Stochastic Nonlinear Schrödinger Equation 1. A priori estimates

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Abstract

We consider a non-linear Schrödinger equation with a small real coefficient δ in front of the Laplacian. The equation is forced by a random forcing which is a white noise in time and is smooth in the space-variable x from a unit cube; Dirichlet boundary conditions are assumed on the cub's boundary. We prove that the equation has a unique solution which vanishes at $t = 0$. This solution is almost certainly smooth in x and k-th moment of its m-th Sobolev norm in x is bounded by $C_{m,k}\ \delta^{-km-k/2}$. The proof is based on a lemma which can be treated as a stochastic maximum principle.

Introduction. We consider the nonlinear Schrödinger equation, forced by a random force ζ^ω:

$$\dot{v} = \delta \triangle v - i|v|^2 v + \zeta^\omega(t,x), \tag{0.1}$$

$$v = v(t,x), \quad t \geq 0, \quad x \in \mathbb{R}^n, \qquad v|_{t=0} = 0. \tag{0.2}$$

Solution v is a complex function, odd periodic in x:

$$v(t, x_1, \dots, x_n) = v(t, \dots, x_j + 2, \dots) = -v(t, \dots, -x_j, \dots), \quad j = 1, \dots, n; \tag{0.3}$$

dimension $n = 1, 2$ or 3 and the dissipation δ is $0 < \delta \leq 1$. The boundary condition implies that v vanishes at the boundary of the cube of the half-periods $\{0 \leq x_j \leq 1\}$.

In [K1, K2] we conside the problem (0.1) - (0.3) with a forcing ζ which is a random field, smooth in x and stationary mixing in t. [1] There we examine the quantities E_m, equal to the squared Sobolev norms $\|v(t,\cdot)\|_m^2$ of a solution v, averaged in ensemble and locally averaged in time and prove that

$$C^{-1}\delta^{-3m/17+4} \leq E_m^{1/2} \leq \delta^{-3m/2-1}, \tag{0.4}$$

where in the first inequality m has to be ≥ 6. In [K2] we reformulate (0.4) as estimates for the space-scale of the solution v and use them to study averaged spectral characteristics of

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[1] Publications [K1, K2] deal with more general equations and allow the coefficient δ to be complex

v. Thus we obtaine estimates for the spectrum of v, related to the Kolmogorov-Obukhov law from the theory of turbulence. It was clear for us that the estimates (0.4) are not optimal (as well as their spectral counterparts) and it was plausible that better estimates might be available for solutions of a stochastic nonlinear Schrödinger (SNLS) equation, which is an equation (0.1) where the random field ζ^ω is a white noise in time. To check these hopes we choose for the object of our next research the SNLS equation with the forcing ζ^ω of the form

$$\zeta^\omega(t,x) = \eta^\omega(t,x)\dot{w}(t), \tag{0.5}$$

where $w(t)$ is a Wiener process and η^ω is an adapted process, continuous in (t,x) and smooth in x. [2] The first step to study (0.1)-(0.3), (0.5) is to prove existence and uniqueness of a solution v and to estimate its norms. By analogy with deterministic PDEs and with equation (0.1) forced by a smooth in x bounded random field ζ we thought that this will be a routine work, forming an introductory part of a larger research. To our surprise it was not so and a proof of unique solvability of the SNLS equation and derivation of corresponding a priori estimates occupies the whole of this paper.

Our main result is the following theorem, proved in Section 4:

Theorem. *The SNLS equation (0.1)-(0.3), (0.5) has a unique solution $v^\omega(t,x)$. This solution is a.s. continuous in (t,x) and smooth in x. For any real numbers $t \geq 0, q \geq 1$ and any integer $m \geq 0$ it satisfies the estimates*

$$\mathbf{E} \sup_{t \leq s \leq t+\delta^{-1}} \sup_x |v(s,x)|^q \leq C_q \delta^{-q/2}, \qquad \mathbf{E}\|v(t)\|_m^q \leq C_{q,m}\delta^{-qm-q/2}.$$

In the theorem $\|\cdot\|_m$ stands for the norm of the Sobolev space $H^m = H^m_{op}(\mathbb{R}^n, \mathbb{C})$ of odd periodic complex function on $\mathbb{R}^n$:

$$\|u\|_m^2 = \int_{K^n} \sum_{|\alpha|=m} |\partial_x^\alpha u|^2 dx. \tag{0.6}$$

Proof of the theorem is based on the following result, related to the maximum principle for parabolic equations: if $u^\omega(t,x)$ is a real odd periodic solution for the linear SPDE

$$\dot{u}(t,x) - \triangle u(t,x) = f^\omega(t,x)\dot{w}(t), \quad u(0,x) = 0,$$

where f^ω is an adapted process such that $|f^\omega| \leq 1$, then for any $T \geq 0$ and $q \geq 1$ we have $\mathbf{E}\,|\sup |u|_{[T,T+1]\times\mathbb{R}^n}|^q \leq C_q$. We prove this estimate in the Appendix.

Due to the theorem and the usual arguments by Krylov-Bogoliubov, the stochastic differential equation, defined by the SNLS in a space of odd periodic functions, has an invariant measure, supported by the space of smooth functions (see Section 5).

