t\Lambda} \left(i\rho\nabla\mathcal{H}(\Psi_{t\Lambda}(v)) \right) dt = i\rho\nabla_v \int_0^{2\pi} \mathcal{H}(\Psi_{t\Lambda}(v)) dt = i\rho\nabla_v \mathcal{H}^{\text{res}}(v),$$

as $\Psi_{\theta}^* \equiv \Psi_{-\theta}$, and where we used (2.19). \square

Clearly $\mathcal{H}^{\text{res}}(0) = 0$. Since $\mathcal{H}(u) \leq -C\|u\|_0^{2q_*+2}$ by the Hölder inequality and since the transformations $\Psi_{t\Lambda}$ preserve $\|u\|_0$, then

$$\mathcal{H}^{\text{res}}(u) \leq -C\|u\|_0^{2q_*+2} \quad \forall u.$$

The resonant Hamiltonian \mathcal{H}^{res} has symmetries, given by some rotations $\Psi_m, m \in \mathbb{R}^\infty$:

Lemma 2.8. *i) Let $\mathbf{1} = (1, 1, \dots)$. Then $\mathcal{H}^{\text{res}}(\Psi_{t\mathbf{1}}v) = \text{const}$ (i.e., it does not depend on t);
ii) Let \mathcal{M}^l the l -th component of the sequence $(\mathbf{k}(1), \mathbf{k}(2), \dots)$, $l = 1, \dots, d$ (see (1.1)). Then $\mathcal{H}^{\text{res}}(\Psi_{t\mathcal{M}^l}v) = \text{const}$, for each l .
iii) $\mathcal{H}^{\text{res}}(\Psi_{t\Lambda}v) = \text{const}$.*

Proof. i) By (1.19) we have

$$\mathcal{H}^{\text{res}}(\Psi_{t\mathbf{1}}v) = \int_0^{2\pi} \mathcal{H}(\Psi_{t'\Lambda}(\Psi_{t\mathbf{1}}v)) dt' = \int_0^{2\pi} \mathcal{H}(\Psi_{t\mathbf{1}}(\Psi_{t'\Lambda}v)) dt'.$$

Let us denote $\Psi_{t\mathbf{1}}(\Psi_{t'\Lambda}v) = v(t; t')$. Then $(d/dt)v(t; t') = iv$. The flow of this hamiltonian equation commutes with that of the equation with the Hamiltonian \mathcal{H} .¹⁰ So $\mathcal{H}(v(t; t'))$ is independent from t for each t' , and i) follows since $\mathcal{H}^{\text{res}}(\Psi_{t\mathbf{1}}v) = \int \mathcal{H}(v(t; t')) dt'$.

ii) Proof is the same since the transformations $\Psi_{t\mathcal{M}^l}, t \in \mathbb{R}$, are the flow of the momentum Hamiltonian $M^l(u) = \frac{1}{2} \sum_{j=1}^{\infty} \mathbf{k}^l(j) |u_j|^2$, which commutes with \mathcal{H} .

iii) It is a straightforward consequence of the definition (1.19) of the resonant averaging. \square

Since the transformations $\Psi_{t\mathbf{1}}$ form the flow of the Hamiltonian $H_0(v) = \frac{1}{2} \sum |v_j|^2 = \frac{1}{2} |v|_{h^0}^2$, the transformations $\Psi_{t\Lambda}$ – the flow of $H_1(v) = \frac{1}{2} \sum \lambda_j |v_j|^2$, and the transformations $\Psi_{t\mathcal{M}^l}, t \in \mathbb{R}$ – the flow of the momentum Hamiltonian, we may recast the assertions of the last lemma as follows:

$$\{\mathcal{H}^{\text{res}}, H_0\} = 0, \quad \{\mathcal{H}^{\text{res}}, H_1\} = 0, \quad \{\mathcal{H}^{\text{res}}, M^l\} = 0 \quad \forall l. \quad (2.32)$$

¹⁰This follows from the fact that the functional $\frac{1}{2}|v|_{h^0}^2$ is an integral of motion for the Hamiltonian \mathcal{H} , which becomes obvious if we note that in the u -representation \mathcal{H} has the form (0.2) and $\frac{1}{2}|v|_{h^0}^2$ is $\frac{1}{2} \int |u|^2(x) dx$.

Here $\{\cdot, \cdot\}$ signifies the Poisson bracket. As the transformations $\Psi_m, m \in \mathbb{R}^\infty$, are symplectic, then the symmetries in the lemma above preserve the hamiltonian vector field R^0 and commute with it. In particular, since $\Psi_{t\Lambda} = e^{-it\Delta}$, then the spectral spaces E_λ of the operator $-\Delta$,

$$E_\lambda = \text{span} \{e^j : \lambda_j = \lambda\},$$

are invariant for the flow-maps of R^0 .

Since the transformations $\Psi_m, m \in \mathbb{R}^\infty$, obviously preserve the vector field R^1 as well as the law of the random force in (2.25) (see the proof of the lemma below), then those Ψ_m which are symmetries of R^0 (equivalently, which are symmetries of the Hamiltonian \mathcal{H}^{res}), preserve weak solutions of (2.25). So we have:

Lemma 2.9. *If $v(\tau)$ is a solution of equation (2.25) and $m \in \mathbb{R}^\infty$ be either a vector $m = t\mathbf{1}, t \in \mathbb{R}$, or a vector $m = t\Lambda$, or $m = t\mathcal{M}^l, l = 1, \dots, d$, then $\Psi_m v(\tau)$ also is a weak solution.*

Proof. Denote $\Psi_m v(\tau) = v'(\tau)$. Applying Ψ_m to eq. (2.25), using Lemma 2.8 and exploiting the invariance of the operator R^1 with respect to Ψ_m , we get

$$dv'_k = (\Psi_m R(v(\tau)))_k d\tau + e^{im_k} b_k d\beta^k = (R(v'(\tau)))_k + b_k (e^{im_k} d\beta^k).$$

Since $\{e^{im_k} \beta^k(\tau), k \geq 1\}$ is another set of standard independent Wiener processes, then $v'(\tau)$ is a weak solution of (2.25). \square

Corollary 2.10. *If μ is a stationary measure for equation (2.25) and a vector m is as in Lemma 2.9, then the measure $\Psi_m \circ \mu$ also is stationary.*

The next lemma characterises the increments of $R^0(v)$ in the space h^0 . It will be needed below to study the ergodic properties of the effective equation:

Lemma 2.11. *Let $p > d/2$. Then for any $v_1, v_2 \in h^p$ we have*

$$|R^0(v) - R^0(w)|_{h^0} \leq C(|v|_{h^p} + |w|_{h^p})^{2q^*} |v - w|_{h^0}.$$

Proof. Repeating the proof of the Lipschitz property of R^0 in the space h (see (2.21)) and using the notation of that proof, i.e. denoting $\Psi_{t\Lambda} v = v_t, \hat{v} = \mathcal{F}^{-1}v$, and similar for the vector w , we get that

$$\begin{aligned} |R^0(v) - R^0(w)|_{h^0} &\leq \int_0^{2\pi} \left| \Psi_{-t\Lambda} (P^0(\Psi_{t\Lambda} v) - P^0(\Psi_{t\Lambda} w)) \right|_{h^0} dt \\ &= \int_0^{2\pi} |P^0(v_t) - P^0(w_t)|_{h^0} dt = \int_0^{2\pi} \left\| |\hat{v}_t|^{2q^*} \hat{v}_t - |\hat{w}_t|^{2q^*} \hat{w}_t \right\|_0 dt \\ &\leq C \int_0^{2\pi} (|\hat{v}_t|_{L^\infty} + |\hat{w}_t|_{L^\infty})^{2q^*} \|\hat{v}_t - \hat{w}_t\|_0 dt \leq C_1 (|v|_{h^p} + |w|_{h^p})^{2q^*} |v - w|_{h^0}. \end{aligned}$$

\square

3 Explicit calculation

We intend here to calculate explicitly the effective equation (2.25), keeping track of the dependence on the size L of the torus. To do that, it is convenient to use the natural parametrisation of the exponential basis by vectors $\mathbf{k} \in \mathbb{Z}_L^d$; that is, decompose functions $u(x)$ to Fourier series, $u(x) = \sum_{\mathbf{k} \in \mathbb{Z}_L^d} v_{\mathbf{k}} e^{i\mathbf{k} \cdot x}$. We modify the norms $|\cdot|_{h^p}$ accordingly :

$$\|u\|_p^2 = (2\pi L)^d \sum_{\mathbf{k} \in \mathbb{Z}_L^d} \left(|\mathbf{k}| \vee \frac{1}{L} \right)^{2p} |v_{\mathbf{k}}|^2 =: |v|_{h^p}^2 .$$

Now, as in the Introduction, the eigenvalues of the minus-Laplacian are $\lambda_{\mathbf{k}} = |\mathbf{k}|^2$ and the damping coefficients $\gamma_{\mathbf{k}} = f(\lambda_{\mathbf{k}})$.

In the v -coordinates the nonlinearity becomes the mapping $v \mapsto P^0(v)$, whose \mathbf{k} -th component is

$$P_{\mathbf{k}}^0(v) = -i\rho \sum_{\mathbf{k}_1, \dots, \mathbf{k}_{2q_*+1} \in \mathbb{Z}_L^d} v_{\mathbf{k}_1} \cdots v_{\mathbf{k}_{q_*+1}} \bar{v}_{\mathbf{k}_{q_*+2}} \cdots \bar{v}_{\mathbf{k}_{2q_*+1}} \delta_{q_*+2 \dots 2q_*+1}^{1 \dots q_*+1} \mathbf{k}$$

(see (0.8)). Accordingly,

$$v_{\mathbf{k}} \cdot P_{\mathbf{k}}^0 = \rho \sum_{\mathbf{k}_1, \dots, \mathbf{k}_{2q_*+1} \in \mathbb{Z}_L^d} \text{Im} (v_{\mathbf{k}_1} \cdots v_{\mathbf{k}_{q_*+1}} \bar{v}_{\mathbf{k}_{q_*+2}} \cdots \bar{v}_{\mathbf{k}_{2q_*+1}} \bar{v}_{\mathbf{k}}) \delta_{q_*+2 \dots 2q_*+1}^{1 \dots q_*+1} \mathbf{k} . \quad (3.1)$$

In order to calculate the resonant average, we first notice that $v_{\mathbf{k}} \cdot P_{\mathbf{k}}^0$ can be written as a series (1.17), where $|C_{pql}| \leq 1$ and $|q| + |p| + |l| = 2q_* + 2$. In this case the sum in the l.h.s. of (1.18) is bounded by

$$C \left(\sum_{\mathbf{k} \in \mathbb{Z}_L^d} |v_{\mathbf{k}}| \right)^{2q_*+2} \leq C_1(L) |v|_p^{q_*+1} \left(\sum_{\mathbf{k} \in \mathbb{Z}_L^d} |\mathbf{k}|^{-2p} \right)^{q_*+1} .$$

So the condition (1.18) is met if $2p > d$.

Since the order of the resonance $m = 2q_* + 2$, then $\langle v_{\mathbf{k}} \cdot P_{\mathbf{k}}^0 \rangle_{\Lambda}(v)$ equals

$$\rho \sum_{\mathbf{k}_1, \dots, \mathbf{k}_{2q_*+1} \in \mathbb{Z}_L^d} \text{Im} (v_{\mathbf{k}_1} \cdots v_{\mathbf{k}_{q_*+1}} \bar{v}_{\mathbf{k}_{q_*+2}} \cdots \bar{v}_{\mathbf{k}_{2q_*+1}} \bar{v}_{\mathbf{k}}) \delta_{q_*+2 \dots 2q_*+1}^{1 \dots q_*+1} \mathbf{k} \delta(\lambda_{q_*+2 \dots 2q_*+1}^{1 \dots q_*+1} \mathbf{k}) ,$$

(see (0.21)). This follows from (3.1) and (1.19) if one notes that appearing there restriction $(q-l) \cdot \Lambda = 0$ is now replaced by the factor $\delta(\lambda_{q_*+2 \dots 2q_*+1}^{1 \dots q_*+1} \mathbf{k})$. In a similar way, we see that the quantity $R_{\mathbf{k}}^0$, entering equation (2.25), takes the form

$$R_{\mathbf{k}}^0(v) = -i\rho \sum_{\mathbf{k}_1, \dots, \mathbf{k}_{2q_*+1} \in \mathbb{Z}_L^d} v_{\mathbf{k}_1} \cdots v_{\mathbf{k}_{q_*+1}} \bar{v}_{\mathbf{k}_{q_*+2}} \cdots \bar{v}_{\mathbf{k}_{2q_*+1}} \delta_{q_*+2 \dots 2q_*+1}^{1 \dots q_*+1} \mathbf{k} \delta(\lambda_{q_*+2 \dots 2q_*+1}^{1 \dots q_*+1} \mathbf{k}) .$$

