Spectral Properties of Solutions for Nonlinear PDEs in the Turbulent Regime

Sergei B. Kuksin

Heriot-Watt University, Edinburgh
and
Steklov Mathematical Institute, Moscow

In *GAFA 9 (1999), 141-184.*

Abstract

We consider non-linear Schrödinger equations with small complex coefficient of size δ in front of the Laplacian. The space-variable belongs to the unit n-cube ($n \leq 3$) and Dirichlet boundary conditions are assumed on the cube's boundary. The equations are studied in the turbulent regime which means that $\delta \ll 1$ and supremum-norms of the solutions we consider are at least of order one. We prove that space-scales of the solutions are bounded from below and from above by some finite positive degrees of δ and show that this result implies non-trivial restrictions on spectra of the solutions, related to the Kolmogorov–Obukhov five-thirds law (these restrictions are less specific than the 5/3-law, but they apply to a much wider class of solutions). Our approach is rather general and is applicable to many other nonlinear PDEs in the turbulent regime. Unfortunately, it does not apply to the Navier–Stokes equations.

Introduction

Classical theory of turbulence studies properties of velocity field of incompressible fluid with high Reynolds numbers. The motion of the fluid can be free or forced. In the first case the velocity slowly decays with time and its characteristics should be studied while it remains of the same order of magnitude (see [Kol1 , F]). In the second case it is usually assumed that the forcing is a space-time dependent random field, stationary in time and smooth in space [Kol2, F]. Of the main interest for the theory of turbulence are averaged spectral characteristics of the velocity field, where in the second case one averages in ensemble and in the first case – in time (possibly, also in random initial data). Assuming that the flow is space-periodic with period 2 (this assumption is not very popular among physicists, but it simplifies the mathematics, so we accept it), we write the velocity field $u(t,x)$ as Fourier series, $u(t,x) = \sum_{s\in\mathbb{Z}^n} \hat{u}_s(t) e^{\pi i s\cdot x}$. We denote by $E_s = \langle |\hat{u}_s|^2 \rangle$ the squared norm of a Fourier coefficient, averaged in time and (or) in ensemble (so E_s is the energy corresponding to a wave vector $s \in \mathbb{Z}^n$), and denote

$$\mathcal{E}_r = \frac{1}{2C} \sum_{r-C \le |s| \le r+C} E_s \tag{1}$$

(without specifying how big is the constant $C > 1$). Roughly, $\mathcal{E}_r$ is the energy corresponding to a wave number r (i.e., to all wave vectors s from the sphere $\{|s| = r\}$), so $\mathcal{E}_{|s|} \sim E_s |s|^{n-1}$. The function $\mathcal{E}_r$ represents the energy spectrum of the flow $u(t, x)$, cf. [LL] and [F], p.53.

One of the most famous predictions of the Kolmogorov theory of turbulence [Kol1, Kol2] is the Kolmogorov–Obukhov five-thirds law (see [LL, §33] and [F]) which claims that $\mathcal{E}_r$ behaves as $const \cdot r^{-5/3}$ for the wave numbers r from the *inertial range* $r_0 < r < r_1$ and decays faster than any negative degree of r for $r \ge r_1$ (the law does not say much about the *energy range* $0 < r < r_0$ and we do not discuss it in this work), see Fig. 1.

Figure 1

The threshold r_1 is of order $R^{3/4}$, where R is the Reynolds number of the flow. Its inverse $\lambda_1 = r_1^{-1} \sim R^{-3/4}$ is called Kolmogorov's inner scale of the turbulent flow.

The Kolmogorov–Obukhov law is a heuristic result. Our goal in this work is to prove some theorems which — in a sense — imply that dependence of $\mathcal{E}_r$ on r is of the same nature as it is shown in Fig. 1.

Classically the fluid flow (which hosts the turbulence) is described by the Navier–Stokes (NS) equations. Unfortunately, these equations are very difficult for mathematical analysis. In this work we replace them by nonlinear Schrödinger (NLS) equations (cf. section 8.1 where we discuss other PDEs). These equations, first, model NS equations and, second, describe turbulence which occur in some less classical physical situations [Z, LO, MMT]. * We study both non-forced and forced NLS equations:

$$-i\dot{u} = -\delta\nu\Delta u + |u|^{2p} u\,, \quad p \in \mathbb{N}\,, \tag{2}$$

$$-i\dot{u} = -\delta\nu\Delta u + |u|^2 u + \zeta^\omega(t,x)\,. \tag{3}$$

In the equations δ is a small real parameter, $0 < \delta \ll 1$, and ν is a unit complex number with non-negative real and imaginary parts. Usually we shall supplement the equations by odd 2-periodic boundary conditions:

$$\begin{aligned} u(t,x) &= u(t,x_1,\dots,x_j+2,\dots,x_n) \\ &= -u(t,x_1,\dots,-x_j,\dots,x_n),\ j=1,\dots,n\,, \end{aligned} \tag{4}$$

where $n \leq 3$. Clearly, any function which satisfies (4) vanishes at the boundary of the cube K^n of half-periods, $K^n = \{0 \leq x_j \leq 1\}$.

We study the equations in the turbulent regime: the parameter $\delta > 0$ is very small and supremum-norms of the solutions we consider are at least of order one. To satisfy the second restriction we study solutions of the non-forced equation (2) with smooth initial data of order one,

$$u(0,x) = u^0(x) \sim 1\,, \tag{5}$$

while $|u|$ remain of the same order, and study equation (3) with an order one forcing ζ (stationary in t, smooth and odd periodic in x).

Solutions which we consider are complex functions $u = u_\delta(t,x)$ (or random fields $u = u_\delta^\omega(t,x)$), smooth in x and depending on the small parameter δ. Critical for their analysis is a notion of the space-scale of a function $f_\delta(x)$ or $f_\delta(t,x)$, which we introduce in section 1 and study in sections 2,3. For a smooth in x function $f_\delta(x)$ we define its space-scale $\ell_x(f)$ as $\ell_x(f) = \delta^\gamma$ where $\gamma = \gamma(f)$ is supremum over all real numbers $\tilde{\gamma}$ such that

$$|f_\delta|_{C^m} \equiv \max_{|\alpha|\leq m} \sup_x |\partial_x^\alpha f_\delta(x)| \geq \delta^{-\tilde{\gamma} m}\,. \tag{6}$$

The inequality is assumed to hold for all m bigger than some m_0 (which depends on $\tilde{\gamma}$ and f) and for all $\delta < \delta_{\tilde{\gamma} m}$. The space-scale ℓ_x possesses many natural properties (see section 1) and seems to be a new notion, cf. [CDT] and [BGO] for definitions of other space-scales related to the problem of turbulence.

If a function f is time-dependent i.e., $f = f_\delta(t,x)$ where $t \in L = [T_1,T_2]$ ($T_1 \geq -\infty$ and $T_2 \leq +\infty$ may depend on δ), then the time should be incorporated into a definition.

* For turbulence in NLS equations relations similar to the 5/3-law are known as Kolmogorov's asymptotics [Z, MMT]. We do not present them here since, first, they are not yet as received as the 5/3-law is and, second, usually they are attributed to the so-called weak turbulence which we do not touch in this paper.

We define the space-scale $\ell_x^L\big(f_\delta(t,x)\big)$ as $\ell_x^L = \delta^{\gamma_L}$, where $\gamma_L(f)$ is supremum over all real numbers $\tilde\gamma$ such that the estimate

$$\frac{1}{|L'|}\int_{L'} |f_\delta(t,\cdot)|^2_{C^m}\,dt \ge \delta^{-2\tilde\gamma m} \tag{7}$$

holds for all sufficiently big m, all sufficiently long segments $L' \subset [T_1, T_2]$ and for $\delta < \delta_{\tilde\gamma m}$.

If $f = f_\delta^\omega(t,x)$ is a random field, then we define its averaged space-scale $\ell_x^{L,\mathrm{E}}(f)$ as $\ell_x^{L,\mathrm{E}} = \delta^{\gamma_L^{\mathrm{E}}}$, where γ_L^{E} is supremum over all real numbers $\tilde\gamma$ such that

$$\frac{1}{|L'|}\int_{L'} \mathbf{E}\,|f_\delta(t,\cdot)|^2_{C^m}\,dt \ge \delta^{-2\tilde\gamma m}\,, \tag{8}$$

for m, L' and δ as above.

We call a function $f_\delta(x)$ *short-scale* if $0 < \gamma(f) < \infty$. Short-scale time-dependent functions and random fields are defined similarly.

The space-scales ℓ_x, ℓ_x^L and $\ell_x^{L,\mathrm{E}}$ are norm-independent: in section 1 we show that they will not change if in (6) – (8) we replace the C^m-norm by, say, the norm of the Sobolev space $H^m(K^n;\mathbb{C})$. What is important is the number of derivatives, not the norm we use to measure them.

In sections 2, 3 we study spectral properties of short-scale functions $f_\delta(x)$ and $f_\delta(t,x)$ (in Appendix 2 – of random fields $f_\delta^\omega(t,x)$). Roughly, we show that any short-scale function $f_\delta(x) = \sum \hat f_{\delta s} e^{\pi i s\cdot x}$ is mostly supported by the Fourier modes $\hat f_{\delta s} e^{\pi i s\cdot x}$ with $|s| \lesssim \ell_x(f)^{-1}$ but not by the modes with $|s| \ll \ell_x(f)^{-1}$ and that most part of the Sobolev H^m-norm of this function with a sufficiently big m is carried by the modes with $|s| \sim \ell_x(f)^{-1}$.

After these preliminaries we pass in section 4 to problem (2), (5) and study its solution $u_\delta(t,x)$ for $0 \le t \le \delta^{-b}$ with any $b \ge 1/3$. Here our main result is

Theorem A. If oscillation of the function $\big|u^0(x)\big|$ on the cube $K^n = \{0 \le x_j \le 1\}$ is at least one and $u(t,x)$ is a solution of (2), (5), then

$$\sup_{0\le t\le \delta^{-1/3}} |u_\delta(t,\cdot)|_{C^m(K^n)} \ge C_m \delta^{-m\kappa} \quad \forall\,\delta \in (0,1)\,,$$

for any $m \ge 2$ and any $\kappa < 1/3$.

(Cf. Appendix 3, where we treat this result in terms of a dynamical system defined by equation (2) in a function space of odd periodic functions.)

We stress that in Theorem A no boundary conditions for u_δ on ∂K^n are assumed. In particular, u_δ may be the restriction to $[0,\infty)\times K^n$ of any function which solves equation (2) in the half-space $[0,\infty)\times\mathbb{R}^n$, see section 8.2.

Next we show that if u_δ is an odd periodic solution (i.e. it solves the problem (2), (4), (5)), then for $m \ge 4$ $u_\delta(t,x)$ satisfies estimate (7) where $L' = [0,\delta^{-b}]$, $b \ge 1/3$, and

$\gamma \to 1/3$ when $m \to \infty$. This implies an upper bound to the space-scale of any solution u_δ of (2), (4), (5): $\ell_x^L(u_\delta) \le \delta^{1/3}$.

Theorem A and the upper bound for the space-scales of solutions are non-linear results. Indeed, if in equation (2) $p = 0$, then for any solution $u_{0\delta}(t,x)$ for the linear problem (2), (4), (5) with smooth $u^0(x)$ we have $|u_{0\delta}(t,\cdot)|_m \le C_m \delta^{-1}$ and $\ell_x^L(u_{0\delta}) \ge \delta^0 = 1$.

To get a lower bound for the space-scale $\ell_x^L(u_\delta)$, an upper estimate for solutions is needed. In order to obtain a good bound we assume that the coefficient in front of the Laplacian in (2) is pure imaginary, so the equation takes the form

$$\dot{u} = \delta \Delta u + i|u|^{2p} u \ . \tag{9}$$

We prove that solutions of (9) satisfy the maximum principle and that for any solution u_δ of (9), (4), (5) we have $\|u_\delta(t,\cdot)\|_m \le C_m \delta^{-m/2}$ $(m \ge 1)$. Accordingly, $\ell_x^L(u_\delta) \ge \delta^{1/2}$ and

$$\delta^{1/2} \le \ell_x^L(u_\delta) = \delta^{\gamma_L} \le \delta^{1/3} \ , \tag{10}$$

if $L = [0, \delta^{-b}]$, $b \ge 1/3$. We stress that the estimates (10) do not depend on the dimension $n \le 3$ and the degree $2p+1$ of the nonlinearity.

The estimates (10) can be easily rescaled to the case when $u(t,x)$ is an odd $2M$-periodic solution of (9) such that $\operatorname{osc} u(0,x) \sim U$, where M and U are degrees of δ, see (4.15) and section 8.2.

We do not know how sharp the estimates (10) are; in particular, we do not know if the exponent $\gamma_L(u_\delta)$ equals to a universal u^0-independent constant from the segment $[1/3, 1/2]$. The first estimate in (10) is natural: it well agrees with scaling of equation (9) (since $\delta \Delta u \sim |u|^{2p} u \sim 1$, then two differentiations increase a solution u by the factor δ^{-1} and m of them — by the factor $\delta^{-m/2}$), as well as with WKB-type results for this equation (see [JLM]). We have no intuitive arguments which would suggest that the second estimate in (10) is sharp.

Next in section 4 we use (10) to study the spectrum of a solution u_δ, using general theorems from sections 2, 3. We write u_δ as Fourier series $\sum \hat{u}_{\delta s}(t) e^{\pi i s \cdot x}$ and consider the quantities E_s,

$$E_s = \delta^b \int_0^{\delta^{-b}} |\hat{u}_{\delta s}(t)|^2 \, dt \ ,$$

$b \ge 1/3$. We prove that E_s decays faster than any negative degree of $|s|$ for $|s| > \delta^{-1/2}$ and behaves like δ to a *finite* degree being averaged along the layer $\{\delta^{-1/3} \le |s| \le \delta^{-1/2}\}$, or the layer $\{\delta^{-\gamma_L+\varepsilon} \le |s| \le \delta^{-\gamma_L-\varepsilon}\}$. Due to these results, a graph of the energy spectrum $\mathcal{E}_r$ defined as in (1) roughly has the form shown on Fig. 2.

Figure 2

Comparing Fig. 2 with Fig. 1 we see that Kolmogorov's scale λ_1 equals $\ell_x^L(u_\delta)$, if the former is defined (the latter is well defined for any function $u_\delta(t,x)$). The solution u_δ is of order one and its space-period equals two. So a "Reynolds number" for this solution is $\sim \delta^{-1}$ and $\lambda_1 \sim R^{-3/4} \sim \delta^{3/4}$ in contradiction with estimates (10). First explanation for this contradiction is obvious: estimates (10) were obtained not for solutions of NS equations but for solutions of NLS. Second explanation is due to A.N. Kolmogorov himself: in [Kol 2] he pointed out that for his theory to hold, the fluid must be forced by a random forcing. We model the randomly forced turbulence by solutions of equation (3), which we discuss in section 7. To simplify presentation we consider solutions of (3), (4) with zero initial conditions:

$$u(0,x) = 0\,. \tag{11}$$

For a random field u_δ^ω which solves (3), (4), (11) we get in section 7.2 (using our previous work [K2]) an estimate from below:

Theorem B. For any $m \geq 6$, $t_0 \geq 0$ and $L \geq \delta^{-1}$, a solution u_δ^ω of (3), (4), (11) satisfies the estimate $\left(\frac{1}{L}\int_{t_0}^{t_0+L} \mathbf{E}\,\|u_\delta^\omega(t,\cdot)\|_{H^m}^2\,dt\right)^{1/2} \geq C_m \delta^{-\frac{3}{17}m+3.5}$.

Hence, the space-scale $\ell_x^{L,\mathrm{E}}(u) = \delta^{\gamma_L^{\mathrm{E}}}$ of the solution u is $\leq \delta^{3/17}$. Again, this is a non-linear phenomenon since for solutions of the linear equation with the term $|u|^2 u$ replaced by u, the space scale is $\geq \delta^0 = 1$.

In section 7.3 we estimate the solution u_δ^ω from above. To do this we assume that $\nu = i$ in equation (3). Under this assumption we prove that for all $t \geq 0$ and $m \geq 1$ we have $\mathbf{E}\,\|u_\delta^\omega(t,\cdot)\|_{H^m}^2 \leq C_m \delta^{-3m-2}$. Thus, $\ell_x^{L,\mathrm{E}} \geq \delta^{3/2}$ and

$$\delta^{3/2} \leq \ell_x^{L,\mathrm{E}}(u_\delta^\omega) \leq \delta^{3/17} \tag{12}$$

for a solution u_δ^ω of the problem (3), (4), (11) with $\nu = i$. Again, we stress universal nature of the estimates: they do not depend on the dimension $n \leq 3$ and on the specific choice of the order-one random field $\zeta^\omega(t,x)$ (smooth in x, stationary in t).

Writing a solution u_δ^ω as $\sum \hat{u}_{\delta s}^\omega(t) e^{\pi i s\cdot x}$, we define the energy E_s^L of a wave vector s by averaging $|\hat{u}_{\delta s}^\omega|^2$ in ensemble and in time along a finite segment L:

$$E_s^L = \frac{1}{|L|}\int_L \mathbf{E}\,|\hat{u}_{\delta s}^\omega(t)|^2\,dt\,.$$

Assuming that $\delta^{-1} \leq |L| < \infty$, we extract from (12) the following information on the energies E_s^L:

i) for any $\gamma' \geq 3/2$ and any M we have $\sum_{|s|\geq\delta^{-\gamma'}} E_s^L \leq \delta^{M\gamma'}$ if $\delta < \delta_{M\gamma'}$;

ii) for any $\varepsilon > 0$ there exist finite $c(\varepsilon)$ and $C(\varepsilon)$ such that the average of E_s^L in s along the layer $\mathfrak{A} = \{\delta^{-\frac{3}{17}+\varepsilon} \leq |s| \leq \delta^{-\frac{3}{2}-\varepsilon}\}$ is $\geq \delta^c$ and $\leq \delta^C$, for all $\delta < \delta_\varepsilon$;

iii) most part of the averaged squared Sobolev norm $\frac{1}{L}\int_L \mathbf{E}\,\|u\|_m^2\,dt = \sum \langle s\rangle^{2m} E_s$ is carried by wave vectors s from the layer $\mathfrak{A}$.

Thus, the graph of the energy spectrum $\mathcal{E}$ has a form, similar to the one on Fig. 2 (with the segment $[\delta^{-1/3}, \delta^{-1/2}]$ replaced by the bigger one $[\delta^{-3/17}, \delta^{-3/2}]$). As we see, it resembles the Kolmogorov–Obukhov shape, shown on Fig. 1.