The theorem implies better than in (0.4) upper bound for the quantity $E_m(t) = \mathbf{E}\|v(t)\|_m^2$:

$$E_m(t)^{1/2} \leq C_m \delta^{-m-1/2} \quad \text{for any} \;\; t.$$

In a forthcoming publication we shall present a lower bound for $E_m(t)$ and shall study spectral properties of solutions v, following [K2].

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[2] Our arguments generalize to equations (0.1) with $\zeta = \sum_{j=1}^\infty \eta_j \dot{w}_j$, where the random fields η_j and the independent Wiener processes w_j are as above and $\sum |\eta_j| < \infty$.

Notations. By C, C_1 etc. we denote different constants, independent of δ. By $\|\cdot\|_m$, $m \in \mathbb{N}$, – the norms in the Sobolev spaces H^m (see (0.6)) and by $|\cdot|_p$, $1 \le p \le \infty$, – the norms in L_p-spaces. We consider random fields (r.f.'s) $u^\omega(t,x)$, depending on time t and space x. Often we treat them as random processes in spaces of x-dependent functions and write as $u^\omega(t,\cdot)$ or $u^\omega(t)$ (e.g. we may write that u^ω is a random process $u^\omega(t) \in H^m$). We say that a r.f. or a random process is continuous (or smooth, etc) if it has a modification which is almost surely (a.s.) continuous (or smooth, etc).

1 Preliminaries on SPDEs

In this paper we discuss SPDEs of the form

$$\dot{u}(t,x) - \sigma\triangle u(t,x) + F(u(t,x)) = \zeta^\omega(t,x), \quad t > 0,\ x \in \mathbb{R}^n, \tag{1.1}$$

where $n = 1, 2$ or 3, $\sigma > 0$ and $u(t,x)$ is a complex function which satisfies the odd periodic boundary conditions:

$$u(t, x_1, \dots, x_n) = u(t, \dots, x_j + 2, \dots) = -u(t, \dots, -x_j, \dots),\ j = 1, \dots, n. \tag{1.2}$$

These boundary conditions are assumed everywhere below, unless other conditions are specified. The equation (1.1) will be studied in Sobolev spaces $H^m = H^m_{op}(\mathbb{R}^n; \mathbb{C})$, formed by odd 2-periodic complex functions

$$H^m_{op}(\mathbb{R}^n; \mathbb{C}) = \{u \in H^m_{loc}(\mathbb{R}^n; \mathbb{C}) \mid u \text{ satisfies } (1.2)\}.$$

The spaces are given the homogeneous Hilbert norms $\|\cdot\|_m$ as in (0.6) i.e.,

$$\|u\|_m^2 = \langle u, u\rangle_m, \quad \langle u, v\rangle_m = \mathrm{Re}\ 2^{-n} \sum_{|\alpha|=m} \int_0^2 \cdots \int_0^2 (\partial_x^\alpha u(x) \partial_x^\alpha \bar{v}(x) dx_1 \dots dx_n.$$

We note that odd periodic functions $u(x)$ vanish at the boundary of the cube of half-periods:

$$u(x)\mid_{\partial K^n} = 0, \quad K^n = \{x \mid 0 \le x_j \le 1\}.$$

Accordingly, we treat them as periodic functions on $\mathbb{R}^n$, or as functions defined on the torus $\mathbb{T}^n = \mathbb{R}^n/2\mathbb{Z}^n$, or as functions on K^n which satisfy Dirichlet boundary conditions.

The nonlinearity F in (1.1) is assumed to define a locally Lipschitz or uniformly Lipschitz map of a space H^m to itself. That is,

$$\|F(u) - F(v)\|_m \le C\big(\|u\|_m \vee \|v\|_m\big)\|u - v\|_m, \tag{1.3}$$

or

$$\|F(u) - F(v)\|_m \le C\|u - v\|_m \quad \forall\, u, v \in H^m, \tag{1.4}$$

where $a \vee b$ signifies the maximum of two numbers.

Example 1. Let $f(r)$ be a smooth real-valued function and (1.1) has the form

$$\dot{u} = \sigma\triangle u + if(|u|^2)u + \zeta^\omega(t,x),$$

where the noise ζ^ω is as above. The nonlinearity $if(|u|^2)u$ defines a map $H^m \to H^m$ which is smooth and locally Lipschitz if $m \ge 2$ since $n \le 3$.

Example 2. Now we cut out the nonlinearity for big $\|u\|_m$ to get the equation:

$$\dot{u} = \sigma \triangle u + i\varphi(\|u\|_m) f(|u|^2)u + \zeta^\omega(t,x), \tag{1.5}$$

where $\varphi \in C_0^\infty(\mathbb{R})$. The cut nonlinearity defines a map $H^M \to H^M$ which is smooth locally Lipschitz if $M \geq m \geq 2$ and is globally Lipschitz if $M = m \geq 2$.