Taking into account that $R_{\mathbf{k}}^1 = -\gamma_{\mathbf{k}} v_{\mathbf{k}}$, we finally arrive at an explicit formula for the effective equation (2.25):

$$\begin{aligned} dv_{\mathbf{k}} = & \left(-\gamma_{\mathbf{k}} v_{\mathbf{k}} \right. \\ & \left. - i\rho \sum_{\mathbf{k}_1, \dots, \mathbf{k}_{2q_*+1} \in \mathbb{Z}_L^d} v_{\mathbf{k}_1} \cdots v_{\mathbf{k}_{q_*+1}} \bar{v}_{\mathbf{k}_{q_*+2}} \cdots \bar{v}_{\mathbf{k}_{2q_*+1}} \delta_{q_*+2 \dots 2q_*+1}^{1 \dots q_*+1} \mathbf{k} \delta(\lambda_{q_*+2 \dots 2q_*+1}^{1 \dots q_*+1} \mathbf{k}) \right) d\tau \\ & + b_{\mathbf{k}} d\beta^{\mathbf{k}} , \quad \mathbf{k} \in \mathbb{Z}_L^d . \end{aligned} \quad (3.2)$$

Due to (2.31),

$$R_k^0(v) = i\rho \nabla_{v_k} \mathcal{H}^{\text{res}}(v) = 2i\rho \frac{\partial}{\partial \bar{v}_k} \mathcal{H}^{\text{res}}(v).$$

Therefore eq. (3.2) can be written as the damped–driven hamiltonian system (0.18).

Examples. a) If $q_* = 1$, then (3.2) reads

$$dv_{\mathbf{k}} = \left(-\gamma_{\mathbf{k}} v_{\mathbf{k}} - i\rho \sum_{\mathbf{k}, \mathbf{k}', \mathbf{k}'' \in \mathbb{Z}_L^d} v_{\mathbf{k}} v_{\mathbf{k}'} \bar{v}_{\mathbf{k}''} \delta_{\mathbf{k}+\mathbf{k}', \mathbf{k}''+r} \delta_{\lambda_{\mathbf{k}}+\lambda_{\mathbf{k}'}, \lambda_{\mathbf{k}''}+\lambda_{\mathbf{k}}} \right) d\tau + b_{\mathbf{k}} d\beta^{\mathbf{k}},$$

where $\mathbf{k} \in \mathbb{Z}_L^d$. If $f(t) = t + 1$, then this equation looks similar to the CGL equation

$$\dot{u} - \Delta u + u = i|u|^2 u + \frac{d}{d\tau} \sum b_{\mathbf{k}} \beta^{\mathbf{k}}(\tau) e^{i\mathbf{k} \cdot x},$$

written in the Fourier coefficients. The latter equation possesses nice analytical properties; e.g. its stationary measures is unique for any d , see [KN13].

b) Our results remain true if the Hamiltonian \mathcal{H} , corresponding to the nonlinearity in (0.5), has variable coefficients. In particular, let $d = 1$ and the nonlinearity in (0.5) is replaced by $-ip(x)|u|^2 u$ with a sufficiently smooth function $p(x)$. Then the effective equation is

$$dv_k = \left(-\gamma_k v_k - i \sum_{k_1, k_2, k_3, k_4 \in \mathbb{Z}_L} v_{k_1} v_{k_2} \bar{v}_{k_3} p_{k_4} \delta_{k_1+k_2+k_4, k_3+k} \delta_{k_1^2+k_2^2, k_3^2+k^2} \right) d\tau + b_k d\beta^k,$$

where $k_L \in \mathbb{Z}_d$ and p_k 's are the Fourier coefficients of $p(x)$.

4 Main results

4.1 Averaging theorem for the initial-value problem.

We recall that r is a fixed even integer such that $r \geq \frac{d}{2} + 1$, and abbreviate

$$h^r = h, \quad C([0, T], h) = \mathcal{H}_a.$$

We provide \mathcal{H}_a with the Borel σ -algebra and the natural filtration of the sigma-algebras $\{\mathcal{F}_t, 0 \leq t \leq T\}$.

Let $v^\nu(\tau)$ be a solution of (2.1) such that $v^\nu(0) = v_0 = \mathcal{F}(u_0) \in h^r$, consider the corresponding process $a^\nu(\tau)$. Due to (2.28), the process a^ν satisfies obvious analogies of the estimates (1.3), (1.5) and (1.6). Since $(R + \mathcal{R})(a)$ is the nonlinearity $P(v)$, written in the a -variables, then

$$|(R + \mathcal{R})(a)(\tau)|_h = |P(v)(\tau)|_h \leq C|v(\tau)|_h^{q_*+1} = C|a(\tau)|_h^{q_*+1}.$$

Therefore all moments of $|(R + \mathcal{R})(a)|_{\mathcal{H}_a}$ are finite, and we get from eq. (2.30) that $\mathbf{E}|a^\nu|_{C^{1/3}([0, T], h)} \leq \bar{C}$, uniformly in ν . Now arguing as when proving Lemma 2.4 we get that the set of laws $\mathcal{D}(a^\nu(\cdot))$, $0 < \nu \leq 1$, is tight in \mathcal{H}_a . Consider any limiting measure, corresponding to the laws $\mathcal{D}(a^\nu(\cdot))$:

$$\mathcal{D}(a^{\nu_\ell}(\cdot)) \rightharpoonup \mathcal{Q}_a^0 \quad \text{as } \nu_\ell \rightarrow 0. \quad (4.1)$$

Theorem 4.1. *There exists a unique weak solution $a(\tau)$ of effective equation (2.25), satisfying $a(0) = v_0$ a.s. The law of $a(\cdot)$ in the space \mathcal{H}_a coincides with \mathcal{Q}_a^0 . The convergence (4.1) holds as $\nu \rightarrow 0$.*

The proof of the theorem is presented at the end of this section.

Let \mathcal{Q}^0 be a measure in $\mathcal{H}_{I,V}$ as in (2.15). Since $(I, V)(v^\nu(\cdot)) = (I, V)(a^\nu(\cdot))$ for any $\nu > 0$ then re-denoting $a(\tau)$ by $v(\tau)$ we derive a corollary from the previous theorem:

Theorem 4.2. *There exists a unique weak solution $v(\tau)$ of effective equation (2.25), satisfying $v(0) = v_0$ a.s. The law of $(I, V)(v(\cdot))$ in the space $\mathcal{H}_{I,V}$ coincides with \mathcal{Q}^0 and the convergence (2.15) holds as $\nu \rightarrow 0$. Moreover, for any vectors $\tilde{s}_1, \dots, \tilde{s}_m \in \mathbb{Z}_0^\infty$, perpendicular to Λ , we have the convergence*

$$\mathcal{D}(I, V^{\tilde{s}_1}, \dots, V^{\tilde{s}_m})(v^\nu(\cdot)) \rightharpoonup \mathcal{D}(I, V^{\tilde{s}_1}, \dots, V^{\tilde{s}_m})(v(\cdot)).$$

By this result the Cauchy problem for the effective equation has a weak solution. Using Lemma 2.5 and the Yamada-Watanabe argument (see [KS91, Yor74, MR99]) we get that the equation is well posed:

Corollary 4.3. *For any $v_0 \in h^r$, eq. (2.25) has a unique strong and a unique weak solution $v(\tau)$ such that $v(0) = v_0$. Its law satisfies (2.14).*

Now consider $\varphi(v^\nu(\tau)) \cdot \tilde{s} = \varphi(V^{\tilde{s}}(v^\nu(\tau))) \in S^1$. Since $\varphi(V)$ is a discontinuous function of $V \in \mathbb{C}$, then to pass to a limit as $\nu \rightarrow 0$ we do the following. We identify S^1 with $\{v \in \mathbb{R}^2 : |v| = 1\}$, denote $[\tilde{s}] = N$, and approximate the discontinuous function $V^N = (V_1, \dots, V_N) \mapsto \varphi(V^{\tilde{s}})$ by continuous functions

$$V^N \mapsto f_\delta([I(V^N)]) \varphi(V^{\tilde{s}}) \in \mathbb{R}^2, \quad [I] = \min_{1 \leq k \leq N} I_k, \quad 0 < \delta \ll 1.$$

where f_δ is continuous, $0 \leq f_\delta \leq 1$, $f_\delta(t) = 0$ for $t \leq \delta/2$ and $f_\delta = 1$ for $t \geq \delta$.

For any measure μ_τ in a complete metric space, which weakly continuously depends on τ , and any $\tau_1 < \tau_2$ we will denote

$$\langle \mu_\tau \rangle_{\tau_1}^{\tau_2} = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \mu_\tau d\tau.$$

Then the argument above jointly with Lemma 2.2 imply:

Corollary 4.4. *Let $\tilde{s} \in \mathbb{Z}_0^\infty$ be any non-zero vector, orthogonal to Λ , and let $0 \leq \tau_1 < \tau_2 \leq T$. Then*

$$\langle \mathcal{D}(\varphi(u^\nu(\tau)) \cdot \tilde{s}) \rangle_{\tau_1}^{\tau_2} \rightharpoonup \langle \mathcal{D}(\varphi(v(\tau)) \cdot \tilde{s}) \rangle_{\tau_1}^{\tau_2} \quad \text{as } \nu \rightarrow 0.$$

On the contrary, if $s \cdot \Lambda \neq 0$, then by Proposition 4.10 we get that

$$\langle \mathcal{D}(\varphi(u^\nu(\tau)) \cdot s) \rangle_{\tau_1}^{\tau_2} \rightharpoonup \mathfrak{d}\varphi.$$

More generally, if vectors $\tilde{s}_1, \dots, \tilde{s}_M$ from \mathbb{Z}_0^∞ are perpendicular to Λ and a vector s is not, then

$$\langle \mathcal{D}(I, \varphi \cdot \tilde{s}_1, \dots, \varphi \cdot \tilde{s}_M, \varphi \cdot s)(u^\nu(\tau)) \rangle_{\tau_1}^{\tau_2} \rightharpoonup \langle \mathcal{D}(I, \varphi \cdot \tilde{s}_1, \dots, \varphi \cdot \tilde{s}_M)(v(\tau)) \rangle_{\tau_1}^{\tau_2} \times \mathfrak{d}\varphi.$$

We do not know an equivalent description of the measure \mathcal{Q}^0 only in terms of the slow variables (I, V) of equation (2.1). But the following result holds true:

Proposition 4.5. *Consider the natural process on the space $\mathcal{H}_{I,V}$ with the measure \mathcal{Q}^0 . If for some $N \in \mathbb{N}$ and $\delta_* > 0$, stopping times $0 \leq \tau_1 < \tau_2 \leq T$ satisfy (2.26), then for $\tau \in [\tau_1, \tau_2]$ the process $(I, \Phi^{(N)})(I, V)(\tau)$ is a weak solution of the averaged equations (2.16) and (2.17) $|_{j \leq J}$. Here $\Phi^{(N)} = (\Phi_1, \dots, \Phi_{J(N)})$.*

Since the averaged quantities $\langle v_k \cdot P_k \rangle_\Lambda$ and $\langle iv_k \cdot P_k \rangle_\Lambda$ are functions of I and Φ (see (1.21)), then equations (2.16) and (2.17) $|_{j \leq J}$ form an under-determined system of equations for the variables (I, Φ) .

Proof of Theorem 4.1 The proof follows the Khasminski scheme (see [Kha68, FW03, KP08]). Its crucial step is given by the following lemma:

Lemma 4.6. *For any $k \geq 1$ one has*

$$\mathfrak{A}_k^\nu := \mathbf{E} \max_{0 \leq \tau \leq T} \left| \int_0^\tau \mathcal{R}_k(a^\nu(s), \nu^{-1}s) ds \right| \rightarrow 0 \quad \text{as } \nu \rightarrow 0. \quad (4.2)$$

The lemma is proved below in Section 4.4, following the arguments in [KP08, Kuk13]. Now we derive from it the theorem.

For $\tau \in [0, T]$ consider the processes

$$N_k^{\nu_l} = a_k^{\nu_l}(\tau) - \int_0^\tau R_k(a^{\nu_l}(s)) ds, \quad k \geq 1.$$

Due to (2.30) we can write $N_k^{\nu_l}$ as

$$N_k^{\nu_l}(\tau) = \tilde{N}_k^{\nu_l}(\tau) + \bar{N}_k^{\nu_l}(\tau),$$

where $\tilde{N}_k^{\nu_l}(\tau) = a_k^{\nu_l}(\tau) - \int_0^\tau (R_k(a^{\nu_l}(s)) + \mathcal{R}_k(a^{\nu_l}(s), \nu_l^{-1}s)) ds$ is a \mathcal{Q}_a^0 martingale and the disparity $\bar{N}_k^{\nu_l}$ is

$$\bar{N}_k^{\nu_l}(\tau) = \int_0^\tau \mathcal{R}_k(a^{\nu_l}(s), \nu_l^{-1}s) ds.$$

The convergence $\mathcal{D}(a^{\nu_l}) \rightarrow \mathcal{Q}_a^0$ and Lemma 4.6 imply that the processes

$$N_k(\tau) = a_k(\tau) - \int_0^\tau R_k(a) ds, \quad k \geq 1,$$

are \mathcal{Q}_a^0 martingales (see for details [KP08], Proposition 6.3).