The Kolmogorov theory studies *stationary* turbulence. So to get results really related to the Kolmogorov–Obukhov law, our theorems have to be applied to time-stationary solutions of (3). To do it, in section 7.5 we assume that the shifted solution $u_\delta^\omega(t+\tau, x)$ converges in distribution as $\tau \to \infty$ to a stationary in time random field $U_\delta^\omega(t,x)$, which is an odd periodic in x solution of (3). (We refer the reader to [FM], where similar convergence is proven for the 2D NS-equations, forced by a random field which *is not smooth* in x.) This solution inherits the estimates we got for u_δ^ω. In particular, its space-scale $\ell_x^{L,\mathrm{E}}$ also satisfies (12). Due to the stationarity, the energy E_s of a wave vector s defines as $E_s = \mathbf{E}\left|\hat{U}_{\delta s}^\omega(t)\right|^2$ (for any t). It satisfies the relations i) – iii). Besides, for all δ from a sequence converging to zero, the inverse space-scale $1/\ell_x^{L,\mathrm{E}}(U)$ separates a region of the "very fast" decay of the energies E_s (and of the energy spectrum $\mathcal{E}_r$) from a region of the "moderate" decay:

iv) for any $\gamma' > \gamma_L^{\mathrm{E}}$ and any $M \ge 1$, $\sum_{|s|\ge\delta^{-\gamma'}} E_s \le \delta^{M\gamma'}$ for $\delta \in \{\delta_j \searrow 0\}$ (the sequence depends on γ' and M);

v) for any $\varepsilon > 0$, there exist finite numbers $c(\varepsilon)$ and $C(\varepsilon) = n\gamma_L^{\mathrm{E}} - 2 + o(1)$ such that the average $E_{\mathfrak{A}}$ of E_s along the layer $\mathfrak{A} = \{\delta^{-\gamma_L^{\mathrm{E}}+\varepsilon} \le |s| \le \delta^{-\gamma_L^{\mathrm{E}}-\varepsilon}\}$ satisfies the estimates $\delta^c \le E_{\mathfrak{A}} \le \delta^C$ for $\delta \in \{\delta_j \searrow 0\}$ (the sequence depends on ε).

Thus, if the stationary (in t) turbulent solution U satisfies the Kolmogorov–Obukhov law (maybe with an exponent different from 5/3), then corresponding Kolmogorov's inner scale $\lambda_1 = r_1^{-1}$ *must be equal* to the space-scale $\ell_x^{L,\mathrm{E}}(U)$. In particular, it *must satisfy* estimates (12).

The number N_δ of degrees of freedom of the turbulent flow is $N_\delta \sim r_1^n$ (see [F], p.107, cf. [K2], p.815), thus we get that $\delta^{-3n} \gtrsim N_\delta \gtrsim \delta^{-6n/17}$ for the random NLS equation and $\delta^{-n} \gtrsim N_\delta \gtrsim \delta^{-2n/3}$ in the deterministic case (the extra factor 2 appears in the exponent since a solution u is a pair of real functions; this factor was forgotten in [K2]). This explains why it is so difficult to find turbulent solutions numerically since in the turbulent regime δ is very small, e.g. for developed hydrodynamic turbulence usually $\delta \lesssim 10^{-5}$ or even $\delta \lesssim 10^{-7}$. Cf. [JT, BGO] for techniques to estimate from above the number of degrees of freedom for equations with non-random forcing.

Finally we note that the approach to study the short-scale (or "irregular", or "chaotic") behaviour of solutions for nonlinear PDEs in the turbulent regime which we develop in this work has nothing to do with the hyperbolic behaviour of a corresponding dynamical system in a function space. Indeed, our results apply to odd periodic solutions of equation (2) with $\nu = p = n = 1$. But this equation is integrable and all Lyapunov exponents of its solutions $u(t,\cdot)$ (treated as curves in a space of odd periodic functions) vanish.

Notations. By C, C_1 etc. we denote different positive constants, independent of δ. By $|\cdot|_{C^m}$ — the norm in the classical space C^m of x-dependent functions (see (6)), by $|\cdot|_{H^m} = \|\cdot\|_m$ — the norms in the Sobolev spaces $H^m(K^n;\mathbb{C})$ and $H^m(\mathbb{T}^n;\mathbb{C})$. We often write functions $u(t,x)$ as curves $u(t)$ in a function space, so $|u(t)|_{C^m} = |u(t,\cdot)|_{C^m}$, etc.

Acknowledgements. The paper was partly written and was typed at the Forschungsinstitut für Mathematik, ETH Zürich, and I am thankful to the FIM for their hospitality. I am obliged to Ya.G.Sinai and N.S.Nadirashvili for discussions (FIM, summer 1997) and to Greg Eyink – for numerous enthusiastic explanations of classical theories of turbulence (Tucson, springs 1996,97). I thank Rahel Boller (FIM) for excellent typing the manuscript. This work was supported by the Russian Fund of Fundamental Research, grant 95-01-00609a.

1. Smooth space-scales

1.1. Time-independent functions.

Let O be a bounded domain in $\mathbb{R}^n$ with a smooth boundary ∂O, or an n-dimensional cube, or the torus $\mathbb{T}^n = \mathbb{R}^n/2\mathbb{Z}^n$, or the whole $\mathbb{R}^n$. Let $f_\delta(x) \in \mathbb{R}^N$ be a vector-function, smooth in $x \in O$ and depending on a parameter $\delta \in (0,1)$. For this function we define its *smooth space-scale* $\ell_x(f)$. We call the space-scale ℓ_x *smooth* since its definition uses arbitrarily large derivatives of a function, which is assumed to be smooth. The definition given below is *asymptotical* in $\delta \to 0$ since $\ell_x(f)$ will not change if we redefine f for $\delta \geq \delta_1$ with any positive δ_1. Due to this, if initially $f(x)$ was defined only for $0 < \delta < \delta_1 < 1$, we extend it by zero to $\delta_1 \leq \delta < 1$ and treat as a function of $\delta \in (0,1)$.

Definition 1. For a function $f_\delta(x)$ as above we set $\ell_x(f) = \delta^\gamma$, where $\gamma = \sup \Gamma$ and the set $\Gamma = \Gamma(f) \subset \mathbb{R}$ is formed by all $\tilde\gamma$ such that

$$|f_\delta|_{C^k} \geq \delta^{-\tilde\gamma k} \quad \text{for all} \quad k \geq k_0 \quad \text{and all} \quad \delta < \delta_k \,, \tag{1.1}$$

where k_0 and $\delta_k > 0$ depend on f and $\tilde\gamma$. The number $\gamma = \gamma(f)$ is called the exponent (of the space-scale).

As usual, supremum over the empty set equals $-\infty$ and $\delta^{-\infty} = \infty$, $\delta^{+\infty} = 0$.

The exponent γ can be equivalently defined in the *lim inf*-terms:

$$\gamma = \gamma(f_\delta) = \liminf_{m\to\infty} \liminf_{\delta\to 0} \frac{\ell n \, \|f_\delta\|_m^{1/m}}{\ell n \, \delta^{-1}} \ . \tag{1.2}$$

(The inner limit in (1.2) is called a lower order at zero of the function $\delta \mapsto \|f_\delta\|_m^{1/m}$.)

The set $\Gamma(f)$ in Definition 1 can be empty (example: $f = \exp -\delta^{-1}$) and can coincide with $\mathbb{R}$ (example: $f = \exp \delta^{-1}$). It means that a priori $\ell_x(f)$ is any degree of δ, from $\delta^\infty = 0$ to $\delta^{-\infty} = \infty$. The definition becomes informative if $-\infty < \gamma < \infty$. We are the most interested in the functions f_δ such that $1 > \ell_x(f) > 0$ (equivalently, $0 < \gamma < \infty$) and call them *short-scale* functions.

If $|f_\delta|_{C^0} \ge C\delta^a$, with any $a \ge 0$, then also $|f_\delta|_{C^k} \ge C\delta^a$. Now each negative $\tilde\gamma$ belongs to the set Γ and $\gamma(f) \ge 0$. An upper bound for $\gamma(f)$ follows from the following:

Proposition 1. For any function $f_\delta(x)$, $\gamma(f) < \gamma'$ if and only if there exists $\varepsilon > 0$ such that the relation

$$|f_\delta|_{C^M} \le \delta^{-(\gamma'-\varepsilon)M} \tag{1.3}$$

holds for arbitrarily big M and for $\delta \in \{\delta_{\varepsilon M}(j) \searrow 0\}$ (here and below the latter means "for each element of the sequence $\{\delta_{\varepsilon M}(1), \delta_{\varepsilon M}(2), \dots\}$, where $\delta_{\varepsilon M}(j) \searrow 0$ as $j \to \infty$ ").

Proof. If $\gamma' \le \gamma(u)$, then for any $\varepsilon > 0$ the number $\tilde\gamma = \gamma' - \varepsilon$ belongs to Γ and relation (1.1) with $\tilde\gamma = \gamma' - \varepsilon$ contradicts (1.3) with $M \ge k_0$. On the contrary, if (1.3) fails for each positive ε, then (1.1) holds with each $\tilde\gamma = \gamma' - \varepsilon$, so $\gamma' \le \gamma(u)$. Thus (1.3) exactly means that $\gamma' > \gamma$. □

In particular, choosing $\gamma' = \gamma + \varepsilon'$ with any $\varepsilon' > 0$ and using (1.3) we get that the relation

$$|f_\delta|_{C^M} \le \delta^{-(\gamma+\varepsilon')M} \tag{1.4}$$

holds for some arbitrarily big M and for arbitrarily small $\delta > 0$.

Definition 1 was designed to study short-scale functions $f_\delta(x)$. For "long-scale" ones it is unsatisfactory. For example, a function $f_\delta(x) = F(\delta x)$ "obviously" has a space-scale of order δ^{-1}, but $\ell_x(f_\delta) = \delta^0 = 1$. To cover the long-scale case, the C^k-norms in the Definition 1 have to be replaced by the quasinorms $[\cdot]_{C^k}$, where

$$[u]_{C^k} = \max_{|\alpha|=k} \sup_{x\in O} |\partial_x^\alpha u| \ .$$

We stuck to the definition in the form given above since it possesses many natural properties (and the important property 1) in Proposition 2 below would fail if in Definition 1 one replaces the norms by the quasinorms).

Proposition 2. 1) (Norm-independence). A value of the exponent γ (and of the scale ℓ_x) will not change if in (1.1) C^k-norms are replaced by any norms $\|\cdot\|_k$ such that

$$K_k^{-1} \|u\|_{k-l} \le |u|_{C^k} \le K_k \|u\|_{k+l} \ \ \forall u \in C^\infty(O)$$

for any $k \ge k'$, with some fixed $l \ge 0$ (in particular, the norms of Sobolev spaces $W^{k,p}(O)$ with any $1 \le p < \infty$ can be used). Proposition 1 also remains true with the C^M-norms replaced by the norms $\|\cdot\|_M$.

2) (Invariance with respect to change of x). If $O \to Q$, $x \mapsto y$, is a smooth δ-independent diffeomorphism such that the diffeomorphism and its inverse both have finite C^k-norms for every k, then $\ell_y\big(f_\delta(y)\big) = \ell_x\Big(f_\delta(x) := f_\delta\big(y(x)\big)\Big)$ for any smooth in y function $f_\delta(y)$.

3) (Invariance with respect to change of f). If $|f_\delta(x)| \le C$ for all δ and x and $F \in C^\infty(\mathbb{R}^N, \mathbb{R}^{N_1})$, then $\ell_x\big(F(f)\big) \ge \ell_x(f)$. In particular, if F is a diffeomorphism of $\mathbb{R}^N$, then $\ell_x\big(F(f)\big) = \ell_x(f)$.

4) (Invariance with respect to scaling of f). For any a, $\ell_x(f) = \ell_x(\delta^a f)$.

5) (Invariance with respect to differentiation). For any $f = f_\delta(x)$, $\ell_x(f) \le \ell_x(\nabla f)$. If f is a short-scale function and $|f| \le C\delta^{-a}$ with some $a \ge 0$, then $\ell_x(f) = \ell_x(\nabla f)$.

Proof. 1) For any $\varepsilon > 0$ we have:

$$\|f_\delta\|_k \ge K_{k-l}^{-1} |f_\delta|_{C^{k-l}} \ge K_{k-l}^{-1}\delta^{-(\gamma-\varepsilon)(k-l)} \ge K_{k-l}^{-1}\delta^{-(\gamma-2\varepsilon)k} ,$$

for $k \ge \max\{k_0, l(\gamma/\varepsilon - 1)\}$. It means that a new space-scale defined with the new norm is no longer than ℓ_x since $\gamma - 3\varepsilon$ belongs to the corresponding set Γ. Similar ℓ_x is no longer than the new scale, so they are equal.

2) Since $|f_\delta(x)|_{C^M} \le C_M |f_\delta(y)|_{C^M}$, then using Proposition 1 we see that a bound $\gamma(f(y)) < \gamma'$ implies the bound $\gamma(f(x)) < \gamma'$ (to the function $u(x)$ the proposition should be applied with ε replaced by $\varepsilon/2$). The opposite also holds, so the assertion follows.

3) Proposition 1 yields the assertion since $\gamma(f) < \gamma'$ implies that $\gamma(F(f)) < \gamma'$.

Proof of 4) is obvious.

5) If $\tilde\gamma \in \Gamma(\nabla f)$, then $|f|_{C^{k+1}} \ge |\nabla f|_{C^k} \ge \delta^{-\tilde\gamma k}$. This is bigger than $\delta^{-(\tilde\gamma-\varepsilon)(k+1)}$ for any positive ε, if k is sufficiently big. Hence, $\gamma(f) \ge \sup\Gamma = \gamma(\nabla f)$.

If f is a short-scale function, then $\gamma(f) = \sup\{\tilde\gamma \in \Gamma(f) \mid \tilde\gamma > 0\}$. Let us take any positive $\tilde\gamma \in \Gamma$. Then $|f|_{C^k} \ge \delta^{-\tilde\gamma k}$. Since $|f|_{C^0} \le \delta^{-a}$ by assumption, then $|\nabla f|_{C^{k-1}} \ge \delta^{-\tilde\gamma k}$ if k is sufficiently big and $\tilde\gamma \in \Gamma(\nabla f)$. □

Another nice property of Definition 1 is that in the few cases when "everybody" knows the size of the space scale of a function f, the scale ℓ_x agrees with this knowledge:

Proposition 3. If F is a smooth function on $\mathbb{R}^n$ which is not a polynomial and a function f_δ on a domain $O \subset \mathbb{R}^n$, $0 \in O$, is defined as $f_\delta(x) = F(x\delta^{-a})$ with $a \ge 0$, then $\ell_x(f_\delta) \le \delta^a$. If F and all its derivatives are bounded, than $\ell_x(f_\delta) = \delta^a$.

Proposition 4. If a function $g_\delta(x)$ is analytic in $x \in O$ and can be analytically extended to the complex domain $\{(x+iy) | x \in O, y \in \mathbb{C}^n, |y| < \delta^a\}$, $a \ge 0$, where it is bounded by a δ-independent constant, then $\ell_x(g) \ge \delta^a$.

To prove Proposition 3 we note that $[f_\delta]_{C^k} = \delta^{-ka}\left[F|_{\delta^{-a}O}\right]_{C^k}$, where the second factor is positive (at least, for small δ), and is bounded under the assumptions of the second assertion. To prove Proposition 4 we note that $[g_\delta]_{C^k(O)} \le const\ \delta^{-ka}$ by the Cauchy estimate. So any $\tilde\gamma \in \Gamma$ must be smaller than a and $\gamma \le a$.

1.2. Time-dependent functions

Let a smooth in $x \in O$ function $f_\delta(t,x)$ depends on an additional "time-variable" $t \in [T_1, T_2]$, where T_1 and T_2 such that $-\infty \le T_1 < T_2 \le \infty$ may depend on δ (if $T_2 = \infty$, then the segment should be replaced by $[T_1, \infty)$; similar with $T_1 = -\infty$). Now the time-dependence of f_δ has to be incorporated into the definition of a smooth space-scale. The easiest way to do it is to consider the *shortest space-scale* $\ell_x^{\inf}$, where

$$\ell_x^{\inf}(f_\delta(t,x)) = \delta^{\gamma^{\sup}}, \quad \gamma^{\sup} = \sup_{T_1 \le t \le T_2} \gamma(f_\delta(t,\cdot)) .$$

But it turns out that another space-scale $\ell_x^L(f_\delta)$, defined in terms of averaging rather than in supremum-terms, is more useful. To simplify the definition we impose some restrictions on length of the time-segment: we assume that either

a) $T_2 = \infty$, or

b) $T_2 - T_1 = C\delta^{-b}$ with some $C > 0$ and $b \geq 0$.

Definition 2. For a function $f_\delta(t,x)$ as above we set $\ell_x^L(f_\delta) = \delta^{\gamma_L}$, where $\gamma_L = \sup \Gamma_T$ and the set Γ_T is formed by all $\tilde\gamma$ such that

$$\Big(\frac{1}{|L|}\int_L |f_\delta(t,\cdot)|^2_{C^k}\,dt\Big)^{1/2} \geq \delta^{-\tilde\gamma k} \quad \forall\, k \geq k_{\tilde\gamma},\ \delta < \delta_{k\tilde\gamma}, \tag{1.5}$$

where

$$\begin{aligned} L = [T_1, T_2] \ \text{ if } T_2 - T_1 = C\delta^{-b} \ \text{ and} \\ L \subset [T_1, T_2] \ \text{ is any segment of length } \infty > |L| \geq \delta^{-b} \text{ if } T_2 = \infty\ ; \end{aligned} \tag{1.6}$$

if $T_2 = \infty$, then $b \geq 0$ in (1.6) may depend on f.

Properties 1) – 4) stated in Proposition 2 for the scale ℓ_x remain true for the scale ℓ_x^L. In particular, we shall profit from the first property and shall replace in (1.5) the C^k-norm by the norm $\|\cdot\|_k$ of the Sobolev space $H^k(O;\mathbb{R}^n)$:

$$\Big(\frac{1}{|L|}\int_L \|f_\delta(t,\cdot)\|^2_k dt\Big)^{1/2} \geq \delta^{-\tilde\gamma k} \quad \forall\, k \geq k_{\tilde\gamma}, \delta \leq \delta_{k\tilde\gamma}. \tag{1.5$'$}$$

Proposition 5. If $|L| = C\delta^{-b}$ and L_1 is a subsegment of L such that $|L_1| = C_1\delta^{-b_1}$, then $\ell_x^L(f) \leq \ell_x^{L_1}(f)$.

Proof is obvious since for any non-negative function its average along L is bigger than the average along L_1 times $|L_1|/|L|$. □

The relation

$$\ell_x^L(f) \geq \ell_x^{\inf}(f) \ \text{ if } \ \gamma^{\sup} \geq 0 \ \text{ and } \ |f_\delta(t,x)| \leq C\delta^{-a} \ \text{ for all } \ t, x, \delta \tag{1.7}$$

(a is any real number), is less obvious but it will easily result from the following:

Lemma 1. If $|f_\delta(t,x)| \leq C\delta^{-a}$ for all t, x and $\gamma^{\sup} \geq 0$, then $\gamma^{\sup}$ equals $\sup \Gamma_0$, where the set Γ_0 is formed by all $\tilde\gamma$ such that $\sup_t |f_\delta(t,\cdot)|_{C^k} \geq \delta^{-\tilde\gamma k}$ for $k \geq k_{\tilde\gamma}$ and $\delta < \delta_{k\tilde\gamma}$.

Proof. We should check that $\gamma^{\sup}$ equals $\gamma' := \sup\{\hat\gamma \in \Gamma_0\}$. For any $\varepsilon > 0$ there exists τ_ε such that $\gamma\big(f_\delta(\tau_\varepsilon,\cdot)\big) \geq \gamma^{\sup} - \varepsilon$. Then $|f_\delta(\tau_\varepsilon,\cdot)|_{C^k} \geq \delta^{-(\gamma^{\sup}-2\varepsilon)k}$ for all $k \geq k_\varepsilon$ and small δ. Hence, $\gamma^{\sup} - 2\varepsilon \in \Gamma_0$ for each ε and $\gamma' \geq \gamma^{\sup}$.

To prove the inverse inequality we may assume that $\gamma' > 0$ since $\gamma^{\sup} \geq 0$ by assumption. Then for any positive $\varepsilon < \gamma'$ and arbitrarily big k there exists $t' = t'_{\varepsilon,k}$ such that $|f_\delta(t',\cdot)|_{C^k} \geq \delta^{-(\gamma'-\varepsilon)k}$ for $\delta < \delta_{\varepsilon,k}$. Now by the interpolation inequality (see (A3) in Appendix 1), for $m \geq k$ we have:

$$|f_\delta(t')|_k \leq C\,|f_\delta(t')|_0^{(m-k)/m}\,|f_\delta(t')|_m^{k/m} \leq C_1\delta^{-a(m-k)/m}\,|f_\delta(t')|_m^{k/m}$$

and $|f_\delta(t')|_m^{k/m} \geq C_1^{-1}\delta^{-ak/m}\delta^{-(\gamma'-\varepsilon)k+a}$. Assuming that k is sufficiently big we find that $|f_\delta(t')|_m \geq \delta^{-(\gamma'-2\varepsilon)m}$ for any $m \geq k$. Thus, $\gamma\big(f_\delta(t')\big) \geq \gamma' - 2\varepsilon$ and $\gamma^{\sup} \geq \gamma'$ since ε is arbitrarily small. □

Lemma 1 implies (1.7). Indeed, if $\tilde\gamma \in \Gamma_T$ then by (1.5) $|f_\delta(t)|_k \geq \delta^{-(\tilde\gamma-\varepsilon)k}$ for any $\varepsilon > 0$ and some $t = t_{k\varepsilon} \in L$. Hence, $\tilde\gamma \in \Gamma_0$ and $\gamma^{\sup} \geq \gamma_L$ as states (1.7).

As above, we say that a function $f_\delta(t,x)$ is *short-scale* if $0 < \gamma_L < \infty$.

Example. Let us consider a linear Schrödinger equation under odd periodic boundary conditions:

$$\begin{gathered}\dot u = \delta\Delta u + iu + f(t,x)\ ,\ \ x \in \mathbb{R}^n\ ,\\ u(t,x) = u(t,x_1,\dots,x_j+2,\dots,x_n) = -u(t,x_1,\dots,-x_j,\dots,x_n)\quad \forall\, j\ ,\end{gathered} \tag{1.8}$$

where $f(t,x)$ is a smooth function, odd periodic in x, and such that

$$\|f(t,\cdot)\|_m \leq C_m \quad \forall\, t,\ \forall\, m\ . \tag{1.9}$$

The operator $A = \delta\Delta + i$ is diagonal in the sin-basis $\{\xi_s(x) = \sin \pi s_1 x_1 \dots \sin \pi s_n x_n | s \in \mathbb{N}^n\}$ of the space of odd periodic functions and $A\xi_s = \lambda_s\xi_s$ with $\lambda_s = -\delta\pi^2|s|^2 + i$. Writing the forcing f as $f = \sum f_s(t)\xi_s$ and supplying (1.8) with (for example) zero initial conditions $u(0,x) = 0$, we find a solution $u = u_\delta(t,x)$ in the form $u_\delta(t,x) = \sum u_s(t)\xi_s(x)$, where $u_s(t) = \int_0^t e^{\lambda_s(t-\tau)} f_s(\tau)d\tau$. By (1.9), $|f_s(\tau)| \leq C_N\,|s|^{-N}$ for all s and τ. Hence, $|u_s(t)| \leq C_N\,|s|^{-N}/\big(\pi^2\delta\,|s|^2\big)$ and

$$\|u_\delta(t,\cdot)\|_m^2 \leq \Big(\frac{C_N}{\pi^2\delta}\Big)^2 \sum |s|^{-2N-4+2m} \leq C'_m\delta^{-2}\ ,$$

if we choose $N > m - 2 + n/2$. Thus, $\gamma_L(u_\delta) \leq 0$ and $\ell_x^L(u_\delta) \geq 1$. □

2. Spectral properties of short-scale functions

In this chapter we consider short-scale functions $f_\delta(x)$ and $f_\delta(t,x)$ with $x \in \mathbb{T}^n = \mathbb{R}^n/2\mathbb{Z}^n$ or $x \in \mathbb{R}^n$. We state all results for x-periodic functions since their reformulations for the case $x \in \mathbb{R}^n$ are obvious.

Using Proposition 2, in definitions of the scales ℓ_x and ℓ_x^L we replace C^k-norms by Sobolev norms $\|\cdot\|_k$, where for $u(x) = \sum_{s\in\mathbb{Z}^n} \hat u_s e^{i\pi s\cdot x}$ we set $\|u(x)\|_k^2 = \sum_{s\in\mathbb{Z}^n} \langle s\rangle^{2k}\,|\hat u_s|^2$

with $\langle s \rangle = \max\{1, |s|\}$. Following a popular spectral decomposition approach to study function spaces of Sobolev–Besov type (cf. [Tr]), we define the quantities U_r,

$$U_r = \sum_{2^r \le |s| < 2^{r+1}} |\hat{u}_s|^2 \ , \quad r \ge 0,$$

and the norms $\|\cdot\|_k'$, where

$$(\|u\|_k')^2 = \sum_{r=0}^{\infty} 4^{rk} U_r.$$

Since in the definition of U_r the summation is taken over multi-indexes s such that $4^{rk} \le \langle s \rangle^{2k} \le 4^k 4^{rk}$, then the norm $\|\cdot\|_k'$ is equivalent to $\|\cdot\|_k$:

$$\|u\|_k' \le \|u\|_k \le 2^k \|u\|_k' \ . \tag{2.1}$$

Let $u_\delta(x)$ be a smooth in x function such that $0 \le \gamma = \gamma(u) < \infty$. Using (1.4) and item 1) of Proposition 2 we get that

$$\sum_{r=0}^{\infty} 4^{rM} U_r \le \delta^{-2\gamma_2 M} \tag{2.2}$$

for any $\gamma_2 > \gamma$; the inequality holds for arbitrarily big M and for $\delta \in \{\delta_{\gamma_2 M}(j) \searrow 0\}$. From other hand, by the definition of ℓ_x (and Proposition 2), for any $\gamma_1 < \gamma$ we have:

$$\sum_{r=0}^{\infty} 4^{rm} U_r \ge \delta^{-2\gamma_1 m} \quad \forall m \ge m_{\gamma_1} \ , \ \delta < \delta_{\gamma_1 m} \ . \tag{2.3}$$

In addition to (2.2), (2.3), we assume that L_2-norm of the function u_δ is not too big:

$$\|u\|_0^2 \le C\delta^{-a} \ \ \forall \delta \ , \tag{2.4}$$

with some $a \ge 0$. For any $\rho_1 \le \rho_2 \le \infty$ we define the layer $\mathfrak{A}(\rho_1, \rho_2) \subset \mathbb{Z}^n$ as follows:

$$\mathfrak{A}(\rho_1, \rho_2) = \left\{ s \in \mathbb{Z}^n | \delta^{-\rho_1} \le |s| \le \delta^{-\rho_2} \right\} .$$

Our first theorem states that for large m, H^m-norms of the function u are mostly carried by Fourier modes, corresponding to the wave numbers close to ℓ_x^{-1}:

Theorem 1. Let a function u satisfies (2.4). Then:
1) if $0 \le \gamma = \gamma(u) < \infty$, then for any $\varepsilon > 0$ and $\nu > 0$ the inequality

$$\sum_{s \in \mathfrak{A}} |\hat{u}_{\delta s}|^2 \langle s \rangle^{2m} \ge (1 - \nu) \|u\|_m^2 \tag{2.5}$$

with $\mathfrak{A} = \mathfrak{A}(\gamma - \varepsilon, \gamma + \varepsilon)$ holds for all $m \ge m(\varepsilon)$ and for $\delta \in \{\delta_{\varepsilon m\nu}(j) \searrow 0\}$.

2) If (2.2) with some $\gamma_2 \le \infty$ and (2.3) with some $\gamma_1 \ge 0$ hold for $\delta \le \delta_M$ and $\delta \le \delta_m$ respectively (so $\gamma_1 \le \gamma \le \gamma_2$), then for any $\varepsilon > 0$, $\nu > 0$ the inequality (2.5) is valid with $\mathfrak{A} = \mathfrak{A}(\gamma_1 - \varepsilon, \gamma_2 + \varepsilon)$, for all $m \ge m_\varepsilon$ and $\delta < \delta_{\varepsilon m\nu}$.

We note that in the second assertion of the theorem, assumption (2.2) becomes empty if $\gamma_2 = \infty$. In this case the layer $\mathfrak{A}(\gamma_1 - \varepsilon, \gamma_2 + \varepsilon)$ degenerates to a complement to the ball $\{|s| < \delta^{-\gamma_1+\varepsilon}\}$.

Our next result states that a function u is almost localised to the wave numbers $|s| \lesssim \ell_x^{-1}$, but not to the wave numbers $|s| \ll \ell_x^{-1}$. Sobolev norms are not used now:

Theorem 2. Under the assumptions of item 1) of Theorem 1,

1) i) for any $\gamma' > \gamma$ and any $M \ge 1$,

$$\sum_{|s| \ge \delta^{-\gamma'}} |\hat{u}_{\delta s}|^2 \le C_M \delta^{M\gamma'} \tag{2.6}$$

for $\delta \in \{\delta_{\gamma' M}(j) \searrow 0\}$;

ii) for any $\varepsilon > 0$ the averaging of a squared Fourier coefficient $|\hat{u}_{\delta s}|^2$ along the layer $\mathfrak{A} = \mathfrak{A}(\gamma - \varepsilon, \gamma + \varepsilon)$, defined as $\langle |\hat{u}_{\delta s}|^2 \rangle_{\mathfrak{A}} := |\mathfrak{A}|^{-1} \sum_{s \in \mathfrak{A}} |\hat{u}_{\delta s}|^2$, is such that

$$\delta^c \le \langle |\hat{u}_{\delta s}|^2 \rangle_{\mathfrak{A}} \le \delta^C \tag{2.7}$$

for some finite constants $c = c(\varepsilon)$, $C = C(\varepsilon)$ and for $\delta \in \{\delta_\varepsilon(j) \searrow 0\}$.

2) Under the assumptions of item 2) of Theorem 1, the relation (2.6) holds for any $\gamma' > \gamma_2$ and $\delta < \delta(M)$; the relation (2.7) with $\mathfrak{A} = \mathfrak{A}(\gamma_1 - \varepsilon, \gamma_2 + \varepsilon)$ holds for any $\varepsilon > 0$ and $\delta < \delta(\varepsilon)$.

Amplification. If $\gamma > 0$, then for $\varepsilon \to 0$ and $\delta \in \{\delta_\varepsilon(j) \searrow 0\}$ estimates (2.7) of Theorem 2 can be specified as follows: $\ell_x^{\tilde{c}} \le \langle |\hat{u}_{\delta s}|^2 \rangle_{\mathfrak{A}} \le \ell_x^{n - a/\gamma + o(1)}$, where $\tilde{c} = \tilde{c}(\varepsilon)$ is a finite number.

Now let $f_\delta(t, x)$, $x \in \mathbb{T}^n$, $T_1 \le t \le T_2$ be a time-dependent function and $0 \le \gamma_L(f) < \infty$. Then for $\gamma_1 < \gamma_L$ we have:

$$\frac{1}{|L|} \int_L \|f_\delta(t)\|_m^2 \, dt \ge \delta^{-2\gamma_1 m} \quad \forall m \ge m_{\gamma_1} \ , \ \delta < \delta_{\gamma_1 m} \ , \tag{2.8}$$

where $L \subset [T_1, T_2]$ is any segment as in (1.6) (see (1.5$'$)). If $T_2 - T_1 = C\delta^{-b}$, then for $\gamma_2 > \gamma_L$ we have:

$$\frac{1}{|L|} \int_L \|f_\delta(t)\|_M^2 \, dt \le \delta^{-2\gamma_2 M} \tag{2.9}$$

for arbitrarily big M and $\delta \in \{\delta_{\gamma_2 M}(j) \searrow 0\}$; now $L = [T_1, T_2]$.

If $T_2=\infty$, then (2.9) might hold for some segments $L\subset[T_1,T_2]$ only. So for $T_2=\infty$ we shall usually *assume* that (2.9) holds with some $\gamma_2\ge\gamma_L$ for each segment L such that $\infty>|L|\ge\delta^{-b}$.

For any finite segment $L\subset[T_1,T_2]$ and for a function $f_\delta=\sum\hat f_{\delta s}(t)e^{i\pi s\cdot x}$ we denote:

$$\left|\hat f_{\delta s}\right|_L^2=\frac{1}{|L|}\int_L\left|\hat f_{\delta s}(t)\right|^2dt\,,\qquad \|f_\delta\|_{k,L}^2=\frac{1}{|L|}\int_L\|f_\delta(t)\|_k^2\,dt\,.$$

Then $\|f_\delta\|_{k,L}^2=\sum_{s\in\mathbb{Z}^n}\langle s\rangle^{2k}|\hat f_{\delta s}|_L^2$ — the relation between $\|f_\delta\|_{k,L}^2$ and $\{\left|\hat f_{\delta s}\right|_L^2\}$ is the same as between $\|u_\delta\|_k^2$ and $\{|\hat u_{\delta s}|^2\}$.

For a smooth in x function $f_\delta(t,x)$ with $0\le\gamma_L(f)<\infty$ direct analogies of Theorems 1,2 hold. Let us assume that

$$\|f_\delta\|_{0,L}^2\le C\delta^{-a}\quad\forall\delta \tag{2.10}$$

for some $a\ge0$, where the segment L is as in (1.6).

Theorem 1′. Let (2.10) holds. Then

1) If $T_2-T_1=C\delta^{-b}$ and $0\le\gamma:=\gamma_L(f_\delta)<\infty$, then for any $\varepsilon>0$ and $\nu>0$ the inequality $\sum_{s\in\mathfrak{A}}\left|\hat f_{\delta s}\right|_L^2\langle s\rangle^{2m}\ge(1-\nu)\,\|f_\delta\|_{m,L}^2$ with $\mathfrak{A}=\mathfrak{A}(\gamma-\varepsilon,\gamma+\varepsilon)$ holds for all $m\ge m_\varepsilon$ and $\delta\in\{\delta_{\varepsilon m}(j)\searrow0\}$.

2) Let us assume that (2.8) and (2.9) hold with some $\gamma_1\ge0$ and $\gamma_2\le\infty$ for all $\delta<\delta_m$ and $\delta<\delta_M$, for any segment L as in (1.6) (in particular, $\gamma_1\le\gamma_L(f_\delta)\le\gamma_2$). Then for any $\varepsilon>0$ and $\nu>0$ the inequality in item 1) with $\mathfrak{A}=\mathfrak{A}(\gamma_1-\varepsilon,\gamma_2+\varepsilon)$ is valid for all $m\ge m_\varepsilon$ and $\delta<\delta_{\varepsilon m}$.

Theorem 2′. Let (2.10) holds. Then

1) under the assumptions of the first item of Theorem 1′,

i) for any $\gamma'>\gamma_L$ and any $M\ge1$, $\sum_{|s|\ge\delta^{-\gamma'}}|\hat f_{\delta s}|_L^2\le\delta^{-M\gamma'}$ if $\delta\in\{\delta_{M\gamma'}(j)\searrow0\}$;

ii) for any $\varepsilon>0$ and for $\delta\in\{\delta_\varepsilon(j)\searrow0\}$, averaging of $\left|\hat f_{\delta s}\right|_L^2$ along the layer $\mathfrak{A}=\mathfrak{A}(\gamma_L-\varepsilon,\gamma_L+\varepsilon)$ estimates from below and from above by δ^c and δ^C respectively, with some finite numbers $c=c(\varepsilon)$, $C=C(\varepsilon)$.

2) Under the assumptions of the second item of Theorem 1′, assertion i) with $\gamma'>\gamma_2$ and assertion ii) with $\mathfrak{A}=\mathfrak{A}(\gamma_1-\varepsilon,\gamma_2+\varepsilon)$ both hold for all sufficiently small δ (the segment L should be as in (1.6)).

For $\gamma_L>0$ an obvious version of amplification to Theorem 2 can be used to specify the statement 1ii).

3. Proof of Theorems 1, 2

Proof of Theorem 1. 1) Now relations (2.2), (2.3) hold with $\gamma_2 = \gamma + \varepsilon/2$ and $\gamma_1 = \gamma - \varepsilon/2$. Calculations we present below are valid for any $\gamma_2 > \gamma > \gamma_1$.

By (2.2),

$$U_r \leq 4^{-rM} \delta^{-2\gamma_2 M} \quad \forall r \geq 0 , \tag{3.1}$$

for arbitrarily big M and for $\delta \in \{\delta_{\gamma_2 M}(j) \searrow 0\}$. Assuming that $M \geq m+1$, we estimate a tail of the sum in (2.3) as follows:

$$\sum_{r=R}^{\infty} 4^{rm} U_r \leq \delta^{-2\gamma_2 M} \sum_{r=R}^{\infty} 4^{r(m-M)} \leq 2\delta^{-2\gamma_2 M}\, 4^{R(m-M)} .$$

Let us denote $\rho = \gamma_2 + \varepsilon/2$. Choosing for R the smallest integer such that $4^R \geq \delta^{-2\rho}/4$, we get for the tail the following bound:

$$\sum_{r=R}^{\infty} 4^{rm} U_r \leq 4^{1+M-m} \delta^{2M(\rho-\gamma_2)-2\rho m} .$$

For any given $\mu > 0$ the r.h.s. is smaller than μ times the r.h.s. of (2.3) if $M > m(\rho - \gamma_1)/(\rho - \gamma_2) = 2m(\rho - \gamma_1)/\varepsilon$ and if $\delta \in \{\delta_{\varepsilon M j} \searrow 0\}$ is sufficiently small (in terms of m, M and μ).

Since $\sum U_r = \|u\|_0^2 \leq C\delta^{-a}$ by (2.4), then $\sum_{\{r | 4^r \leq N\}} U_r 4^{rm} \leq N^m C \delta^{-a}$. Choosing $N = \delta^{-2\lambda}$ where $\lambda = \gamma_1 - \varepsilon/2$, we bound the r.h.s by $C\delta^{-2m\lambda - a}$. This is less than μ times the r.h.s. of (2.3) if $m > a/\varepsilon$ and δ is sufficiently small.