Similar to (4.2), we find that

$$\mathbf{E} \max_{0 \leq \tau \leq T} \left| \int_0^\tau \mathcal{R}_k(a^\nu(s), \nu^{-1}s) ds \right|^2 \rightarrow 0 \quad \text{as } \nu \rightarrow 0.$$

Then, using the same arguments as before, we see that the processes $N_{k_1}(\tau)N_{k_2}(\tau) - \int_0^\tau A_{k_1 k_2} ds$ are \mathcal{Q}_a^0 martingales, where $A_{k_1 k_2}$ denotes the diffusion matrix for the system (2.25). That is, \mathcal{Q}_a^0 is a solution of the martingale problem with drift R_k and the diffusion A . Hence, \mathcal{Q}_a^0 is a law of a weak solution of eq. (2.25). Such a solution exists for any $v_0 \in h$. So by Lemma 2.5 and the Yamada -Watanabe argument (see [KS91, Yor74, MR99]), weak and strong solutions for (2.25) both exist and are unique. Hence, the limit in (2.15) does not depend on the sequence $\nu_l \rightarrow 0$, the convergence holds as $\nu \rightarrow 0$, and the theorem is proved. \square

4.2 Averaging theorem for stationary solutions.

Let $v^\nu(\tau)$ be a stationary solution of eq. (2.1) as at the end of Section 1.1.¹¹ Solutions v^ν inherit the a-priori estimates (1.3), (1.5), (1.6), so still the set of laws $\mathcal{D}(I(v^\nu(\cdot)), V(v^\nu(\cdot)))$, $0 < \nu \leq 1$, is tight in $\mathcal{H}_{I,V}$ (cf. Lemma 2.4). Consider any limit

$$\mathcal{D}(I(v^{\nu_\ell}(\cdot)), V(v^{\nu_\ell}(\cdot))) \rightharpoonup \mathcal{Q} \quad \text{as } \nu_\ell \rightarrow 0. \quad (4.3)$$

As before, the measure \mathcal{Q} satisfies (2.14) (with the constants C_n, C', C'' , corresponding to $v_0 = 0$). Moreover, it is stationary in τ .

Theorem 4.7. *There exists a stationary solution $v(\tau)$ of the effective equation (2.25) such that $\mathcal{Q} = \mathcal{D}(I(v(\cdot)), V(v(\cdot)))$.*

Proof. Denote $\mu^\nu = \mathcal{D}v^\nu(\tau)$. Estimate (1.5) with $2m = r$ and $n = 1$ implies that $\int |v|_{h^{r+1}}^2 \mu^\nu(dv) \leq C$ for all ν . So the set of measures μ^ν is tight in \mathcal{H}^r . Replacing, if necessary, the sequence $\{\nu_\ell\}$ by a subsequence, we achieve that

$$\mu^{\nu_\ell} \rightharpoonup \mu^0 \quad \text{as } \nu_\ell \rightarrow 0. \quad (4.4)$$

Clearly $(I, V) \circ \mu^0$ is the marginal distribution for \mathcal{Q} as $\tau = \text{const}$, which we will denote q (i.e., $q = \mathcal{Q} |_{\tau=\text{const}}$).

Let $v^0(\tau), \tau \geq 0$, be a solution for the effective equation (2.25) such that $\mathcal{D}v^0(0) = \mu^0$ (existing by Corollary 4.3 and the estimates on μ^0). Then, for the same reason as in Section 4.1,

$$\mathcal{D}(I, V)(v^0(\tau)) |_{\tau \in [0, T]} = \mathcal{Q},$$

and $\mathcal{D}(I, V)(v^0(\tau)) \equiv q$. We do not know if the solution v^0 is stationary, but from the Bogolyubov-Krylov argument we know that for a suitable sequence $T_j \rightarrow \infty$ we have the convergence

$$\frac{1}{T_j} \int_0^{T_j} \mathcal{D}(v^0(\tau)) d\tau \rightharpoonup m^0,$$

where m^0 is a stationary measure for (2.25). Still we have that $(I, V) \circ m^0 = q$, and the measure m^0 satisfies the same apriori estimates as before. Let $v(\tau)$ be a solution for (2.25) such that $\mathcal{D}v(0) = m^0$. It is stationary and $\mathcal{D}(I, V)(v(\tau)) \equiv q$. Modifying a bit the argument above we get that also $\mathcal{D}(I, V)(v(\cdot)) = \mathcal{Q}$. \square

Writing the convergence (4.3) as $\mathcal{D}(I, V)(v^{\nu_\ell}(\cdot)) \rightharpoonup \mathcal{D}(I, V)(v(\cdot))$, we note that, as in Section 4.1, we also have that

$$\mathcal{D}(I, V^{\bar{s}_1}, \dots, V^{\bar{s}_m})(v^{\nu_\ell}(\tau)) \rightharpoonup \mathcal{D}(I, V^{\bar{s}_1}, \dots, V^{\bar{s}_m})(v(\tau)) = (I, V^{\bar{s}_1}, \dots, V^{\bar{s}_m}) \circ m^0$$

as $\nu_\ell \rightarrow 0$, for any m and any vectors $\bar{s}_1, \dots, \bar{s}_m$, perpendicular to Λ . Since for stationary solutions $v^\nu(\tau)$ we have $\langle \mathcal{D}(v^\nu(\tau)) \rangle_{T_1}^{T_2} = \mathcal{D}(v^\nu(\tau))$, then arguing as when proving Corollary 4.4 we also get that

$$\mathcal{D}(I, \Phi^{\bar{s}_1}, \dots, \Phi^{\bar{s}_m})(v^{\nu_\ell}(\tau)) \rightharpoonup (I, \Phi^{\bar{s}_1}, \dots, \Phi^{\bar{s}_m}) \circ m^0. \quad (4.5)$$

¹¹Under certain restrictions on the equation it is known that its law (i.e. the stationary measure of the equation) is unique, e.g., see [Shi06]. We will not discuss this now.

Moreover if $s \in \mathbb{Z}_0^\infty$ is such that $s \cdot \Lambda \neq 0$, then in view of Proposition 4.10 and the stationarity of the solutions we have

$$\mathcal{D}(I, \Phi^{\bar{s}_1}, \dots, \Phi^{\bar{s}_m}, \Phi^s)(v^{\nu_l}(\tau)) \rightharpoonup ((I, \Phi^{\bar{s}_1}, \dots, \Phi^{\bar{s}_m}) \circ m^0) \times d\theta. \quad (4.6)$$

If eq. (2.25) has a unique stationary measure m^0 , then the convergences above hold as $\nu \rightarrow 0$. But in this case a stronger assertion holds:

Theorem 4.8. *Let v^ν be a stationary solution of equation (2.1), $\mathcal{D}(v^\nu(\tau)) \equiv \mu^\nu$, and assume that the effective equation (2.25) has a unique stationary measure m^0 . Then*

$$\mu^\nu \rightharpoonup m^0 \quad \text{as } \nu \rightarrow 0. \quad (4.7)$$

Proof. i) Consider again the convergence (4.4). We are going to show that the limiting measure μ^0 equals m^0 . Then the limit in (4.4) does not depend on the sequence $\{\nu_l \rightarrow 0\}$, so it holds as $\nu \rightarrow 0$, and (4.7) follows.

ii) Due to Lemma 2.2, $\mu^\nu(\varnothing) = 0 = \mu^0(\varnothing)$, so we may regard μ^ν and μ^0 as measures on $h_I^r \times \mathbb{T}^\infty$. Let us fix any $n \in \mathbb{N}$ and consider measures μ^{ν^n} , μ^{0^n} and m^{0^n} which are images of the measures μ^ν , μ^0 and m^0 under the projection

$$\Pi^n : v \mapsto v^n.$$

We will regard them as measures on $\mathbb{R}_+^n \times \mathbb{T}^n = \{(I^n, \varphi^n)\}$. To prove that $\mu^0 = m^0$ it suffices to verify that $\mu^{0^n} = m^{0^n}$ for each n .