We have seen that

$$\sum_{\delta^{-2\lambda} < 4^r < \delta^{-2\rho}/4} U_r 4^{rm} \geq (1 - 2\mu)\delta^{-2m\gamma_1} , \tag{3.2}$$

if $m \geq m_{\gamma_1}$ as in (2.3), $m > a/\varepsilon$ and $\delta \in \{\delta_{\varepsilon\gamma_1}(j) \searrow 0\}$ is sufficiently small. Summation in the l.h.s. of (3.2) is taken over r such that $\delta^{-\lambda} < 2^r < \delta^{-\rho}/2$. It corresponds to summation over multi-indexes s from a union of the layers $\{2^r \leq |s| \leq 2^{r+1}\}$. This union is contained in the domain $\{\delta^{-\lambda} < |s| \leq \delta^{-\rho}\} = \mathfrak{A}(\lambda, \rho)$ (this set equals $\mathfrak{A}$ as in (2.5) due to our choice of γ_1 and γ_2). Arguing similar, we find that

$$\begin{aligned} 4^{-m} \sum_{s \notin \mathfrak{A}(\gamma_1 - \varepsilon/2, \gamma_2 + \varepsilon/2)} |\hat{u}_{\delta s}|^2 \langle s \rangle^{2m} &\leq \|u\|_m'^2 - \sum_{\delta^{-2\lambda} < 4^r < \delta^{-2\rho}/4} U_r 4^{rm} \\ &\leq 2\mu \|u\|_m'^2 \leq 2\mu \|u\|_m^2 . \end{aligned} \tag{3.3}$$

The first statement of the theorem follows from (3.3) if we choose $\mu = 2^{-2m-1}\nu$ (the estimate (3.2) will be used later).

2) Now the inequality (3.1) with $\gamma_2 \geq \gamma_1$ as in the assumption of this item holds for all sufficiently small δ. Accordingly, the estimate (3.3) is valid for all $\delta < \delta_{\varepsilon m \mu}$. □

Proof of Theorem 2. 1) i) The sum to be estimated is bounded by $\sum_{2^r \geq \frac{1}{2}\delta^{-\gamma'}} U_r$. By (3.1) (with M re-denoted as N) the latter is bounded by

$$\delta^{-2\gamma_2 N} \sum_{2^r \geq \frac{1}{2}\delta^{-\gamma'}} 4^{-rN} \leq 2^{2N} \delta^{-2\gamma_2 N} \delta^{2\gamma' N} = 2^{2N} \delta^{2N(\gamma' - \gamma_2)} .$$

Choosing any $\gamma_2 \in (\gamma, \gamma')$ and $N > M\gamma'/(2(\gamma' - \gamma_2))$ we get the result.

ii) Let us note that the l.h.s. of (3.2) is bounded from above by the sum

$$4^{-m}\delta^{-2m\rho} \sum_{\delta^{-2\lambda} < 4^r < \delta^{-2\rho}/4} U_r \leq 4^{-m}\delta^{-2m\rho} \sum_{s \in \mathfrak{A}} |\hat{u}_{\delta s}|^2$$

where $\mathfrak{A} = \mathfrak{A}(\gamma_1 - \varepsilon/2, \gamma_2 + \varepsilon/2)$. Thus, choosing in (3.2) $\mu = 1/4$ we find that

$$\sum_{s \in \mathfrak{A}} |\hat{u}_{\delta s}|^2 \geq 4^{m-1} \delta^{2m(\rho - \gamma_1)} .$$

It remains to estimate from above the cardinality $|\mathfrak{A}|$. For δ small, $|\mathfrak{A}|$ differs by a factor $1 + o(1)$ from

$$\int_{\delta^{-\gamma_1 + \varepsilon/2} \leq |x| \leq \delta^{-\gamma_2 - \varepsilon/2}} d^n x = C \int_{\delta^{-\gamma_1 + \varepsilon/2}}^{\delta^{-\gamma_2 - \varepsilon/2}} \rho^{n-1} d\rho = C_1 \big(\delta^{-n\gamma_2 - n\varepsilon/2} - \delta^{-n\gamma_1 + n\varepsilon/2} \big) .$$

Thus, $|\mathfrak{A}| \leq C\delta^{-n\gamma_2 - n\varepsilon/2}$ and

$$\langle |\hat{u}_s|^2 \rangle_{\mathfrak{A}} \geq C' \delta^{2m(\rho - \gamma_1) + n\gamma_2 + n\varepsilon/2} . \tag{3.4}$$

This inequality proves the first estimate in (2.7) for $\delta \in \{\delta_\varepsilon(j) \searrow 0\}$.

The second estimate in (2.7) is trivial. Indeed, since $\sum_{s \in \mathfrak{A}} |\hat{u}_{\delta s}|^2 \leq \|u\|_0^2 \leq C\delta^{-a}$ by (2.4) and since $|\mathfrak{A}| \geq C_1 \delta^{-n\gamma_2 - n\varepsilon/2}$ for δ sufficiently small, then

$$\langle |\hat{u}_{\delta s}|^2 \rangle_{\mathfrak{A}} \leq C'' \delta^{n\gamma_2 + n\varepsilon/2 - a} \quad \forall\, \delta < \delta_0 . \tag{3.5}$$

The first case of the theorem is proven.

2) In this case all the calculations presented above hold for $\mathfrak{A} = \mathfrak{A}(\gamma_1 - \varepsilon/2, \gamma_2 + \varepsilon/2)$ (item ii)) and $\gamma' > \gamma_2$ (item i)), for all sufficiently small δ. This proves the result if we re-denote $\varepsilon/2$ by ε. □

Proof of Amplification. For $\gamma_1 = \gamma - \varepsilon/2$ and $\gamma_2 = \gamma + \varepsilon/2 > 0$ the proof of Theorem 2 (namely, estimates (3.4), (3.5)) provides us with the following bounds for the averaging of $|\hat{u}_{\delta s}|^2$:

$$C' \delta^{n\gamma + n\varepsilon + 3m\varepsilon} \leq \langle |\hat{u}_{\delta s}|^2 \rangle_{\mathfrak{A}} \leq C'' \delta^{n\gamma + n\varepsilon - a} = C'' \delta^{\gamma(n - a/\gamma + n\varepsilon/\gamma)} ,$$

so the assertion follows. □

Proofs of Theorems 1′, 2′ are quite similar and we omit them.

4. Nonlinear Schrödinger equation

In this section we start to study space-scales of solutions $u = u_\delta(t,x)$ for the NLS equation

$$-i\dot{u} = -\delta\nu\Delta u + |u|^{2p} u \ , \tag{4.1}$$

where p is a natural number and

$$\delta \in (0,1) \ ; \ \nu = \nu_{Re} + i\nu_{Im} \in \mathbb{C} \ , \ |\nu| = 1 \ , \ \nu_{Re} \geq 0 \ , \ \nu_{Im} \geq 0 \ .$$

The space-variable x belongs to $\mathbb{R}^n$, $n = 1,2,3$; for some of our results to hold we should assume that the solution u satisfies the odd periodic boundary conditions:

$$u(t,x) \equiv u(t,x_1,\dots,x_j+2,\dots,x_n) = -u(t,x_1,\dots,-x_j,\dots,x_n) \ \ \forall j = 1 \, . \tag{4.2}$$

The equation (4.1) will be studied under smooth initial conditions $u^0(x) = u^0_\delta(x)$:

$$u(0,x) = u^0_\delta(x) \in C^\infty(\mathbb{R}^n). \tag{4.3}$$

The function u^0 is assumed to satisfy the boundary conditions (4.2) if the solution u has to meet them.

For $\nu_{Im} > 0$ the equation (4.1) is dissipative and the problem (4.1) – (4.3) has a unique smooth solution (see e.g. [LO]). In the important special case $\nu = i$ the equation takes the form

$$\dot{u} = \delta\Delta u + i\,|u|^{2p}\, u \tag{4.4}$$

(and can be treated as a system of two nonlinear parabolic equations with a diagonal linear part). For $\nu = 1$ there is no dissipation and the equation takes the hamiltonian form:

$$\dot{u} = -i\delta\Delta u + i\,|u|^{2p}\, u \ . \tag{4.5}$$

In this case the L_2-norm of a solution of (4.5), (4.2) preserves:

$$|u(t)|^2_{L_2} \equiv \int_{K^n} |u(t,x)|^2\, dx = \text{const} \quad \text{if} \quad \nu_{Im} = 0.$$

Otherwise $\nu_{Im} > 0$ and the L_2-norm decays,

$$|u(t)|_{L_2} \leq e^{-tn\pi^2\delta\nu_{Im}} \left|u^0\right|_{L_2} \ , \tag{4.6}$$

since multiplying the equation (4.1) by $\bar{u}$, integrating over K^n and taking the imaginary part we get:

$$\frac{1}{2}\,\frac{d}{dt}\,|u(t)|^2_{L_2} \leq -\delta\nu_{Im}\,|\nabla u(t)|^2_{L_2} \leq -n\pi^2\delta\nu_{Im}\,|u(t)|^2_{L_2} \ .$$

(The second inequality follows from an explicit form for the eigenvalues of the operator $-\Delta$ under the odd periodic boundary conditions, cf. the example in section 1).

4.1. Upper estimates for space-scales of solutions

Our first goal is to show that despite the decaying of the L_2-norm by a time $\gg \delta^{-1}$ (see (4.6)), a solution of (4.1), (4.3) with $u^0(x) \sim 1$ develops a short space scale by the time $\delta^{-1/3}$:

Theorem 3. Let $u^0 = u^0_\delta(x)$ be a smooth function such that

$$\sup_{x\in K^n} \left|u^0_\delta(x)\right| \le C\ , \ \operatorname*{osc}_{K^n} \left|u^0_\delta(\cdot)\right| := \sup_{x,y\in K^n} \left|u^0_\delta(x)\right| - \left|u^0_\delta(y)\right| \ge 1\ , \tag{4.7}$$

and $u = u_\delta(t,x)$, $(t,x) \in [0,\infty) \times K^n$, be a smooth solution of (4.1), (4.3). Then for any $\kappa < 1/3$ and $m \ge 2$ there exists a u^0-independent positive constant $\delta_{m\kappa}$ and $T_0 = T_0(u^0,\delta)$, $0 \le T_0 \le T = \delta^{-1/3}$, such that

$$[u_\delta(T_0,\cdot)]_{C^m(K^n)} \ge \delta^{-m\kappa} \tag{4.8}$$

and $\operatorname{osc}_{K^n}|u_\delta(T_0,\cdot)| \ge 1/2$, provided that $\delta < \delta_{m\kappa}$. No boundary conditions for u on ∂K^n are assumed.

Remark 1. The same result remains true for $T = \delta^{-1/3}/C$ with any $C \ge 1$, if $\delta_{m\kappa} > 0$ is modified accordingly. □

Remark 2. Let the function $u^0(x)$ vanishes somewhere in K^n (e.g., it is an odd periodic function). Then $\sup_{K^n} \left|u^0(x)\right| = \operatorname{osc}_{K^n} \left|u^0(x)\right|$. Let us denote this number by U. The assumption (4.7) reads as $C \ge U \ge 1$. Now the first inequality (U is bounded by a δ-independent constant C) is superficial. Indeed, let $U \ge 1$. After we rescale the time t and the solution u as $t = U^{-2p}\tau$, $u = Uv$, the equation (4.1) takes the form $-iv'_\tau = -\delta U^{-2p}\nu\Delta v + |v|^{2p}\, v$. Besides, $\sup|v(0)| = \operatorname{osc}|v(0)| = 1$. Applying Theorem 3 to the equation for v we get that $u(t,x)$ satisfies inequalities (4.8) in the stronger form:

$$\sup_{0\le t\le \delta^{-1/3}U^{-4p/3}} |u_\delta(t)|_{C^m(K^n)} \ge U^{2pm\kappa+1}\delta^{-m\kappa}\ . \tag{4.9}$$

□

By estimate (4.8), $\ell_x^{\inf}(u_\delta) \le \delta^{1/3}$. Theorem 3 also implies an upper bound for the scale $\ell_x^L(u_\delta)$. To get it some preliminary work has to be done.

Since $|u|_{C^m(K^n)} \le C_m\,\|u\|_{m+2} = C_m\,|u|_{H^{m+2}(K^n)}$ (we recall that $n \le 3$), then for any $m \ge 4$ we have:

$$\sup_{0\le t\le \delta^{-1/3}/2} \|u_\delta(t)\|_m \ge \tilde C_m \delta^{-m\tilde\kappa_m},\ \ \tilde\kappa_m = \frac{m-2}{m}\kappa\ , \tag{4.8'}$$

(we use Remark 1). Let us choose $\kappa = \kappa_m \nearrow 1/3$, then also $\tilde{\kappa}_m \nearrow 1/3$. By (4.8′), segment $[0, T']$, $T' = \delta^{-1/3}/2$, contains a point t_* such that

$$\|u(t_*)\|_m \geq E_m := \tilde{C}_m \delta^{-m\tilde{\kappa}_m}.$$

Since

$$\frac{1}{T'-t_*}\int_{t_*}^{T'} \|u(t)\|_m \, dt \geq \|u(t_*)\|_m - \int_{t_*}^{T'} \left| \frac{d}{dt}\|u(t)\|_m \right| dt$$

(the integral in the r.h.s. is well defined since the function $\|u(t)\|_m$ is absolutely continuous and Lipschitz), then

$$\frac{1}{T'-t_*}\int_{t_*}^{T'} \left(\|u(t)\|_m + (T'-t_*)\left|\frac{d}{dt}\|u(t)\|_m\right|\right) dt \geq E_m \,.$$

Similar estimate holds for the integral from 0 to t_* and we get that

$$\frac{1}{T'}\int_0^{T'} \left(\|u(t)\|_m + T'\left|\frac{d}{dt}\|u(t)\|_m\right|\right) dt \geq E_m \,. \tag{4.10}$$

For any real number u we denote $u_+ = \max(0, u)$ and $u_- = -\min(0, u)$ (so $u_\pm \geq 0$ and $u = u_+ - u_-$). We need the following corollary from (4.10):

Lemma 2. If in addition to the assumptions of Theorem 3 we have

$$\|u_\delta^0\|_m \leq C_m',$$

then

$$\frac{1}{T}\int_0^T \left(\|u(t)\|_m + \delta^{-1/3}\big(\frac{d}{dt}\|u(t)\|_m\big)_+\right) dt \geq C_m \delta^{-m\tilde{\kappa}_m}$$

with $T = \delta^{-1/3}$, for all $\delta < \delta_m(\|u^0\|_m)$.

Proof. For short we denote $f(t) = \|u(t)\|_m$. We may assume that $f(t') \leq E_m/3$ for some point $t' \in [T', 2T']$ — otherwise the estimate with $C_m = \tilde{C}_m/3$ is obvious. Since $\|u_\delta^0\|_m \leq C_m'$, then also $f(0) \leq E_m/3$. Hence, $|f(t') - f(0)| = \left|\int_0^{t'} f'(t)dt\right| \leq \frac{2}{3}E_m$, or

$$\left|\int_0^{t'} \big(f'(t)\big)_+ dt - \int_0^{t'} \big(f'(t)\big)_- dt\right| \leq \frac{2}{3}E_m \,.$$

Since $t' \geq T'$, then by (4.10)

$$2\delta^{1/3}\int_0^{t'} \big(f + \frac{1}{2}\delta^{-1/3}(f')_+ + \frac{1}{2}\delta^{-1/3}(f')_-\big) dt \geq E_m \,.$$

Combining the last two estimates we get that

$$2\delta^{1/3}\int_0^{t'}\big(f+\delta^{-1/3}(f')_+\big)dt \geq E_m - \frac{2}{3}E_m = \frac{1}{3}E_m\ ,$$

and the lemma is proven with $C_m = \frac{1}{6}\tilde{C}_m$. □

To make the next step we assume that u is a smooth solution of (4.1) – (4.3), where $u^0(x)$ satisfies (4.7) and $\|u_0\|_m \leq C'_m$. Let us denote by H^m_{op} the subspace of odd periodic functions from $H^m_{loc}(\mathbb{R}^n;\mathbb{C})$ (these functions satisfy (4.2)). We supply the space H^m_{op} with the homogeneous Hilbert norm $\|\cdot\|_m$, where

$$\|u\|^2_m = \langle u,u\rangle_m,\quad \langle u,v\rangle_m = 2^{-n}\int_{\mathbb{T}^n}\big((-\Delta)^m u\big)\bar{v}dx$$

(the factor 2^{-n} stands to normalise the measure dx on $\mathbb{T}^n$). Multiplying (4.1) by $u(t)$ in H^m_{op} we get:

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_m = -\delta\nu_{Im}\|u\|^2_{m+1} + \mathrm{Im}\langle |u|^{2p}u,u\rangle_m \leq \left\||u|^{2p}u\right\|_m \|u\|_m\ .$$

Hence, $(d/dt)\|u\|_m \leq \left\||u|^{2p}u\right\|_m$ and $(d/dt\,\|u\|_m)_+ \leq \left\||u|^{2p}u\right\|_m$. Using Lemma 2 we find that

$$\frac{1}{T}\int_0^T\big(\|u\|_m + \delta^{-1/3}\left\||u|^{2p}u\right\|_m\big)dt \geq C_m\delta^{-m\tilde{\kappa}_m}\ , \tag{4.11}$$

if δ is sufficiently small. Since

$$\left\||u|^{2p}u\right\|_m \leq C_\varepsilon\|u\|_m\|u\|^{2p}_{n/2+\varepsilon/2p} \leq C\,C_\varepsilon\|u\|_m\|u\|_0^{2p-\frac{pn+\varepsilon}{m}}\|u\|_m^{\frac{pn+\varepsilon}{m}}$$

(for the first inequality which holds with any $\varepsilon>0$ see e.g. Appendix 1 in [K1], the second is an interpolation, see Appendix 1.A) and since L_2-norm of the solution u decays (see (4.6)), then we have:

$$\left\||u|^{2p}u\right\|_m \leq C\,C_1\left\|u^0\right\|_0^{2p-\frac{pn+1}{m}}\|u\|_m^{1+\frac{pn+1}{m}} \leq C\,C_1\|u\|_m^{1+\frac{pn+1}{m}}\ .$$

Thus,

$$\frac{1}{T}\int_0^T\big(\|u\|_m + \delta^{-1/3}\|u\|_m^{1+O(m^{-1})}\big)dt \geq C'_m\delta^{-m\tilde{\kappa}_m} =: F_m$$

with $O(m^{-1}) = (pn+1)/m$. Abbreviating $\frac{1}{T}\int_0^T\dots dt$ to $\int\dots dt$, we see that either $\int\|u\|_m\,dt \geq \frac{1}{2}F_m$, or $\int\delta^{-1/3}\|u\|_m^{1+O(m^{-1})}\,dt \geq \frac{1}{2}F_m$. In the second case we use the Hölder inequality with $p=2/\big(1+O(m^{-1})\big)$ to get that $\int\|u\|_m^{1+O(m^{-1})}\,dt \leq \big(\int\|u\|^2_m\,dt\big)^{\frac{1+O(m^{-1})}{2}}\cdot 1$ and

$$\int\|u\|^2_m\,dt \geq \big(\frac{1}{2}F_m\delta^{1/3}\big)^p = C_m\delta^{-2m\frac{\tilde{\kappa}_m}{1+O(m^{-1})}} =: C_m\delta^{-2m\kappa'_m}\ ,$$

where $\kappa'_m \nearrow 1/3$. The case $\int \|u\|_m \, dt \geq \frac{1}{2} F_m$ implies a better estimate and we get:

Theorem 4. Let $u_\delta(t,x)$ be a smooth solution of the problem (4.1) – (4.3), where $\|u_\delta^0\|_m \leq C'_m$ and $\operatorname{osc}_{K^n} |u^0| \geq 1$. Then the following estimates hold for $m \geq 4$ with some $\kappa'_m \nearrow \frac{1}{3}$ and $C_m > 0$:

$$\delta^{1/3} \int_0^{\delta^{-1/3}} \|u\|_m^2 \, dt \geq C_m \delta^{-2m\kappa'_m} \quad \text{if} \ \ \delta \leq \delta_m \, .$$

Due to this result, $\ell_x^{[0,\delta^{-1/3}]}(u_\delta) \leq \delta^{1/3}$. Using Proposition 5 we find that $\ell_x^L(u_\delta) \leq \delta^{1/3}$ for any $L = [0, \delta^{-b}]$, $b \geq \frac{1}{3}$.