Let us denote $\mathcal{A}(\Lambda^n) =: \mathcal{A}^n$, and let the vectors $\zeta^1, \dots, \zeta^n \in \mathbb{Z}^n$ and the unimodular matrix R be as in Lemma 1.1 with $\mathcal{A} = \mathcal{A}^n$. Let $L = L_{\mathcal{A}^n} : \mathbb{T}^n \rightarrow \mathbb{T}^{n-1}$ be the operator in (1.9), i.e.

$$L : \mathbb{T}^n \ni \varphi^n \mapsto (\varphi^n \cdot \zeta^1, \dots, \varphi^n \cdot \zeta^{n-1})^T \in \mathbb{T}^{n-1}. \quad (4.8)$$

Writing $R^T(\varphi^n) = (y_1, \dots, y_n)^T = (\mathbf{y}, y_n)^T$, where $\mathbf{y} = (y_1, \dots, y_{n-1})^T$, we have $L(\varphi^n) = \mathbf{y}$. We will denote by π^1 the natural projection $y \mapsto \mathbf{y}$.

For further purposes we make the following observation. Let μ be a Borel measure on h . Consider its images under rotations $\Psi_{t\Lambda}$ and projections Π^n . In the (I, φ) -variables the mapping $\Psi_{t\Lambda}$ becomes $\text{id} \times (\cdot + t\Lambda)$, so

$$\Pi^n \circ (\Psi_{t\Lambda} \circ \mu) = (\text{id} \times (\cdot + t\Lambda^n)) \circ \Pi^n \circ \mu$$

(where $\Pi^n \circ \mu$ is written in the (I^n, φ^n) -variables). By (1.8) the transformation R^T of \mathbb{T}^n conjugates the translation by the vector $t\Lambda^n$ with the translation by $t e^n$. Therefore,

$$\mathcal{R}^T \circ \Pi^n \circ (\Psi_{t\Lambda} \circ \mu) = (\text{id} \times (\cdot + t e^n)) \circ \mathcal{R}^T \circ \Pi^n \circ \mu, \quad (4.9)$$

where $\mathcal{R} = \text{id} \times R^T$.

iii) Let us apply to the measures μ^{ν^n} , μ^{0^n} , m^{0^n} the transformation \mathcal{R}^T :

$$N^{\nu^n} = \mathcal{R}^T \circ \mu^{\nu^n}, \quad N^{0^n} = \mathcal{R}^T \circ \mu^{0^n}, \quad M^{0^n} = \mathcal{R}^T \circ m^{0^n}. \quad (4.10)$$

Recall that by (4.4), $N^{\nu_l^n} \rightharpoonup N^{0^n}$ as $\nu_l \rightarrow 0$. Our first goal is to calculate the limiting measure N^{0^n} . To do this let us disintegrate N^{ν^n} and N^{0^n} with respect to the mapping

$$\text{id} \times \pi^1 : \mathbb{R}_+^n \times \mathbb{T}^n \rightarrow \mathbb{R}_+^n \times \mathbb{T}^{n-1}, \quad (I^n, (\mathbf{y}, y_n)^T) \mapsto (I^n, \mathbf{y}).$$

That is (see [Dud02], Section 10.2), write them as

$$N^{\nu n} = N_{I^n, \mathbf{y}}^{\nu n}(dy_n) p^{\nu n}(dI^n d\mathbf{y}), \quad N^{0n} = N_{I^n, \mathbf{y}}^{0n}(dy_n) p^{0n}(dI^n d\mathbf{y}),$$

where $p^{\nu n} = (\text{id} \times \pi^1) \circ N^{\nu n}$ and $p^{0n} = (\text{id} \times \pi^1) \circ N^{0n}$. Since $\mathbf{y} = L(\varphi^n)$, then $p^{\nu n} = \mathcal{D}(I^n \times (L \circ \varphi^n))(v^{\nu n}(\tau))$. As each vector ζ^j in (4.8) is perpendicular to Λ^n , then in view of (4.5) we have

$$p^{0n} = \lim_{\nu_i \rightarrow 0} \mathcal{D}(I^n \times (L \circ \varphi^n))(v^{\nu_i n}(\tau)) = (I^n \times (L \circ \varphi^n)) \circ m^{0n}. \quad (4.11)$$

To calculate N^{0n} it remains to find the fiber-measures $N_{I^n, \mathbf{y}}^{0n}$. To do this let us take any bounded continuous function f on $\mathbb{R}_+^n \times \mathbb{T}^{n-1} \times S^1$ and consider $\langle N^{\nu n}, f \rangle = \mathbf{E}f(I^n, \mathbf{y}, y_n)(v^{\nu}(\tau))$. Since $\mathbf{y}(v) = L(\varphi^n)$ and $y_n(v) = v \cdot \eta^n$, where the vector η^n is not perpendicular to Λ , then by (4.6)

$$\langle N^{\nu n}, f \rangle \rightarrow \int f(I^n, \mathbf{y}, y_n) \left((I^n \times (L \circ \varphi^n)) m^{0n} \right) (dI^n d\mathbf{y}) d\bar{y}_n.$$

From other hand, by (4.4)

$$\langle N^{\nu_i n}, f \rangle \rightarrow \langle N^{0n}, f \rangle = \int f(I^n, \mathbf{y}, y_n) N_{I^n, \mathbf{y}}^{0n}(dy_n) p^{0n}(dI^n d\mathbf{y}).$$