4.2. Equations with pure dissipative linear part

The results of Theorem 4 are not too far from optimal, at least when $\nu_{Re} = 0$ and the equation (4.1) takes the form (4.4):

Theorem 5. For any smooth function $u^0(x)$ the problem (4.2) – (4.4) has a unique smooth solution $u_\delta(t,x)$. This solution satisfies the following estimates:

$$|u_\delta(t,\cdot)|_{C^0} \leq \min\left(\left|u^0\right|_{C^0}, K_1 e^{-\delta t K} \left|u^0\right|_{C^0}\right), \quad K_1 = 2^{n/2}, \ K = \pi^2 n/4 \, , \tag{4.12}$$

and *

$$\|u_\delta(t,\cdot)\|_m \leq C_m \delta^{-m/2} e^{-\delta t K(1+mp)} \, , \tag{4.13}$$

for any $m \geq 0$ and $\delta > 0$.

The theorem is proven is section 6.

The last two theorems and (1.7) show that if $u_\delta(t,x)$, $t \in L = [0, \delta^{-b}]$, is a solution for the problem (4.2) – (4.4) and $|u^0|_{C^0} = 1$, then

$$\delta^{1/2} \leq \ell_x^{\inf}(u_\delta) \leq \ell_x^L(u_\delta) \leq \delta^{1/3} \quad \text{provided that} \quad b \geq \frac{1}{3} \tag{4.14}$$

(we recall that $|u^0|_{C^0} = \operatorname{osc} |u^0|$ for any odd periodic function u^0).

Now let us assume that a function $u(t,x)$ is a solution for (4.4) which is odd $2M$-periodic (rather than 2-periodic as before) and $\sup |u^0(x)| = U$. Let both M and U be of the form $C\delta^d$. The substitution $x = My, t = U^{-2p}\tau, u = Uv$ implies for the odd 2-periodic function $v(\tau, y)$ equation (4.4) with δ replaced by $\delta' = \delta M^{-2} U^{-2p}$. Applying to v estimates (4.14), we find that the solution $u(t,x)$ with $t \in L = [0, \delta^{-b}]$, where $\delta^{-b} \geq \delta^{-1/3} M^{2/3} U^{-4p/3}$, is such that

$$\delta^{1/2} U^{-p} \leq \ell_x^L(u) \leq \left(\delta M U^{-2p}\right)^{1/3} , \tag{4.15}$$

---

* see the footnote on p.28

provided that $\delta' < 1$. (The first estimate in (4.15) means that $\ell_x^L \geq \delta^{1/2-d_1p}$, where $U = C_1\delta^{d_1}$. Similar with the second.)

If we choose $t > N\delta^{-1}\ell n\,\delta^{-1}$ with sufficiently big N, then the third factor in the r.h.s. of (4.13) becomes smaller than $\delta^{LK(1+mp)}$, where L grows to infinity with N. Since $p \geq 1$, then for any $c \geq 0$, $\|u_\delta(t)\|_m \leq C_m\delta^{cm}$ if $t > N(c)\delta^{-1}\ell n\,\delta^{-1}$. It means that in contrast to (4.14), $\ell_x^L(u_\delta) \geq \delta^{-c}$ if $L = [T_1, T_2]$ with $T_2 > T_1 \gg \delta^{-1}\ell n\,\delta^{-1}$.

Now Theorems $1', 2'$ are applicable to study spectral properties of solutions for problem (4.2) – (4.4). In particular, the following results hold true:

Corollary 2. Let $u_\delta(t,x) = \sum \hat{u}_{\delta s}(t)e^{i\pi s\cdot x}$, where $t \in L = [0, \delta^{-b}]$ with $b \geq 1/3$, be a smooth solution of the problem (4.2) – (4.4) with $|u^0|_{C^0} = 1$. Then:

1) for any $\gamma' > 1/2$ and any M, $\sum_{|s|\geq\delta^{-\gamma'}} |\hat{u}_{\delta s}|_L^2 \leq \delta^{M\gamma'}$ if $\delta \leq \delta_M$;

2) for any $\varepsilon > 0$,

$$\delta^c \leq |\mathfrak{A}(1/3-\varepsilon, 1/2+\varepsilon)|^{-1} \sum_{s\in\mathfrak{A}(1/3-\varepsilon,1/2+\varepsilon)} |\hat{u}_{\delta s}|_L^2 \leq \delta^C$$

with some finite $c(\varepsilon)$, $C(\varepsilon)$ and for each $\delta < \delta_\varepsilon$;

3) $\gamma_L(u_\delta) \in [1/3, 1/2]$ and

$$(\ell_x^L)^{\tilde{c}(\varepsilon)} \leq \left|\mathfrak{A}(\ell_x^L - \varepsilon, \ell_x^L + \varepsilon)\right|^{-1} \sum_{s\in\mathfrak{A}(\ell_x^L-\varepsilon,\ell_x^L+\varepsilon)} |\hat{u}_{\delta s}|_L^2 \leq (\ell_x^L)^{n+o(1)}$$

with some finite $\tilde{c}(\varepsilon)$ and for each $\delta \in \{\delta_\varepsilon(j) \searrow 0\}$.

5. Proof of Theorem 3

5.1. The polar coordinate representation

We write a solution $u = u_\delta(t,x)$ of equation (4.1) as $u = re^{i\varphi}$, $r \geq 0$. The phase $\varphi(t,x)$ is defined modulo $2\pi\mathbb{Z}$; we fix any its choice, continuous in t and x outside the zero-set $\Sigma = r^{-1}(0)$.

Proofs of Theorems 3 and 5 use equations for the real functions r and φ:

Lemma 3. If $u = re^{i\varphi}$ satisfies (4.1), then for $(t,x) \notin \Sigma$ we have:

$$\dot{r} = \delta\,\mathrm{Im}\,\nu\Psi = \delta\nu_{Im}\Psi_{Re} + \delta\nu_{Re}\Psi_{Im}\ , \tag{5.1}$$

$$\dot{\varphi} = r^{2p} - \frac{\delta}{r}\mathrm{Re}\,\nu\Psi = r^{2p} + \frac{\delta}{r}\nu_{Im}\Psi_{Im} - \frac{\delta}{r}\nu_{Re}\Psi_{Re}\ , \tag{5.2}$$

where $\Psi = \Psi_{Re} + i\Psi_{Im}$ with $\Psi_{Re} = \Delta r - r\,|\nabla\varphi|^2$ and $\Psi_{Im} = r\Delta\varphi + 2\nabla r\cdot\nabla\varphi$.

Proof. Since

$$\Delta u = e^{i\varphi}(\Delta r - r\,|\nabla\varphi|^2) + ie^{i\varphi}(r\Delta\varphi + 2\nabla r\cdot\nabla\varphi) = e^{i\varphi}\Psi,$$

then we can rewrite (4.1) as

$$\dot r e^{i\varphi} + i\dot\varphi r e^{i\varphi} \equiv \dot u = -i\delta\nu\Psi e^{i\varphi} + ir^{2p+1}e^{i\varphi}.$$

Comparing coefficients in front of $e^{i\varphi}$ and $ie^{i\varphi}$ in the l.h.s. and the r.h.s. we get that $\dot r = \delta\,\mathrm{Im}\,\nu\Psi$ and $\dot\varphi r = r^{2p+1} - \delta\,\mathrm{Re}\,\nu\Psi$. □

In particular, if u solves (4.5), then $\nu = i$ and the equations for r and φ take the forms:

$$\dot r = \delta\Delta r - \delta r\,|\nabla\varphi|^2\ , \tag{5.3}$$

$$\dot\varphi = r^{2p} + \delta\Delta\varphi + 2\frac{\delta}{r}\nabla r\cdot\nabla\varphi\ . \tag{5.4}$$

5.2. Proof of the theorem

Till the end of the proof we fix any $m \geq 2$. We may assume that $[u^0]_{C^m(K^n)} < \delta^{-m\kappa}$ — otherwise we have nothing to prove. Then by (4.7) and the interpolation inequality (A4) (see Appendix 1.B),

$$|\nabla u^0(x)| \leq C\delta^{-\kappa} \quad \forall x\in K^n.$$

By (4.7), $\sup|u^0| \geq 1+\min|u^0|$. Let $x'\in K^n$ be any point such that $|u^0(x')| \geq 1+\min|u^0|$. Using again (4.7) we find a cube $K\subset K^n$ such that $x'\in K$ and

$$C \geq \left|u^0(x)\right| \geq \frac12 \quad \forall x\in K\ , \quad \operatorname*{osc}_K \left|u^0\right| = \frac12\ . \tag{5.5}$$

Since $\mathrm{osc}_K\left|u^0\right| = 1/2$ and $\nabla|u^0| \leq C\delta^{-\kappa}$, then

$$\sqrt n \geq \mathrm{diam}\,K \geq \frac{\delta^\kappa}{2C}\ . \tag{5.6}$$

First we shall show that

$$\sup_{0\leq t\leq T}[u(t)]_{C^m(K)} \geq \delta^{-m\kappa} \quad \forall\,\delta<\delta_{m\kappa}, \tag{5.7}$$

where $T=\delta^{-1/3}$. To prove (5.7) we assume the opposite, i.e., that for some $\delta<\delta_{m\kappa}$ we have

$$[u(t)]_{C^m(K)} \leq R^m \quad \forall\, t\in[0,T]\ , \tag{5.8}$$

where $R=\delta^{-\kappa}$, and derive from (5.8) a contradiction. Below we abbreviate $[\cdot]_{C^m(K)}$ to $[\cdot]_m$; we note that $[\cdot]_0 = |\cdot|_{C^0}$.

We write u and u^0 as $u = re^{i\varphi}$ and $u^0 = r^0 e^{i\varphi^0}$. Let $T_1 \le T$ be the biggest number such that

$$\left| r(t,x) - r^0(x) \right| \le \frac{1}{4} \quad \forall\, 0 \le t \le T_1\,, \quad x \in K\,. \tag{5.9}$$

By (5.5) and (5.9),

$$C_1 \ge r(t,x) = |u(t,x)| \ge \frac{1}{4} \quad \forall\, 0 \le t \le T_1\,,\ x \in K\,. \tag{5.10}$$

Interpolating the first inequality with (5.8) (see Appendix 1.B3) and using (5.6) we get that

$$[u(t)]_k \le C_k\big((\operatorname{diam} K)^{-k}[u(t)]_0 + [u(t)]_0^{(m-k)/m}[u(t)]_m^{k/m}\big) \le CR^k$$

for $t \le T_1$ and $0 \le k \le m$. Since $|u(t,x)| \ge 1/4$ (see (5.10)), then also

$$[r(t)]_k \le CR^k \quad \forall t \le T_1\,,\quad 0 \le k \le m\,, \tag{5.11}$$

and

$$[\nabla\varphi(t)]_{k-1} \le CR^k \quad \forall t \le T_1\,,\quad 1 \le k \le m\,. \tag{5.12}$$

Lemma 4. If $P(t,x) = r^a \partial_x^{\alpha_1} r \ldots \partial_x^{\beta_1}\varphi \ldots$, where a is an integer and the dots stand for finite products of similar derivatives with $|\alpha_j| = \ell_j$ $(0 \le \ell_j \le m)$ and $|\beta_j| = r_j$ $(1 \le r_j \le m)$, then for $0 \le t \le T_1$ we have $|P(t)|_0 \le CR^\mu$, where $\mu = \sum \ell_j + \sum r_j$.

Proof. We should show that

$$\left| r^a \partial_x^{\alpha_1} r \ldots \partial_x^{\beta_1}\varphi \ldots \right| \le CR^\mu \tag{5.13}$$

for $0 \le t \le T_1$ and $x \in K$. Since, first, the total number of derivatives in the l.h.s. of (5.13) equals μ and any single derivative contributes the factor CR to an upper estimate for the sup-norm (see (5.11), (5.12)), and, second, both r and r^{-1} are bounded by a constant (see (5.10)), then (5.13) follows. □

In the domain $[0, T_1] \times K$ the functions r and φ satisfy equations (5.1), (5.2). Using the last lemma we estimate the r.h.s.'s of these equations as follows:

$$[\delta \operatorname{Im} \nu \Psi(t)]_0 \le C\delta R^2\,, \tag{5.14}$$

$$\left[\frac{\delta}{r} \operatorname{Re} \nu \Psi(t)\right]_0 \le C\delta R^2\,. \tag{5.15}$$

By (5.1) and (5.14),

$$\left| r(t,x) - r^0(x) \right| \le CT_1 \delta R^2 \le C\delta^{2(\frac{1}{3}-\kappa)}\,. \tag{5.16}$$

Since $\kappa < 1/3$, then the r.h.s. is smaller than $1/5$ if $\delta < \delta_m$ with sufficiently small δ_m. It shows that

$$T_1 = T\,. \tag{5.17}$$

Hence, estimates (5.8) – (5.16) hold for $0 \le t \le T$.

By (5.5) there exist points $x_1, x_2 \in K$ such that $(r^0)^{2p}(x_1) - (r^0)^{2p}(x_2) \ge 2C_* > 0$. Hence, $r^{2p}(t, x_1) - r^{2p}(t, x_2) \ge C_*$ for $0 \le t \le T$ (see (5.16) and (5.10)). Using equation (5.2) and the last estimate we get that

$$\left|\left(\varphi(T, x_1) - \varphi(T, x_2)\right) - \left(\varphi^0(x_1) - \varphi^0(x_2)\right)\right| \ge T\left(C_* - 2 \sup_{t \le T, x \in K} \left|\frac{\delta}{r}\mathrm{Re}\,\nu\Psi\right|\right). \qquad (5.18)$$

By (5.15), $C_* - 2\sup\left|\frac{\delta}{r}\mathrm{Re}\,\nu\Psi\right| \ge C_* - 2C\delta R^2 \ge C_*/2$ if δ is small. From other hand, by (5.6) and (5.12) the l.h.s. in (5.18) is bounded by $2\sqrt{n}CR$ and we arrive at the inequality

$$2\sqrt{n}CR = 2\sqrt{n}C\delta^{-\kappa} \ge TC_*/2 = \delta^{-1/3}C_*/2\,.$$

Since $\kappa < 1/3$, then we got a contradiction, provided that δ is sufficiently small. This contradiction proves (5.7).

By (5.9) and (5.17) we get that $\mathrm{osc}_K|u(T_0, \cdot)| \ge 1/2$. So also $\mathrm{osc}_{K^n}|u(T_0, \cdot)| \ge 1/2$ and the theorem is proven. □

Remark. Let $u_\delta(t, x)$ be an odd periodic solution of (4.1) such that $|u^0|_{C^0(K^n)} = 1$. Then $\mathrm{osc}\,|u^0| = 1$ and Theorem 3 is applicable. Let $T_0 \le \delta^{-1/3}$ be the first moment when (4.8) holds. We claim that

$$\frac{1}{2} \le |u_\delta(t)|_{C^0(K^n)} \le \frac{3}{2} \quad \forall\, 0 \le t \le T_0. \qquad (5.19)$$

Indeed, the first inequality is proven already. To prove the second we assume that it is violated and find a point $(t_0, x_0), t_0 \le T_0$, such that $|u_\delta(t_0, x_0)| = 3/2$ and $|u_\delta(t)|_{C^0} < 3/2$ for $t < t_0$. Next we find any point $t_1 \in [0, t_0]$ where $|u_\delta(t_1, x_0)| = 1$ and consider equation (4.1) for $t \ge t_1$ with $u_\delta(t_1, \cdot)$ as a new initial condition. Since $3/2 \ge \mathrm{osc}\,|u_\delta(t_1)| = |u_\delta(t_1)|_{C^0} \ge 1$, then we can apply to this initial-value problem our proof of Theorem 3. Doing this we choose $x' = x_0$, construct a new cube $K \ni x'$ and find $T_0' \le t_1 + \delta^{-1/3}$ such that $|u(T_0')|_{C^m(K)} = R^m$. By (5.9), $|u(t, x_0)| \le 5/4$ for $0 \le t \le T_0'$. Since we must have $T_0' \ge T_0$, then $|u_\delta(t_0, x_0)| \le 5/4$. This contradiction proves the second estimate in (5.19). □

6. Proof of Theorem 5

To prove (4.12) we write $u = re^{i\varphi}$. The function $r \ge 0$ is continuous in the cylinder $\Pi = [0, \infty) \times K^n$; outside the zero-set $\Sigma = r^{-1}(0)$ it is smooth and satisfies equation (5.3):

$$\dot{r} = \delta\Delta r - \delta r \left|\nabla\varphi\right|^2, \quad (t, x) \in \Pi\backslash\Sigma\,.$$

The function

$$\xi(t, x) = \left|u^0\right|_{C^0} e^{-\delta t n\pi^2/4} \prod_{j=1}^{n} \sqrt{2}\cos\frac{\pi}{2}(x_j - \frac{1}{2})$$

is positive in Π and solves there the equation $\dot{\xi} = \delta\Delta\xi$. Besides, $\xi \geq |u^0|_{C^0}$ in $\{0\} \times K^n$. Now let us consider the function $h = \xi - r$. In $\Pi\backslash\Sigma$ we have: $\dot{h} = \delta\Delta h + \delta r\,|\nabla\varphi|^2$. Since h is nonnegative on the boundary

$$\partial(\Pi\backslash\Sigma) = \big([0,\infty) \times \partial K^n\big) \cup \partial\Sigma \cup \big(\{0\} \times K^n\big)$$

($\partial\Sigma$ is the boundary of Σ in Π) and since $\delta r\,|\nabla\varphi|^2 \geq 0$, then by the maximum principle $h \geq 0$ in $\Pi\backslash\Sigma$. (We apply the principle to a parabolic equation in arbitrary domain. In the form we need it can be found e.g. in [La].). It means that everywhere in Π we have:

$$r(t,x) \leq \xi(t,x) \leq 2^{n/2}\left|u^0\right|_{C^0} e^{-\delta t n\pi^2/4}\,.$$

Replacing in these arguments $\xi(t,x)$ by the constant function equal $|u^0|_{C^0}$ we get that $|u(t,x)| \leq |u^0|_{C^0}$ everywhere in Π. So (4.12) is proven.

Now we prove (4.13). For $m = 0$ the estimate follows from (4.12) (or from (4.6)). To prove it for $m \geq 1$ we apply to (4.4) any operator $\partial^\alpha/\partial x^\alpha$ with $|\alpha| = m$, multiply the equation by $\partial^\alpha\bar{u}/\partial x^\alpha$, integrate over $\mathbb{T}^n$ against the measure $2^{-n}dx$ and take imaginary part of the result. Summing up the obtained relations with all $|\alpha| = m$ we get:

$$\frac{d}{dt}\,\|u\|_m^2 \leq -\delta\,\|u\|_{m+1}^2 + 2^{-n}\sum_{|\alpha|=m}\;\sum_{\alpha_1+\cdots+\alpha_{2p+1}=\alpha}\int |u_1|\cdots|u_{2p+2}|\,dx\,, \tag{6.1}$$

where $u_j = \partial^{\alpha_j}u/\partial x^{\alpha_j}$ for $j \leq 2p+1$ and $u_{2p+2} = \partial^\alpha u/\partial x^\alpha$. Next we estimate any term I in the sum in the r.h.s. of (6.1), $I = \int |u_1|\cdots|u_{2p+2}|\,dx$. By the Hölder estimate,

$$I \leq |u_1|_{L_{r_1}}\cdots|u_{2p+2}|_{L_{r_{2p+2}}}\,, \tag{6.2}$$

where $r_j = \frac{2(m+1)}{m_j}$ for $j \leq 2p+1$ and $r_{2p+2} = \frac{2(m+1)}{m+2}$. Let us apply to each factor in the r.h.s. of (6.2) the Gagliardo–Nirenberg inequality [Ni]:

$$|u_j|_{L_{r_j}} \leq C\big(|u|_{L_\infty}^{1-a_j}\,\|u\|_{m+1}^{a_j} + |u|_{L_\infty}\big) \leq C_1\,|u|_{L_\infty}^{1-a_j}\,\|u\|_{m+1}^{a_j} \tag{6.3}$$

(the second inequality holds since $|u|_{L_\infty} \leq C\,\|u\|_2 \leq C\,\|u\|_{m+1}$). The exponent a_j is such that

$$a_j \geq \frac{m_j - n/r_j}{(m+1) - n/2}\,,\quad a_j \geq \frac{m_j}{m+1} \tag{6.4}$$

with $m_j = |\alpha_j|$ (so $m_1 + \cdots + m_{2p+1} = m_{2p+2} = m$). We take $a_j = \frac{m_j}{m+1}$ (this choice satisfies (6.4)) and get from (6.2), (6.3) that

$$I \leq C\,|u|_{L_\infty}^{2p+2-A}\,\|u\|_{m+1}^A\,,\quad A = a_1 + \cdots + a_{2p+2} = \frac{2m}{m+1}\,. \tag{6.5}$$

Now (4.12), (6.1) and (6.5) imply the following differential inequality:

$$\frac{d}{dt}\,\|u\|_m^2 \leq -\delta\,\|u\|_{m+1}^2 + C_m e^{-\delta t K(2p+2-A)}\,\|u\|_{m+1}^A\,.$$

Hence, * the norm $\|u\|_m$ decreases with t if $\delta \|u\|_{m+1}^2 \geq C_m e^{-\delta t K(2p+2-A)} \|u\|_{m+1}^A$, i.e., if

$$\|u\|_{m+1} \geq C'_m \delta^{-1/(2-A)} e^{-\delta t K \frac{2p+2-A}{2-A}} .$$

If $\|u\|_m$ increases, than the last inequality must hold with the opposite sign and

$$\begin{aligned} \|u\|_m &\leq \|u\|_0^{1/(m+1)} \|u\|_{m+1}^{m/(m+1)} \leq C_m e^{-\delta t K \frac{1}{m+1}} \delta^{-m/2} e^{-\delta t K \frac{m(pm+p+1)}{m+1}} \\ &= C_m \delta^{-m/2} e^{-\delta t K(1+pm)} . \end{aligned} \tag{6.6}$$

Since (6.6) holds for $t = 0$ (if δ is sufficiently small), then $\|u\|_m$ never can surpass the r.h.s. of (6.6) and (4.13) follows. □

7. Randomly forced NLS equations

In this section we discuss the NLS equation (4.1) forced by a random force and estimate space-scales of its solutions. Now a solution is a random field and the randomness has to be incorporated into a definition of the space-scale. We do this in the easiest way and just replace in (1.5′) the squared function norm by its expectation. We denote thus defined averaged space-scale as $\ell_x^{L,\mathrm{E}} = \delta^{\gamma_{\mathrm{E}}^L}$. See Appendix 2 for the exact definition and for main properties of the scale $\ell_x^{L,\mathrm{E}}$.

7.1. Preliminaries

We consider perturbations of equation (4.1) with $p = 1$ and $\nu = i$ (the first assumption is made to simplify the presentation while the second is needed for Theorem 7 below, but not for Theorem 6):

$$\dot{u} - \delta \Delta u - i |u|^2 u = \zeta^\omega(t, x) . \tag{7.1}$$

Everywhere in this section solutions u are assumed to be odd periodic in x.

The force ζ in (7.1) is a random field with an underlying probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where the measure $\mathbf{P}$ is assumed to be complete. This random field is assumed to admit a realization such that for a.a. ω the function ζ^ω is measurable in (t, x), smooth odd periodic in x, bounded locally uniformly in t and uniformly in x. Besides, for any x the random process $\zeta^\omega(\cdot, x)$ is stationary with integrable correlation and all moments of the random variable $|\zeta^\omega(t, \cdot)|_{C^m}$ are finite for any m. More specifically this means that ζ satisfies assumptions (H1), (H2) from [K2] together with the following one:

---

* **Correction** (made in November 2001). The arguments below are wrong. They become correct if we replace $e^{-\delta t K(2p+2-A)}$ by one. So, $\|u(t, \cdot)\|_m \leq C_m \delta^{-m/2}$ for all t. Interpolating (4.12) and the last inequality with m replace by $M \gg 1$, we find that

$$\|u_\delta(t, \cdot)\|_m \leq C'_m \delta^{-m/2} e^{-\delta t K'}, \tag{4.13′}$$

for any fixed $K' < K$.

(H3) *(finite moments)*. For any m and M,

$$\mathbf{E}\big(|\zeta^\omega(t)|^M_{C^m}\big) \le C_{m,M} \ . \tag{7.2}$$

Examples 1, 3 and 4, given in section 6 of [K2], satisfy the assumptions (H1) – (H3) (as well as Example 2, if there the coefficients z_s are of the form $z_s = o\big(|s|^{-M}\big)$ for any M). For the reader's convenience we repeat here Example 1 from [K2]:

Example. Let $\zeta^\omega(t,x) = \zeta^\omega(t)w(x)$, where $\zeta^\omega(t)$ is a stationary Gaussian process (real or complex) with an integrable correlation and a spectral density which is positive at zero. The complex function $w(x)$ is smooth odd periodic and its gradient, restricted to ∂K^n, does not vanish identically.

Validity of the assumptions (H1), (H2) for the process ζ^ω is checked in [K2]; (H3) is now obvious. □

We supplement the equation by zero initial conditions:

$$u(0,x) = 0 \ . \tag{7.3}$$

(Our results remain true if (7.3) is replaced by some mild restrictions on the random function $u^\omega(0,x)$. For example, by the following: $|u^\omega(0,\cdot)|_{C^0} \le C$ a.s. (= almost surely with respect to the measure $\mathbf{P}$) and $\mathbf{E}\,|u^\omega(0,\cdot)|^2_{H^m} \le C_m\delta^{-3m}$ for $m \ge 1$).

We define an (odd periodic) strong random solution $u = u^\omega_\delta(t,x)$ ($t \ge 0, x \in \mathbb{R}^n$) of (7.1), (7.3) as an $\mathcal{F}$-measurable random field such that for all ω outside a zero-measure set $\Omega_u \in \mathcal{F}$ the function $u^\omega_\delta(\cdot,\cdot)$ is a strong odd periodic solution of (7.1), (7.3). Namely,

i) u^u_δ is Lipschitz in (t,x) (locally uniformly in t, uniformly in x) and C^2-smooth in x;

ii) it satisfies (7.1), (7.3) (the derivative $\dot u(t,x)$ is defined Lebesgue – almost everywhere by the Rademacher theorem).

If $\zeta^\omega(\cdot,\cdot)$ was a continuous function, then we would define a strong solution in the usual way, i.e., as a C^1-smooth in time, C^2-smooth in space function which satisfies the equation and the initial conditions. But the class of discontinuous in time forces ζ is too important and convenient to be neglected, so we stuck to the definition i), ii).

By our assumptions, the function $\zeta^\omega(\cdot,\cdot)$ is bounded locally in t and is smooth in x for $\omega \notin \Omega_0$ with a zero-measure subset Ω_0. For any fixed $\omega \notin \Omega_0$,

1) the equation (7.1) (considered as a deterministic equation with the smooth in x r.h.s. $\zeta = \zeta^\omega$) has a unique solution $u(t,x)$ which meets (7.3). We denote it as $u = \mathcal{U}(\zeta)$.

For any fixed $T > 0$ we set $Q = [0,T] \times K^n$.

2) The restriction $u|_Q$, in the norm of the space $\mathcal{H}_2 := L_\infty\big([0,T], C^2(K^n)\big)$, continuously depends on $\zeta|_Q$ in the norm of the space $\mathcal{H}_\ell$ with some $\ell \ge 2$ (one can take $\ell = 4$, but this is irrelevant for us now),

3) $u|_Q$ does not depend on $\zeta|_{[T,\infty)\times K^n}$ and $\zeta|_{(-\infty,0]\times K^n}$.

— These are commonplace results which follow, for example, from [LO], or can be obtained by means of deterministic versions of the estimates we use below to prove Theorem 7.

Now we define a random field u^ω as $u^\omega = \mathcal{U}(\zeta^\omega)$ if $\omega \notin \Omega_0$ and $u^\omega = 0$ if $\omega \in \Omega_0$. Since the map $\mathcal{U}$ is continuous, then u^ω is $\mathcal{F}$-measurable, so it is a strong random solution of (7.1), (7.3) with $\Omega_u = \Omega_0$. By 1), the solution is a.s. uniquely defined.

Remark. By 3) the strong random solution u^ω is progressively measurable. That is, any random variable $u^\omega(t,x)$ is measurable with respect to the σ-algebra $\mathcal{F}_{[0,t]}$ which is generated by the random variables $\zeta^\omega(\tau,y)$ with $0 \le \tau \le t$ and $y \in K^n$. □

Our goal in next subsections is to study strong random solutions u_δ^ω, which from now on we call just "solutions".

7.2. Lower estimates

In [K2] (see there Theorem 6 and estimate (5.8)) we prove the following result:

Theorem 6. For any $t_0 \ge 0$, $L \ge \delta^{-1}$ and $m \ge 6$ a solution $u = u_\delta^\omega(t,x)$ for the problem (7.1), (7.3) with $0 < \delta \le 1$ satisfies the following estimate with a probability $\ge 1/2$:

$$\frac{1}{L}\int_{t_0}^{t_0+L}\left(\|u\|_m + \delta^{-1}\left\||u|^2 u\right\|_m\right)dt \ge C_m^{-1}\delta^{-\frac{3}{17}(m-2)}\,. \tag{7.4}$$

To get from (7.4) a lower bound for $\|u\|_m^2$, we need an upper bound for $|u(t)|_{C^0}$:

Lemma 5. The following estimate holds for all $t \ge 0$ with any $m \in \mathbb{N}$:

$$\mathbf{E}\,|u_\delta^\omega(t,\cdot)|_0^m \le C_m\delta^{-m} \quad \text{if} \quad 0 < \delta \le 1\,. \tag{7.5}$$

Proof. As in the proof of estimate (4.12) in Theorem 5, we write the solution u_δ^ω in the polar form: $u_\delta^\omega = r^\omega e^{i\varphi^\omega}$. The function $r = r^\omega \ge 0$ is as smooth as u outside the zero-set $\Sigma = \Sigma^\omega = u^{-1}(0) \subset \Pi = [0,\infty) \times K^n$ (we note that Σ contains $\{0\} \times K^n$). In $\Pi \backslash \Sigma$ it satisfies the equation

$$\dot r = \delta\Delta r - \delta r\,|\nabla\varphi|^2 + \mathrm{Re}\big(\zeta^\omega e^{-i\varphi}\big),$$

cf. Lemma 3. We consider the function $\mu(x) = \prod_{j=1}^n \cos\frac{\pi}{2}\left(x_j - \frac{1}{2}\right)$ (it is $\ge 2^{-n/2}$ in K^n), denote by $\eta^\omega(t)$ the random process $\eta^\omega(t) = |\zeta^\omega(t)|_{C^0}$ and consider the following equation in Π:

$$\dot\xi = \delta\Delta\xi + 2^{n/2}\eta^\omega(t)\mu(x)\,.$$

Since $\Delta\mu = -n\pi^2\mu/4$, then the function

$$\xi(t,x) = 2^{n/2}\mu(x)\int_0^t \eta^\omega(\tau)e^{-\delta n\pi^2(t-\tau)/4}d\tau$$

(defined for a.a. ω) is a positive in Π solution for this equation. Because the assumption (H3),

$$\mathbf{E}\,|\xi(t)|_{C^0}^m \le C_m \delta^{-m} \quad \forall\, m \ge 1\ .$$

Indeed, to prove this estimate, say, for $m=2$ we write $\xi(t,x)^2$ as

$$2^n \mu(x)^2 \int_0^t \int_0^t \eta^\omega(\tau_1)\eta^\omega(\tau_2) e^{-\delta n \pi^2 \big((t_1-\tau_1)+(t_2-\tau_2)\big)/4} d\tau_1 d\tau_2\ .$$

Using (H3) we get that $\mathbf{E}\eta^\omega(\tau_1)\eta^\omega(\tau_2) \le C_{0,2}$. Hence, expectation of the C^0-norm of the integral is bounded by $C\delta^{-2}$, as stated.

Now let us consider the domain $Q = Q^\omega = \Pi \backslash \Sigma^\omega$ and the function $h = \xi - r$. Since $\xi > 0$ in Π and r vanishes on ∂Q, then there $h \ge 0$. In Q it satisfies the parabolic equation:

$$\dot h = \delta \Delta h + \left(\delta r\, |\nabla \varphi|^2 + \eta(t) 2^{n/2} \mu - \operatorname{Re} \zeta e^{-i\varphi}\right)\ .$$

Since $r \ge 0$ and $\operatorname{Re}\zeta e^{-i\varphi} \le \eta$, then the term in the brackets is non-negative and by the maximum principle [La] $h \ge 0$ in Q for a.a ω.

Thus, a.s. $r \le \xi$ everywhere in Π and (7.5) follows from the estimate for the moments of $|\xi(t)|_{C^0}$. □

Using (7.5) we can estimate expectation of the cubic in u term in the l.h.s. of (7.4). Indeed, $\left\| |u|^2 u \right\|_m^2$ is bounded by the sum of all terms J of the form

$$J = \int |u_1|^2 |u_2|^2 |u_3|^2 \, dx\ ,\ \ u_j = \frac{\partial^{\alpha_j} u}{\partial x^{\alpha_j}}\ , |\alpha_j| = m_j\ ,$$

where $m_1 + m_2 + m_3 = m$. By the Hölder estimate, $J \le \prod_{j=1}^3 |u_j|_{L_{r_j}}^2$ where $r_j = 2m/m_j$. By the Gagliardo–Nirenberg inequality (see (6.3) with $m+1$ replaced by m), for a.a. ω we have: $|u_j|_{L_{r_j}} \le C\, |u|_{C^0}^{1-a_j} \|u\|_m^{a_j}$, where $a_j \ge (m_j - n/r_j)/(m - n/2)$ and $a_j \ge m_j/m$. Choosing $a_j = m_j/m$ we get that

$$J \le C\, |u|_{C^0}^{2(3-a_1-a_2-a_3)} \|u\|_m^{2(a_1+a_2+a_3)} = C\, |u|_{C^0}^4 \|u\|_m^2\ .$$

Accordingly,

$$\left\| |u|^2 u \right\|_m \le C\, |u|_{C^0}^2 \|u\|_m\ . \tag{7.6}$$

Abbreviating $\frac{1}{L}\int_{t_0}^{t_0+L} \cdots dt$ to $\int \cdots dt$, we have from (7.6) and Lemma 5 (using the Hölder estimate) that

$$\begin{aligned}\int \mathbf{E} \left\| |u|^2 u \right\|_m dt &\le \int \mathbf{E}\, |u|_{C^0}^2 \|u\|_m \, dt \\ &\le \Big(\int \mathbf{E}\, |u|_{C^0}^4 \, dt\Big)^{1/2} \Big(\int \mathbf{E}\, \|u\|_m^2 \, dt\Big)^{1/2} \le C\delta^{-2} \Big(\int \mathbf{E}\, \|u\|_m^2 \, dt\Big)^{1/2}\ .\end{aligned}$$

Besides, $\int \mathbf{E}\,\|u\|_m\,dt \le \left(\int \mathbf{E}\,\|u\|_m^2\,dt\right)^{1/2}$. By Theorem 6, expectation of the l.h.s. of (7.4) is bigger than one-half of the r.h.s.. Thus,

$$C_1\delta^{-3}\Big(\int \mathbf{E}\,\|u\|_m^2\,dt\Big)^{1/2} \ge \mathbf{E}\left(\text{l.h.s. of (7.4)}\right) \ge \frac{1}{2}C_m^{-1}\delta^{-\frac{3}{17}m+\frac{6}{17}}\ ,$$

and

$$\Big(\frac{1}{L}\int_{t_0}^{t_0+L} \mathbf{E}\,\|u\|_m^2\,dt\Big)^{1/2} \ge \frac{1}{4}C_m^{-1}\delta^{-\frac{3}{17}m+3.5} \quad \text{for} \;\; m\ge 6\ ,\;\; L\ge\delta^{-1}\ .$$

It means that the averaged space-scale of the solution u obeys the estimate $\ell_x^{[0,\infty),E}(u) \le \delta^{3/17}$, i.e., $\gamma_E^{[0,\infty)} \ge \frac{3}{17}$.

7.3. Upper estimates

For solutions of the problem (7.1), (7.3) we have an a priori bound, similar to the estimate we got for solutions of (4.4):

Theorem 7. For $t\ge 0$ and any $m\ge 1$, an odd periodic solution for the problem (7.1), (7.3) with $0<\delta\le 1$ satisfies the estimate $\mathbf{E}\,\|u_\delta^\omega(t,\cdot)\|_m^2 \le C_m\delta^{-3m-2}$.