Since $p^{0n} = (I^n \times (L \circ \varphi^n)) \circ m^{0n}$, then we get from the two convergences above that for p^{0n} -a.a. pairs (I^n, \mathbf{y}) we have $N_{I^n, \mathbf{y}}^{0n} = d\bar{y}_n$. Accordingly,

$$N^{0n} = d\bar{y}_n \times p^{0n}(dI^n d\mathbf{y}).$$

iv) Consider the measure M^{0n} . Due to (4.11) its disintegration with respect to the mapping $\text{id} \times \pi^1$ may be written as

$$M^{0n} = M_{I^n, \mathbf{y}}^{0n}(dy_n) p^{0n}(dI^n d\mathbf{y}) \quad (4.12)$$

with some unknown fiber-measures $M_{I^n, \mathbf{y}}^{0n}$. Now consider the rotated measure $\Psi_{t\Lambda} \circ m^0$, $t \geq 0$, and its n -dimensional projection. By (4.9),

$$\mathcal{R}^T \circ \Pi^n \circ \Psi_{t\Lambda} \circ m^0 = (\text{id} \times l_t) \circ \mathcal{R}^T \circ m^{0n},$$

where $l_t(\mathbf{y}, y_n) = (\mathbf{y}, y_n + t)$. Due to (4.10) and (4.12), the measure in the r.h.s. equals

$$M_{I^n, \mathbf{y}}^{0n}(dy_n + t) p^{0n}(dI^n d\mathbf{y}).$$

But by Corollary 2.10, the measure in the l.h.s. does not depend on t . So $M_{I^n, \mathbf{y}}^{0n}(dy_n) \equiv M_{I^n, \mathbf{y}}^{0n}(dy_n + t)$ is a translation-invariant measure on S^1 , and it must be equal to $d\bar{y}_n$. Accordingly,

$$M^{0n} = d\bar{y}_n \times p^{0n}(dI^n d\mathbf{y}) = N^{0n}.$$

v) We have established that $N^{\nu_i n} \rightarrow M^{0n}$ as $\nu_i \rightarrow 0$. So $N^{\nu_i n} \rightarrow m^{0n}$, which completes the proof. \square

4.3 Mixing in the effective equations

We start with the case when the function $f(\lambda)$ has a linear growth. For simplicity of notation we suppose that $f(\lambda) = \lambda + 1$. We also are forced to assume that $q_* = 1$.

The effective equation (2.25)=(0.18) with $q_* = 1$ looks similar to the equation (0.6) $_{\nu=\infty, q_*=1}$, studied in [KN13]. It turns out that the two equations indeed are similar, at least for $d \leq 3$, and that the proof of the mixing in Section 4 of [KN13], based on an abstract theorem from [KS12], applies to (2.25) with minimal changes. Indeed, the crucial step in [KN13] in order to apply the result from [KS12] is to establish for solutions of the equation the exponential estimate of the form

$$\mathbf{P}\left\{\sup_{t \geq 0} \left(\int_0^t |u(s)|_{L^\infty}^2 ds - Kt \right) \geq \sigma\right\} \leq C' \exp(c_1 |u_0|_{L^\infty}^2 - c_2 \sigma), \quad \forall \sigma > 0, \quad (4.13)$$

with suitable constants K, C', c_1 and c_2 . This estimate is important to study the mixing since it allows to control divergence of trajectories $u_1(t)$ and $u_2(t)$, corresponding to the same realisation of the random force, through the inequality¹²

$$|u_1(t) - u_2(t)|_{L_2} \leq |u_1(0) - u_2(0)|_{L_2} \exp\left(C \int_0^t (|u_1(s)|_{L^\infty}^2 + |u_2(s)|_{L^\infty}^2) ds\right). \quad (4.14)$$

For eq. (2.25) an analogy of (4.13) follows by applying the Ito formula to $[v]_1^2 = H_0(v) + H_1(v)$ (see (2.32)), since due to (2.32) we have that

$$d[v(\tau)]_1^2 + 2 \int_0^\tau [v(s)]_2^2 ds = 4\tau \mathcal{B} + 2 \sum_{j=1}^{\infty} (\lambda_j + 1) (v_j(\tau) \cdot d\mathcal{B}^j(\tau)),$$

where we denote $[v]_2^2 = \sum (\lambda_j + 1)^2 |v_j|^2$ and $\mathcal{B} = \sum (\lambda_j + 1) b_j^2$. Applying to this relation the supermartingale inequality in the standard way (e.g., see in [KN13, KS12]), we get that

$$\mathbf{P}\left\{\sup_{\tau \geq 0} \left(\int_0^\tau [v(s)]_2^2 ds - 2\mathcal{B}\tau \right) \geq \sigma\right\} \leq C' \exp(c_1 |v_0|_1^2 - c_2 \sigma), \quad \forall \sigma > 0.$$

If $d \leq 3$, then by Lemma 2.11 the divergence of two solutions for (2.25) with the same ω satisfies

$$|v_1(\tau) - v_2(\tau)|_{h^0} \leq |v_1(0) - v_2(0)|_{h^0} \exp\left(C \int_0^\tau ([v_1(s)]_2^2 + [v_2(s)]_2^2) ds\right).$$

This last two estimates allow to repeat literally for equation (2.25) the reduction to Theorem 3.1.3 from [KS12], made in [KN13], and prove

Theorem 4.9. *Let $q_* = 1$, $f(\lambda) = \lambda + 1$ and $d \leq 3$. Then the effective equation (2.25) has a unique stationary measure μ and is mixing. That is, every its solution $v(\tau)$ satisfies $\mathcal{D}(v(\tau)) \rightarrow \mu$ as $\tau \rightarrow \infty$.*

The presented proof uses that the nonlinearity in the effective equation is at most cubic. It also applies to the effective equations for eq. (0.6), where the Hamiltonian \mathcal{H} is one of the

¹²To match (4.13) and (4.14) we use crucially that $q_* \leq 1$.

two functions \mathcal{H}^3 with cubic densities as at the end of Section 0.1 (in this case the argument works if $d \leq 6$). The proof without changes applies to equation (0.6), where $q_* = 1$, $d \leq 3$ and $f(\lambda)$ grows super-linearly. The argument also may be adjusted to the case when $q_* = 1$, d is any and $f(\lambda) = c_1 + \lambda^{c_d}$, where c_d is sufficiently big. Based on the similarity with the equation (0.6) $_{\nu=\infty, q_*=1}$, studied in [KN13] for any space-dimension, we conjecture that for $q_* = 1$ and $f(\lambda) = \lambda + 1$ the effective equation is well-posed and mixing for any d . But it is unknown how to prove the mixing for equations with $q_* \geq 2$ (in any space-dimension).

4.4 Proof of Lemma 4.6

For this proof we adopt a notation from [KP08]. Namely, we denote by $\varkappa(t)$ various functions of t such that $\varkappa \rightarrow 0$ as $t \rightarrow \infty$, and denote by $\varkappa_\infty(t)$ functions, satisfying $\varkappa(t) = o(t^{-N})$ for each N . We write $\varkappa(t, M)$ to indicate that $\varkappa(t)$ depends on a parameter M . Besides for events Q and O and a random variable f we write $\mathbf{P}_O(Q) = \mathbf{P}(O \cap Q)$ and $\mathbf{E}_O(f) = \mathbf{E}(\chi_O f)$. Below M stands for a suitable function of ν such that $M(\nu) \rightarrow \infty$ as $\nu \rightarrow 0$, but

$$\nu M^n \rightarrow 0 \quad \text{as } \nu \rightarrow 0, \quad \forall n.$$

Denote by $\Omega_M = \Omega_M^\nu$ the event $\Omega_M = \{\sup_{0 \leq \tau \leq T} |a^\nu(\tau)|_{h^r} \leq M\}$. Then, by (1.6), $\mathbf{P}(\Omega_M^c) \leq \varkappa_\infty(M)$ uniformly in ν , so that one has $\mathfrak{A}_k^\nu \leq \varkappa_\infty(M) + \mathfrak{A}_{k,M}^\nu$, where we have defined

$$\mathfrak{A}_{k,M}^\nu := \mathbf{E}_{\Omega_M} \max_{0 \leq \tau \leq T} \left| \int_0^\tau \mathcal{R}_k(a^\nu(s), \nu^{-1}s) ds \right|. \quad (4.15)$$

So it remains to estimate $\mathfrak{A}_{k,M}^\nu$.

Consider a partition of $[0, T]$ by the points

$$\tau_n = nL, \quad 0 \leq n \leq K \sim T/L.$$

where τ_K is the last point τ_n in $[0, T]$. The diameter L of the partition is $L = \sqrt{\nu}$. Denoting

$$\eta_l = \int_{\tau_l}^{\tau_{l+1}} \mathcal{R}_k(a^\nu(s), \nu^{-1}s) ds, \quad 0 \leq l \leq K-1, \quad (4.16)$$

we see that

$$\mathfrak{A}_{k,M}^\nu \leq LC(M) + \mathbf{E}_{\Omega_M} \sum_{l=0}^{K-1} |\eta_l|, \quad (4.17)$$

since for $\omega \in \Omega_M$ the integrand in (4.16) is smaller than a suitable $C(M)$ (see Lemma 2.1 and (2.21)). For any l let us consider the event

$$\mathcal{F}_l = \left\{ \sup_{\tau_l \leq \tau \leq \tau_{l+1}} |a^\nu(\tau) - a^\nu(\tau_l)|_h \geq P_1(M)L^{1/3} \right\},$$

where $P_1(M)$ is a suitable polynomial. It is not hard to verify using the Doob inequality that for a suitable choice of P_1 the probability of $\mathbf{P}(\mathcal{F}_l)$ is less than $\varkappa_\infty(L^{-1}; M)$ (cf. [KP08]). One gets

$$\sum_{l=0}^{K-1} |\mathbf{E}_{\Omega_M} |\eta_l| - \mathbf{E}_{\Omega_M \setminus \mathcal{F}_l} |\eta_l| | \leq C(M)L \sum_{l=0}^{K-1} \mathbf{P}(\mathcal{F}_l) \leq C(M)\varkappa_\infty(L^{-1}; M), \quad (4.18)$$

so that it remains to estimate $\sum \mathbf{E}_{\Omega_M \setminus \mathcal{F}_l} |\eta_l|$.

We have

$$|\eta_l| \leq \left| \int_{\tau_l}^{\tau_{l+1}} (\mathcal{R}_k(a^\nu(s), \nu^{-1}s) - \mathcal{R}_k(a^\nu(\tau_l), \nu^{-1}s)) ds \right| \\ + \left| \int_{\tau_l}^{\tau_{l+1}} (\mathcal{R}_k(a^\nu(\tau_l), \nu^{-1}s)) ds \right| =: \Upsilon_l^1 + \Upsilon_l^2 .$$

By the regularity of the integrand and the definition of \mathcal{F}_l

$$\sum_l \mathbf{E}_{\Omega_M \setminus \mathcal{F}_l} \Upsilon_l^1 \leq \varkappa(L^{-1/3}; M) = \varkappa(\nu^{-1/6}; M) . \quad (4.19)$$

So it remains to estimate the expectation of $\sum \Upsilon_l^2$. Denoting $t = \nu\tau$ and making use of (2.29) we write Υ_l^2 as

$$\Upsilon_l^2 = L \left| \frac{\nu}{L} \int_0^{\nu^{-1}L} \sum_{\substack{p,q,l \in \mathbb{Z}_{+0}^\infty \\ q-l-e^k \notin \mathcal{A}(\Lambda, m) \\ |q|+|l|+1 \leq m}} P_k^{0pql}(a) \exp(-it(\Lambda \cdot (q-l-e^k))) dt \right| \\ \leq LC(M) \frac{\nu}{L} \sup_{\substack{p,q,l \in \mathbb{Z}_{+0}^\infty \\ q-l-e^k \notin \mathcal{A}(\Lambda, m) \\ |q|+|l|+1 \leq m}} \frac{1}{\Lambda \cdot (q-l-e^k)} \leq L\varkappa(\nu^{-1}L; M) ,$$

because the supremum in the second line is bounded by one, since both Λ and $q-l-e^k$ are integer vectors. Therefore

$$\sum_l \mathbf{E}_{\Omega_M \setminus \mathcal{F}_l} \Upsilon_l^2 \leq \varkappa(\nu^{-1/2}; M) . \quad (4.20)$$

Now (4.15), (4.17), (4.18), (4.19) and (4.20) imply that

$$\mathfrak{A}_k^\nu \leq \varkappa_\infty(M) + \varkappa(\nu^{-1/2}; M) + \varkappa_\infty(\nu^{-1}; M) + \varkappa(\nu^{-1/6}; M) + \varkappa(\nu^{-1/2}; M) .$$

Choosing first M large and then ν small, we make the r.h.s. above arbitrarily small. This proves the lemma. \square

An argument similar to the previous one (see Proposition 4.7 of [KM13b]) implies the following assertion:

Proposition 4.10. *Let $s \in \mathbb{Z}_0^\infty$ be such that $s \cdot \Lambda \neq 0$ and $G : \mathbb{R}_+^M \times \mathbb{T}^{J(M)} \times S^1 \rightarrow \mathbb{R}$ be a bounded Lipschitz-continuous function, for some $M \geq 1$. Then*

$$\mathfrak{B}^\nu := \mathbf{E} \max_{0 \leq \tau \leq T} \left| \int_0^\tau \left(G(I^{\nu M}(l), \Phi^{\nu(M)}(l), s \cdot \varphi^\nu(l)) - \int_{S^1} G(I^{\nu M}(l), \Phi^{\nu(M)}(l), \theta) d\theta \right) dl \right| \rightarrow 0 \quad \text{as } \nu \rightarrow 0 .$$

In particular, taking for G Lipschitz functions on S^1 we get that $\langle \mathcal{D}(s \cdot \varphi^\nu(l)) \rangle_0^t \rightarrow d\theta$ as $\nu \rightarrow 0$, for any $t > 0$.

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