Proof. Arguing as in section 6, we apply to (7.1) a differentiation $\partial^\alpha/\partial x^\alpha$, $|\alpha|=m$, and take the L^2-scalar product with $\partial^\alpha u/\partial x^\alpha$. Since a.s. $\int \partial^\alpha\zeta/\partial x^\alpha \partial^\alpha\bar u/\partial x^\alpha\,dx = (-1)^m\int \partial^{2\alpha}\zeta/\partial x^{2\alpha}\bar u\,dx$, then summing up all these relations we get that

$$\frac{1}{2}\frac{d}{dt}\,\|u\|_m^2 \le -\delta\,\|u\|_{m+1}^2 + 2^{-n}\sum_{|\alpha|=m}\;\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\int |u_1|\cdots|u_4|\,dx + C\,|u|_{C^0}\,|\zeta|_{C^{2m}}\ ,$$

where $u_j=\partial^{\alpha_j}u/\partial x^{\alpha_j}$ for $j\le 3$ and $u_4=\partial^\alpha u/\partial x^\alpha$. As in section 6, we get the differential inequality:

$$\frac{d}{dt}\,\|u\|_m^2 \le -2\delta\,\|u\|_{m+1}^2 + C\,|u|_{C^0}^{4-A}\,\|u\|_{m+1}^A + C\,|u|_{C^0}\,|\zeta|_{C^{2m}}\ , \tag{7.7}$$

where $A=2m/(m+1)$. Let us denote $\mathbf{E}\,\|u(t)\|_m^2 = f_m(t)$. Applying the Jung inequality to the term $C\,|u|_{C^0}^{4-A}\,\|u\|_{m+1}^A$, averaging the estimate (7.7) and using (7.5) we get:

$$\frac{d}{dt}f_m(t) \le -2\delta f_{m+1} + C_\delta\mathbf{E}\,|u|_{C^0}^{(4-A)(m+1)} + 2\delta f_{m+1} + C\mathbf{E}\,|u|_{C^0} \le C'_\delta\ .$$

Thus, $f_m(t)$ is a bounded Lipschitz function.

(To justify our calculations, smooth in x bounded approximations to the solutions $u=u_\delta^\omega$ should be used. This can be done, for example, like that: By assumption (H1) (see [K2]), for any $T<\infty$ and $\gamma>0$ there exists a set $\Omega_\gamma\in\mathcal{F}$ such that $\mathbf{P}(\Omega\backslash\Omega_\gamma)\le\gamma$ and $|\zeta^\omega(t)|_{C^m}\le C_{m\gamma}$ for $\omega\in\Omega_\gamma$ and $0\le t\le T$. We redefine ζ^ω to be zero outside Ω_γ and denote by $u_\gamma=u_{\gamma\delta}^\omega$ a corresponding solution. For $\omega\in\Omega_\gamma$ and $0\le t\le T$ the new solution u_γ equals u. The norms $|u_\gamma(t)|_m$ are bounded uniformly in $0 \le t \le T$ and $\omega \in \Omega$. Thus, for u_γ and $f_{\gamma m}$ all our calculations are justified and imply the estimates we discuss, with some γ-independent constants. Since $u_\gamma = u$ in Ω_γ, then the limiting functions $f_m(t)$ inherit the estimates.)

After we know that the functions f_m are Lipschitz, we average (7.7) using the Hölder estimate and (7.5):

$$\frac{d}{dt} f_m \le -2\delta f_{m+1} + \mathbf{E}\, |u(t)|_{C^0}^{4-\frac{2m}{m+1}} \|u(t)\|_{m+1}^{\frac{2m}{m+1}} + C_m \mathbf{E}\, |u|_{C^0}$$
$$\le -2\delta f_{m+1} + C_m \delta^{-\frac{2m+4}{m+1}} f_{m+1}^{\frac{m}{m+1}} + C_m \delta^{-1} .$$

We see that $f_m(t)$ decreases if $\delta f_{m+1} \ge C\delta^{-1}$ and $\delta f_{m+1} \ge C\delta^{-\frac{2m+4}{m+1}} f_{m+1}^{\frac{m}{m+1}}$. That is, if $\|u\|_{m+1} \ge C_1 \delta^{-3(m+1)-2}$. Hence, if f_m increases, then the last inequality must be reversed and

$$f_m \le f_0^{1/(m+1)} f_{m+1}^{m/(m+1)} \le C\delta^{-3m-2} .$$

Since initially $f_m(0) = 0$, then $f_m(t)$ never can surpass the r.h.s. of the last inequality and the theorem's assertion follows. □

Theorem 7 implies that $\ell_x^{[0,\infty),E}(u) \ge \delta^{3/2}$. Thus,

$$\frac{3}{2} \ge \gamma_E^{[0,\infty)}(u) \ge \frac{3}{17}. \tag{7.8}$$

— the solution u is short-scale.

We emphasise that the estimates (7.8) do not depend on the space-dimension n and on the specific choice of the random field ζ which satisfies (H1)–(H3).

Remark. Estimates similar to (7.8) hold for solutions for the NLS equation forced by a small force $\delta^c \zeta$ with $\frac{3}{4} \ge c \ge 0$ and ζ as in (7.2). Now an analogy of Theorem 6 follows from [K2, Theorem 4]. □

7.4. Spectral properties of solutions

Theorems 6, 7 jointly with the theorems from Appendix 2 imply that the solution u_δ^ω is carried by "high but not too high" modes. To state the result, for any finite segment $L \subset [0, \infty)$ we denote:

$$\|u_\delta^\omega\|_{mL}^2 = \frac{1}{|L|} \int_L \mathbf{E}\, \|u_\delta^\omega(t,\cdot)\|_m^2 \, dt , \qquad |\hat{u}_{\delta s}^\omega|_L^2 = \frac{1}{|L|} \int_L \mathbf{E}\, |\hat{u}_{\delta s}^\omega(t)|^2 \, dt .$$

Applying to u item 3) of Theorem A2.2 we get:

Corollary 3. 1) For any $\gamma' > \frac{3}{2}$ and any $M > 0$ we have:

$$\sum_{|s| \ge \delta^{-\gamma'}} |\hat{u}_{\delta s}^\omega|_L^2 \le \delta^{M\gamma'} , \tag{7.9}$$

for any segment L such that $|L| \geq \delta^{-1}$ and for all $\delta < \delta_{M\gamma'}$.

2) For any $\varepsilon > 0$ there exist finite numbers c and C such that

$$\delta^c \leq \frac{1}{|\mathfrak{A}|} \sum_{s \in \mathfrak{A}} |\hat{u}^{\omega}_{\delta s}|^2_L \leq \delta^C \ , \ \mathfrak{A} = \mathfrak{A}\big(\delta^{-\frac{3}{17}+\varepsilon}, \delta^{-\frac{3}{2}-\varepsilon}\big) \ , \tag{7.10}$$

for any $|L| \geq \delta^{-1}$ and all $\delta < \delta_\varepsilon$. □

Applying to u Theorem A.2 we get that most part of the averaged squared Sobolev norm $\|u\|^2_{mL} = \sum \langle s \rangle^{2m} |\hat{u}_s|^2_L$ is supported by the modes $\hat{u}_s$ with wave vectors s from the layer $\mathfrak{A}$ as in Corollary 3 (if $m \geq m_\varepsilon$ and $|L| \geq \delta^{-1}$).

Remark. Applying Theorems 6, 7 to estimate spectral properties of solutions, we can argue slightly differently. Namely, we can fix any finite segment $L \subset [0, \infty)$, $|L| \geq \delta^{-1}$, and consider $\ell^{L,\mathrm{E}}_x(u) = \delta^{\gamma^E_L}$. The exponent γ^{E}_L satisfies the same estimates (7.8). Now, applying item 3) of Theorem A2.2 we get again the assertions of Corollary 3, but applying item 2) we get that (7.9) with any $\gamma' > \gamma^{\mathrm{E}}_L$ and (7.10) with $\mathfrak{A}$ replaced by $\mathfrak{A}(\gamma^E_L - \varepsilon, \gamma^E_L + \varepsilon)$ hold for δ from corresponding subsequences. □

7.5 On stationary solutions

Let as *assume* that a solution u^{ω}_{δ} of (7.1), (7.3) converges with time to a stationary in t, odd periodic in x solution of (7.1). Namely, we assume that the shifted random field $u^{\omega}_{\delta}(t + \tau, x)$ when $\tau \to \infty$ converges in distribution to a stationary in t, odd periodic in x random field $U^{\omega}_{\delta}(t, x)$ which solves (7.1) and that $u^{\omega}_{\delta}(t + \tau, \cdot)$ converges to $U_\delta(t, \cdot)$ in distribution in any Sobolev space of odd periodic functions of x. (It is not our goal in this work to *prove* this convergence; cf. [FM], Theorem 3.1.) The solution U describes stationary turbulence in equation (7.1).

Passing to limit in (7.6) we see that the same relation holds for the stationary field U. Hence, $\mathbf{E}\,\|U(t)\|^2_{H^m} \geq C'_m \delta^{-6m/17+7}$ for any t. Passing to the limit in Theorem 7 we get that $\mathbf{E}\,\|U(t)\|^2_{H^m} \leq C_m \delta^{-3m-2}$. These lower and upper bounds show that the exponent γ_E of the space-scale of the stationary solution U also satisfies estimates (7.8).

For Fourier coefficients $\hat{U}^{\omega}_{\delta s}(t)$ of the process U, time-ensemble averages $\left|\hat{U}^{\omega}_{\delta s}\right|^2_L$ equal to expectations $\mathbf{E}\left|\hat{U}^{\omega}_{\delta s}(t)\right|^2$ (which are time-independent). These quantities represent energies of the corresponding wave vectors for the random field U; we denote them E_s. The energy spectrum $\{E_s\}$ satisfies the estimates given in Corollary 3. Besides, since $\mathbf{E}\,\|U(t)\|^2_0 \leq C\delta^{-2}$, then we can apply to U item 1) of Theorem A2.2 and the amplification to get better estimates, valid for all δ from a subsequence, converging to zero:

i) for any $\gamma' > \gamma_E$ and any $M \geq 1$, $\sum_{|s| \geq \delta^{-\gamma'}} E_s \leq \delta^{-M\gamma'}$ for $\delta \in \{\delta_{\gamma' M}(j) \searrow 0\}$;

ii) for any $\varepsilon > 0$, there exist finite numbers $c(\varepsilon)$ and $C(\varepsilon) = n + 2/n + o(1)$ such that $\delta^c \leq |\mathfrak{A}|^{-1} \sum_{s \in \mathfrak{A}} E_s \leq \delta^C$, where $\mathfrak{A} = \mathfrak{A}(\gamma^{\mathrm{E}}_L - \varepsilon, \gamma^{\mathrm{E}}_L + \varepsilon)$, for $\delta \in \{\delta_\varepsilon(j) \searrow 0\}$. Cf. the Remark at the end of section 7.4.

8.Generalisations

8.1. Other equations

It seems that now mathematicians working on the problem of turbulence agree that the *decaying turbulence* is a property of solutions of quasilinear PDEs of the form

$$\langle\text{non-linear homogeneous hamiltonian PDE}\rangle + \langle\text{small linear damping}\rangle = 0\,; \tag{8.1}$$

solutions of (8.1) have to be studied while they remain much bigger than the damping. The *stationary* (or *non-decaying*) turbulence is a property of solutions of equations of the form

$$\begin{aligned}\langle\text{non-linear homogeneous hamiltonian PDE}\rangle+\langle\text{small linear damping}\rangle =\\ = \langle\text{order-one forcing}\rangle,\end{aligned} \tag{8.2}$$

where the forcing is a random field, smooth in space and stationary in time. (The stationarity assumption can be replaced by a weaker restriction.) Notorious, both for their importance and difficulty, examples of (8.1), (8.2) are given by the NS equations, free and forced respectively.

The approach to estimate space-scales of solutions of the NLS equations and to obtain for them a weak analogy of the Kolmogorov-Obukhov law which we develop in [K2] and in this work is rather general. It applies to (8.1), (8.2) if we know that

i) some functional norms of nonzero solutions of the non-linear hamiltonian PDE grow with time at least linearly (e.g., they blow up in finite time).

Practically (due to the non-linear homogeneity), i) follows from

ii) the non-linear hamiltonian PDE is "integrable" in the sense that its solutions are given by explicit (or half-explicit) formulas, and the equation has continuous spectrum.

Example 1. The assumption i) (and ii)) follows if the hamiltonian equation takes the form $\dot u(t,x) = V(u(t,x))$, where u varies in a symplectic linear space $(\mathbb{R}^{2M},\omega_2)$ and V is an integrable nonlinear homogeneous hamiltonian vector field in $\mathbb{R}^{2M}$. The NLS equations correspond to the hamiltonian equations $\dot u = i|u|^{2p}u$ in the symplectic space $(\mathbb{C},(i/2)\,dz\wedge d\bar z)$. Nonlinear wave equations $\ddot u = \delta\Delta u - u^{2k+1}$ correspond to the integrable equations $\dot u = v, \dot v = -u^{2k+1}$ and can be treated similar. □

Example 2. The free and forced Burgers equations

$$\dot u - \delta u_{xx} + \frac{\partial}{\partial x}u^2 = 0\,, \qquad u \text{ is odd 2-periodic in } x\in\mathbb{R}^1\,, \tag{8.3}$$

$$\dot u - \delta u_{xx} + \frac{\partial}{\partial x}u^2 = \zeta^\omega(t,x)\,, \quad u \text{ is odd 2-periodic in } x\in\mathbb{R}^1\,, \tag{8.4}$$

correspond to the Hopf equation

$$\dot u + \frac{\partial}{\partial x}u^2 = 0\,, \quad u \text{ is odd 2-periodic in } x\in\mathbb{R}^1. \tag{8.5}$$

This equation is hamiltonian and can be integrated by the method of characteristics, so it satisfies ii). After a finite time any nonzero solution of (8.5) develops a shock and its C^1-norm blows up, so i) also holds. Turbulence in (8.4) (see [EKMS]) and especially in (8.3) with random initial conditions (see references in [F], p.142) was intensively studied since these equations can be integrated by means of the Hopf-Cole transformation. Our approach is applicable to generalisations of these equations (the dimension $n = 1$ and/or the degree two of the nonlinearity can be increased, the Laplacian can be replaced by any elliptic operator). □

Although our techniques suit the randomly forced Burgers equation (8.4) as well as the NLS equation (3), in some important respects the "turbulent limits" $\delta \to 0$ for these two equations are rather different. Indeed, when $\delta \to 0$, solutions of (8.4) converge to *viscous solutions* of the forced Hopf equation:

$$\dot{u} + \frac{\partial}{\partial x} u^2 = \zeta^\omega(t, x) .$$

The limiting viscous dynamics is well defined and has a unique invariant measure, supported by the space of functions of bounded variation. Properties of this measure describe some limiting (as $\delta \to 0$) properties of solutions for (8.4), see [EKMS]. In the NLS case, the limiting equation is

$$-i\dot{u} = |u|^2 u + \zeta^\omega(t, x) .$$

For any x, its solutions $u^\omega(t, x)$ grow with t in the diffusive way (i.e., as $\sqrt{t}$). So the random process $t \to u^\omega(t, \cdot)$ escapes to infinity and has no invariant measure. The random fields $u^\omega_\delta(t, x)$ do not converge to a limiting stationary solution, but their distribution averaged in time satisfies the universal (ζ-independent) estimates which we study in this work.

We *can not* apply our techniques to the NS equations since for NS the underlying non-linear hamiltonian PDE is the Euler equation

$$\dot{u} + (u \cdot \nabla) u + \nabla p = 0, \quad \mathrm{div}\, \mathrm{u} = 0.$$

It is unknown if solutions for this equation satisfy i) or ii).

8.2. Local in space results

Let $u(t, x)$ be any solution for the equation (2) in the half-space $[0, \infty) \times \mathbb{R}^n$ $(n \leq 3)$, such that $u(0, x) = u^0(x)$. Let K_M be a cube $K_M = \{x_j^0 \leq x_j \leq x_j^0 + M\}$ and

$$\sup_{x \in K_M} \left| u^0(x) \right| \leq C \;, \quad \operatorname*{osc}_{K_M} \left| u^0(x) \right| = U \,,$$

where $M = C_M \delta^{b_M}$ and $U = C_U \delta^{b_U}$. Let us restrict the solution u to the parallelepiped $Q_M = [0, C T_M] \times K_M$, where $T_M = \delta^{-(1+4pb_U - 2b_M)/3}$. Scaling u and $(t, x) \in Q_M$ as we did it to prove (4.15) and applying Theorem 3 we find that

$$\ell_x^{\inf}(u\,|_{Q_M}) \leq \delta^{(1+b_M - 2pb_U)/3} \,,$$

provided that $2b_M + 2pb_U < 1$.

Let us assume that u^0 is a smooth δ-independent function. Then $\mathrm{osc}_{K_1}|u^0| \sim 1$ for any 1-cube K_1 (provided that $u^0 |_{K_1}$ is not a constant) and $\ell_x^{\inf}(u |_{Q_1}) \leq \delta^{1/3}$. If $M = \delta^b$ with $1/(2p-1) > b > 0$ and $K_M \subset K_1$, then $\mathrm{osc}_{K_M}|u^0| \sim \delta^b$ (if some non-degeneracy assumptions hold). Now $T_M = \delta^{-\frac{1+(4p-2)b}{3}} > T_1$ and $\ell_x^{\inf}(u |_{Q_M}) \leq \delta^{\frac{1-(2p-1)b}{3}}$ which is bigger than $\ell_x^{\inf}(u |_{Q_1})$. — We can not recover the short space-scale $\leq \delta^{1/3}$ when study the solution locally.

Appendix 1. Interpolation inequalities

A. Sobolev spaces. Let $\mathcal{O}=\mathbb{T}^n$ or $\mathbb{R}^n$ and $H^m(\mathcal{O};\mathbb{R}^M)$, $m\in\mathbb{R}$, be Sobolev spaces with the norms $\|\cdot\|_m$ as in section 2. Then

$$\|u\|_k \le \|u\|_0^{1-k/m}\,\|u\|_m^{k/m} \quad \text{for} \quad 0\le k\le m\,.$$

The estimate follows from the Hölder one. Indeed, choosing $p=m/(m-k)$, $q=m/k$ and denoting by u_s Fourier coefficients of the function u, we get:

$$\|u\|_k^2=\sum |u_s|^{2/p}\,|u_s|^{2/q}\,\langle s\rangle^{2k}\le \big(\sum |u_s|^2\big)^{1/p}\big(\sum |u_s|^2\,\langle s\rangle^{2m}\big)^{1/q}\,.$$

B. C^k-spaces

B1. $\mathcal{O}=\mathbb{T}^n$ or $\mathbb{R}^n$. We denote $[u]_k=\max_{|\alpha|=k}\sup_x|\partial_x^\alpha u(x)|$ and abbreviate $|u|_{C^m}$ to $|u|_m$. The classical inequality of Hadamard–Landau–Kolmogorov states that

$$[u]_k\le C_{\ell m}[u]_\ell^{\frac{m-k}{m-\ell}}[u]_m^{\frac{k-\ell}{m-\ell}}\ ,\ \ \ell\le k\le m\,, \tag{A1}$$

where for $n=1$ optimal values of the constants C_{km} were obtained by A.N.Kolmogorov in [Kol3]. (We note that the inequality with $n\ge 2$ follows from the one with $n=1$ by induction in k: for $k=\ell$ it is obvious; to prove (A1) for k replaced by $k+1$ we should estimate $[\partial_x^\alpha u]_0$ with $|\alpha|=k+1$. To do it we write $\partial_x^\alpha u$ as $(\partial/\partial x_j)\partial_x^\beta u$ with $|\beta|=k$ and apply the one-dimensional inequality to estimate $[(\partial/\partial x_j)\partial_x^\beta u]_0$ in terms of $[\partial_x^\beta u]_0\le[u]_k$ and $[\partial_x^\beta u]_{m-k}\le[u]_m$. Finally we use the base of induction to estimate $[u]_k$ in terms of $[u]_\ell$ and $[u]_m$).

By (A1) and the Jung inequality (stating that $ab\le a^p/p+b^q/q$ if $1/p+1/q=1$ and $a,b\ge 0$), we get that

$$[u]_k\le C'_{km}\big([u]_0+[u]_m\big)\ ,\ 0\le k\le m\,. \tag{A2}$$

Thus, the norm $[u]_0+[u]_m$ is equivalent to the usual C^m-norm $|u|_m$. Since $(a+b)^\rho\ge(a^\rho+b^\rho)/2$ for $a,b\ge 0$ and $0<\rho\le 1$, then

$$\begin{aligned}|u|_\ell^{\frac{m-k}{m-\ell}}\,|u|_m^{\frac{k-\ell}{m-\ell}} &\ge C\big([u]_0+[u]_\ell\big)^{\frac{m-k}{m-\ell}}\big([u]_0+[u]_m\big)^{\frac{k-\ell}{m-\ell}}\\ &\ge \frac{C}{4}\big([u]_0^{\frac{m-k}{m-\ell}}+[u]_\ell^{\frac{m-k}{m-\ell}}\big)\big([u]_0^{\frac{k-\ell}{m-\ell}}+[u]_m^{\frac{k-\ell}{m-\ell}}\big)\ge\frac{C}{4}\big([u]_0+C_{km}^{-1}[u]_k\big)\end{aligned}$$

and

$$|u|_k\le C'_{km}\,|u|_\ell^{\frac{m-k}{m-\ell}}\,|u|_m^{\frac{k-\ell}{m-\ell}}\ ,\ \ell\le k\le m\,. \tag{A3}$$

B2. $\mathcal{O}=K^n=\{0\le x_j\le 1\}$, or $\mathcal{O}$ is a smooth bounded domain in $\mathbb{R}^n$.

Any function $u\in C^k(\mathcal{O})$ admits a lift to a function $U\in C^k(\mathbb{R}^n)$ such that $|U|_k\le C_k\,|u|_k$. Applying (A3) to the lift we get that this estimate remains valid for $u\in C^m(\mathcal{O})$.

Since $|u|_m = \max\big(|u|_{m-1}\,, [u]_m\big)$, then using (A3) and induction we find that the norm $[\cdot]_0 + [\cdot]_m$ is equivalent to the C^m-norm. Using again (A3) we get that

$$[u]_k \le C''_{km}\big([u]_0 + [u]_0^{\frac{m-k}{m}}\,[u]_m^{\frac{k}{m}}\big)\ , \quad 0 \le k \le m\ . \tag{A4}$$

In difference with the case $\mathcal{O} = \mathbb{R}^n$ (cf. (A1)), the term $[u]_0$ in the r.h.s. of (A4) can not be dropped. A counterexample for $k = 1$ is given by a linear function $u(x)$.

B3. $\mathcal{O} = K^n_\rho = \{0 \le x_j \le \rho\}$, $0 < \rho \le 1$. Denoting $y = x/\rho \in K^n$, we get that $\rho^k[u(x)]_k = [u(y)]_k$. Thus,

$$[u(x)]_k \le C_{km}\big(\rho^{-k}[u(x)]_0 + [u(x)]_0^{\frac{m-k}{m}}\,[u(x)]_m^{\frac{k}{m}}\big)\ ,$$

where C_{km} is a ρ-independent constant.

Appendix 2. Short-scale random fields

Here we consider random fields $f^\omega_\delta(t,x)$ corresponding to a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and present a randomised version of definitions and theorems from sections 1, 2. We assume that $t \in [T_1, T_2]$ where as before, either

a) $T_2 = \infty$ (in this case we replace $[T_1, T_2]$ by $[T_1, T_2)$), or

b) $T_2 - T_1 = C\delta^{-b}$, $b \ge 0$.

The space-domain $\mathcal{O}$ is now either a bounded n-domain, or $\mathcal{O} = \mathbb{T}^n = \mathbb{R}^n/2\mathbb{Z}^n$. The random field $f^\omega_\delta(t,x)$ is assumed to admit a realization such that the function $(t,x) \mapsto f^\omega_\delta(t,x)$ is measurable for all δ and a.a. ω; as a function of x it is smooth for a.a. t.

Now we define the averaged space-scale $\ell_x^{L,\mathrm{E}}(f)$:

Definition. For a random field $f^\omega_\delta(t,x)$ as above we set $\ell_x^{L,\mathrm{E}}(f) = \delta^{\gamma_L^{\mathrm{E}}}$, where $\gamma_L^{\mathrm{E}} = \sup \Gamma^E$ and the set Γ^E is formed by all $\tilde\gamma$ such that

$$\frac{1}{|L|}\int_L \mathbf{E}\,\|f^\omega_\delta(t,\cdot\|_k^2\,dt \ge \delta^{-2\tilde\gamma k} \quad \forall\, k \ge k_{\tilde\gamma}\ ,\ \delta \le \delta_{k,\tilde\gamma}\ . \tag{A5}$$

The segment $L \subset [T_1, T_2]$ is as in (1.6).

If the process f^ω_δ is stationary in time, then $\mathbf{E}\,\|f^\omega_\delta(t,\cdot)\|_k^2$ is a time-independent quantity and (A5) means that

$$\mathbf{E}\,\|f^\omega_\delta(t,\cdot)\|_k^2 \ge \delta^{-2\tilde\gamma k} \quad \forall\, k \ge k_{\tilde\gamma}\ ,\ \delta \le \delta_{k,\tilde\gamma}\ .$$

Accordingly, in this case $\gamma' > \gamma_L^{\mathrm{E}}$ if for sufficiently small ε one has $\mathbf{E}\,\|f^\omega_\delta(t,\cdot)\|_M^2 \le \delta^{-2(\tilde\gamma-\varepsilon)M}$ for arbitrarily large M and $\delta \in \{\delta_{\varepsilon M}(j) \searrow 0\}$.

Obvious reformulations of the statements of proposition 2 hold true for the averaged space-scale $\ell_x^{L,\mathrm{E}}(f)$, if $\mathcal{O} = \mathbb{T}^n$. To state the results we write the random field f^ω_δ in the

Fourier representation, $f_\delta^\omega(t,x) = \sum_{s\in\mathbb{Z}^n} \hat f_{\delta s}^\omega(t) e^{\pi i s\cdot x}$. For any finite segment $L\subset[T_1,T_2]$, we consider time-ensemble averaged squares of Sobolev norms and of Fourier coefficients:

$$\|f_\delta^\omega\|_{mL}^2 = \frac{1}{|L|}\int_L \mathbf{E}\,\|f_\delta^\omega(t)\|_m^2\,dt\,, \quad \left|\hat f_{\delta s}^\omega\right|_L^2 = \frac{1}{|L|}\int_L \mathbf{E}\left|\hat f_{\delta s}^\omega(t)\right|^2 dt\,.$$

(If f is stationary in time, then the time-averaging can be dropped i.e., $\|f_\delta^\omega\|_m^2 = \mathbf{E}\,\|f_\delta^\omega\|_m^2$ for any t, etc). The following reformulations of Theorems $1', 2'$ remain true with the same proofs:

Theorem A 2.1. Let $\infty > \gamma_L^{\mathrm{E}}(f) \ge 0$. Then for any $\nu>0$ and $\varepsilon>0$,

1) if the process f^ω is stationary in time and $\mathbf{E}\,\|f_\delta^\omega(t)\|_0^2 \le C\delta^{-a}$ for some $a\ge 0$, then

$$\sum_{s\in\mathfrak{A}(\gamma_L^{\mathrm{E}}-\varepsilon,\gamma_L^{\mathrm{E}}+\varepsilon)} \mathbf{E}\left|\hat f_{\delta s}^\omega(t)\right|^2 \langle s\rangle^{2m} \ge (1-\nu)\mathbf{E}\,\|f_\delta^\omega(t)\|_m^2$$

for $m\ge m_\varepsilon$ and $\delta\in\{\delta_{\varepsilon m}(j)\searrow 0\}$.

2) If $T_2-T_1 = C\delta^{-b}$ and $\|f_\delta^\omega\|_{0\,[T_1,T_2]}^2 \le C\delta^{-a}$, then

$$\sum_{s\in\mathfrak{A}(\gamma_L^{\mathrm{E}}-\varepsilon,\gamma_L^{\mathrm{E}}+\varepsilon)} \langle s\rangle^{2m}\left|\hat f_{\delta s}^\omega\right|_L^2 \ge (1-\nu)\,\|f_\delta^\omega\|_{mL}^2 \qquad (A6)$$

for $L=[T_1,T_2]$, $m\ge m_\varepsilon$ and $\delta\in\{\delta_{\varepsilon m}(j)\searrow 0\}$.

3) If for some γ_1, γ_2 and for all $m\ge m_0, \delta\le\delta_m$ we have $\delta^{-2\gamma_1 m} \le \|f_\delta^\omega\|_{mL}^2 \le \delta^{-2\gamma_2 m}$, for any segment $L\subset[T_1,T_2]$ as in (1.6), then (A6) with the layer $\mathfrak{A}$ replaced by $\mathfrak{A}(\gamma_1-\varepsilon,\gamma_2+\varepsilon)$ holds for the segments L as above, for $m\ge m_\varepsilon$ and $\delta<\delta_\varepsilon$.

Theorem A 2.2. Let $\infty > \gamma_L^{\mathrm{E}} \ge 0$. Then

1) under the assumptions of item 1) of Theorem A 2.1,

i) for any $\gamma' > \gamma_L^{\mathrm{E}}$ and any $M\ge 1$ we have: $\sum_{|s|\ge\delta^{-\gamma'}} \mathbf{E}\left|\hat f_{\delta s}^\omega(t)\right|^2 \le \delta^{M\gamma'}$ if $\delta\in\{\delta_{\gamma' M}(j)\searrow 0\}$;

ii) for any $\varepsilon>0$ and $\mathfrak{A}=\mathfrak{A}(\gamma_L^{\mathrm{E}}-\varepsilon,\gamma_L^{\mathrm{E}}+\varepsilon)$ we have

$$\delta^c \le |\mathfrak{A}|^{-1}\sum_{s\in\mathfrak{A}} \mathbf{E}\left|\hat f_{\delta s}^\omega(t)\right|^2 \le \delta^C,$$

if $\delta\in\{\delta_\varepsilon(j)\searrow 0\}$ with some finite $c=c(\varepsilon)$ and $C=C(\varepsilon)$.

2) Under the assumptions of item 2) of Theorem A 2.1, assertions i), ii) hold true with $\mathbf{E}\left|\hat f_{\delta s}^\omega(t)\right|^2$ replaced by $\left|\hat f_{\delta s}^\omega\right|_{[T_1,T_2]}^2$.

3) Under the assumptions of item 3), assertion i) hold true with γ_L^{E} replaced by γ_2, for all $\delta < \delta_{\gamma' M}$; the assertion ii) hold true with $\mathfrak{A} = \mathfrak{A}(\gamma_1 - \varepsilon, \gamma_2 + \varepsilon)$, for all $\delta < \delta_\varepsilon$.

The assertions ii) of the last theorem can be specified as in the Amplification to Theorem 2.

Appendix 3. Theorem 3 and a dynamical system, defined by the NLS equation

In this appendix we abbreviate $|\cdot|_{C^m(K^n)}$ to $|\cdot|_m$. Let us consider the NLS equation (4.1) under the odd periodic boundary conditions (4.2):

$$\begin{gathered} -i\dot{u} = -\delta\nu\Delta u + |u|^{2p} u \, , \; \delta > 0 \, , \\ u(t,x) \text{ is odd and 2-periodic in } x \, . \end{gathered} \tag{A7}$$

If $u(t,x)$ is a smooth solution of (A7) such that $u(0,x) = u^0(x)$, $\left|u^0\right|_0 = 1$, then for any $\kappa < 1/3$ by Theorem 3 there exists $T_0 \le \delta^{-1/3}$ such that

$$|u(T_0)|_m \ge \tilde{K}_m \delta^{-m\kappa} \, , \; \tilde{K}_m = (\delta_{m\kappa})^{m\kappa} \tag{A8}$$

(for $\delta < \delta_{m\kappa}$ this follows from (4.8), for $\delta \ge \delta_{m\kappa}$ this is obvious with $T_0 = 0$ since $|u(0)|_m \ge \left|u^0\right|_0 = 1)$. Using the Remark at the end of section 5 we get that

$$\frac{1}{2} \le |u(T_0)|_0 \le \frac{3}{2} \, . \tag{A9}$$

Let $u(t,x)$ be any solution of (A7). Denoting $|u(0)|_0 = U$ we substitute to (A7) $u = Uv$, $t = U^{-2p}\tau$ and get for $v(\tau,x)$ the equation $-iv'_\tau = \delta U^{-2p}\nu\Delta v + |v|^{2p} v$. Applying (A8) to $v(\tau,x)$ we find $T_0 \le U^{-2p}(\delta U^{-2p})^{-1/3} = \delta^{-1/3}U^{-4p/3}$ such that $|u(T_0)|_m \ge \tilde{K}_m U(\delta U^{-2p})^{-m\kappa}$. Since $|u(T_0)|_0 \le \frac{3}{2}U$ by (A9), then for $u = u(T_0)$ we have:

$$|u|_0 \le K_m \delta^\mu \, |u|_m^{1-2p\mu} \, , \tag{A10}$$

where $K_m = \frac{3}{2}\tilde{K}_m^{-1/(2pm\kappa+1)}$ and $\mu = m\kappa/(2pm\kappa+1)$.

Let us denote by $\mathcal{A} = \mathcal{A}_m \subset C^\infty$ the set

$$\mathcal{A} = \{u \in C^\infty \mid u \text{ is odd periodic and satisfies (A10)}\} \, .$$

Following [K1] we call $\mathcal{A}$ the *essential part of the phase-space* of equation (A7) (with respect to the C^m-norm). Since $|u|_0 \le |u|_m$, then $\mathcal{A}$ contains a neighbourhood of the origin:

$$\mathcal{A} \supset \left\{u(x) \mid |u|_m \le K_m^{\frac{1}{2p\mu}} \delta^{\frac{1}{2p}} \right\} \, . \tag{A11}$$

Our arguments show that $\mathcal{A}$ is a recursion subset for a dynamical system which (A7) defines in the space of smooth odd periodic functions: Let $u(t) = u(t, \cdot)$ be a smooth solution of (A7), t_0 be any real number and $U_0 = |u(t_0)|_0$.

Theorem A3 (cf. [K1]). There exists $T_0 \le t_0 + \delta^{-1/3} U_0^{-4p/3}$ such that $u(T_0) \in \mathcal{A}$ and $\frac{1}{2} U_0 \le |u(T_0)|_0 \le \frac{3}{2} U_0$.

If $\nu = i$, then the equation takes the form (4.4) and by (4.13), $|u(t)|_m \to 0$ for any its solution. Using (A11) and (4.13) we get that $u(t) \in \mathcal{A}$ if $t \ge t_* = t_0 + C(|u|_m)\delta^{-1}\ell n\, \delta^{-1}$. The theorem shows that the solution will visit the set $\mathcal{A}$ much earlier, before its norm decays.

If $\nu = 1$, then the equation (A7) takes the hamiltonian form (4.5) and $N := |u(t)|_{L_2}$ is a time-independent constant. Since $|u(t)|_0 \ge N$, then by the theorem the solution will visit $\mathcal{A}$ during any time-interval longer than $\delta^{-1/3} N^{-4p/3}$. We do not know if a solution of the hamiltonian equation (A7) $|_{\nu=1}$ can get stuck in $\mathcal{A}$ for all t bigger than some t_1. — This would be the case if $\|u(t)\|_m \to \infty$ when $t \to \infty$, but we do not know if this convergence is possible (or typical) for solutions of a hamiltonian NLS equation, see [Bour].

References

[Bour] Bourgain J.: On the growth in time of higher Sobolev norms of smooth solutions of Hamiltonian PDE; International Mathematics Research Notes, No. 6 (1996), 277 – 304.

[BGO] Bartuccelli M.V., Gibbon J.D., Oliver M.: Length scales in solutions of the complex Ginzburg–Landau equation; Physica 89 D (1996), 267–286.

[CDT] Constantin P., Doering Ch., Titi E.: Rigorous estimates of small scales in turbulent flow; J. of Math. Physics 37 (1996), 6142 – 6156.

[EKMS] E W., Khanin K., Mazel A., Sinai Ya.: Invariant measures for Burgers equation with stochastic forcing, submitted to Annals of Mathematics.

[F] Frisch U.: Turbulence; Cambridge University Press, Cambridge 1995.

[FM] Flandoli F., Maslowski B.: Ergodicity of the 2-D Navier–Stokes equation under random perturbations; Comm. Math. Phys. 172 (1995), 119 – 141.

[JLM] Jin S., Levermore C.D., McLaughlin D.W.: The semiclassical limit of the defocusing NLS hierarchy; Comm. Pure Applied Math. 52 (1999).

[JT] Jones D.A., Titi E.: Upper bounds on the number of determing modes, nodes and volume elements for the Navier-Stokes equations; Indiana University Math. J. 42 (1993), 875–887.

[K1] Kuksin S.B.: Oscillations in space-periodic nonlinear Schrödinger equations; Geometric and Functional Analysis 7 (1997), 338 – 363.

[K2] Kuksin S.B.: On turbulence in nonlinear Schrödinger equation; Geometric and Functional Analysis 7 (1997), 783 – 822.

[Kol1] Kolmogorov A.N.: The local structure of turbulence in incompressible viscous fluid for very large Reynolds number. Paper 45 in "Selected works of A.N. Kolmogorov, vol. 1 " . Moscow, Nauka 1985. Engl. translation: Kluwer, 1991.

[Kol2] Kolmogorov A.N.: The equation of turbulent motion in an incompressible viscous fluid. Paper 48 in the same book.

[Kol3] Kolmogorov A.N.: On inequalities for supremums of successive derivatives of a function on an infinite interval. Paper 40 in the same book.

[La] Landis E.M.: Second order equations of elliptic and parabolic type. Moscow, Nauka 1971. Engl. translation: Transl. of Math. Monographs, 171. AMS, Providence, 1997.

[LL] Landau L.D., Lifschitz E.M.: Fluid mechanics, 2nd ed., Pergamon Press, 1987.

[LO] Levermore C.D., Oliver M.: The complex Ginzburg–Landau equation as a model problem. Lectures in Applied Math. 31, 141 – 189 (1996).

[MMT] Majda J.M., McLaughlin D.W., Tabak E.G. : A one-dimensional model for dispersive wave turbulence. J. Nonlinear Sci. vol. 6, 1–36 (1996).

[Ni] Nirenberg L.: An extended interpolation inequality. Ann. Sc. Norm. Sup., Pisa, Ser. 6, vol. 20, 733 – 737 (1966).

[Tr] Triebel H.: Theory of function spaces. Birkhäuser, 1983.

[Z] Zakharov V.E.: Basic plasma physics, Part 4, vol. 2 (A.A. Galeev and R.N. Sudan, eds.), North-Holland, Amsterdam 1984.

e-mail: S.B.Kuksin@ma.hw.ac.uk