The forcing $\zeta^\omega(t,x)$ is a random field corresponding to a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. It is assumed to be white noise in time t and smooth in the space-variable x. To simplify presentation we restrict ourselves to the case which contains the main difficulties and gives rise to the phenomena we are interested in:

$$\zeta^\omega(t,x) = \eta^\omega(t,x)\dot{w}(t). \tag{1.6}$$

Here $w(t)$ is a Wiener process with respect to an increasing system $\{\mathcal{F}_t\}$ of σ-algebras in $\mathcal{F}$, and the complex r.f. η^ω satisfies the following restrictions:

$(H0)$ $|\eta^\omega(t,x)| \leq 1$ for all ω, t and x.

$(H1)$ η^ω is continuous in (t,x), smooth odd periodic in x for a.a. ω and is adapted to the σ algebras $\mathcal{F}_t$. That is for any $t \geq 0$ and $x \in \mathbb{R}^n$ the r.v. $\eta^\omega(t,x)$ is $\mathcal{F}_t$-measurable.

$(H2)$ For any $m, p \in \mathbb{N}$ and any $t \geq 0$,

$$\mathbf{E}\|\eta^\omega(t,\cdot)\|_m^p \leq C(m,p). \tag{1.7}$$

Assuming (1.6) we define the integral of r.h.s. of (1.1) as follows:

$$\int_{t_0}^{t} \zeta^\omega(s,x)\,ds \overset{def}{=} \int_{t_0}^{t} \eta^\omega(s,\cdot)\,dw(s).$$

That is, we treat η^ω as an adapted random process in H^m (the space is given the Borelian σ-algebra) with uniformly bounded second momenta (we refer to (1.7) with $p = 2$) and define the Ito integral $\int \eta^\omega(s)dw(s) \in H^m$ in the usual L_2-way, see [Dyn, Roz]. [3]

The process $\int_0^t \eta dw(s) \in H^m$ is a.s. continuous. We find a null-set Ω_0 (i.e., $\Omega_0 \in \mathcal{F}$ and $\mathbf{P}\Omega_0 = 0$) such that for any $m \in \mathbb{N}$, the processes $t \mapsto \eta^\omega(t) \in H^m$ and $t \mapsto \int_0^t \eta^\omega(s)dw(s) \in H^m$ are continuous for any $\omega \notin \Omega_0$. The bad null-set Ω_0 will be increased during our proofs countable number of times; we shall not control this process explicitly.

The Burkholder-Davis-Gundy (B-D-G) inequality applies to Ito integrals $\int \xi^\omega(s)dw(s)$, where ξ is a random process in some H^m and provides us with the following result:

Lemma 1. *Let an adapted process $\xi^\omega(t) \in H^m$ satisfies (1.7) with $p = 2$ for $0 \leq t \leq T$. Then*

$$\mathbf{E} \sup_{t_0 \leq t \leq t_1} \| \int_{t_0}^{t} \xi^\omega(s)dw(s)\|_m^q \leq C_q \mathbf{E}\Big(\int_{t_0}^{t_1} \|\xi^\omega(s)\|_m^2 ds\Big)^{q/2} \leq \infty$$

for any $0 \leq t_0 \leq t_1 \leq T$ and any $q \geq 1$.

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[3] In [Dyn] the integrand η is assumed to be an adapted vector-process. The arguments presented in this reference, do not use the fact that the vector space where η is valued, is finite dimensional.

In [IW] the inequality is proven for finite-dimensional vector processes with universal constants C_q, independent of the dimension. To get the B-D-G inequality stated in the lemma, the process $\xi^\omega \in H^m$ should be decomposed to a Hilbert basis of the space H^m. Then the finite-dimensional inequality applied to finite-dimensional approximations to the process implies the result after transition to limit in the dimension of the approximation.

Applying to an integral $\int \xi dw(s)$ as above the Kolmogorov criterion (which remains true for processes valued in a Banach space, see [PZ,Ad]), using (H2) and Lemma 1 with a large q we get:

Corollary 1. *Under the assumptions of Lemma 1, let* $\mathbf{E}\|\xi^\omega(t)\|_m^q \le C(m,q)$ *for all* t *and* q*. Then the process* $t \to \int_0^t \xi dw \in H^m$ *is Hölder continuous for any fixed exponent* $\theta < 1/2$*.*

1.1 Notion of a Solution.

Let us supplement the equation (1.1) with initial condition:

$$u(0,x) = u_0^\omega(x) \text{ for a.a. } \omega. \tag{1.8}$$

Definition 1. A random field $u^\omega(t,x)$ is a solution of (1.1), (1.8) in a space $H^m, m \ge 2$, (or, for short, is an H^m-solution) if the process $t \to u^\omega(t,\cdot) \in H^m$ is adapted, continuous and

$$u(t,\cdot) = u_0(\cdot) + \int_0^t (\sigma \triangle u(s,\cdot) + F(u(s,\cdot)) ds + \int_0^t \eta^\omega(s,\cdot) dw(s), \tag{1.9}$$

for any $t \ge 0$ and a.a. ω.

The first integral in the r.h.s. of (1.9) is a curve in H^{m-2} which depends on the parameter ω, the second is an Ito integral. The l.h.s. and the r.h.s. of (1.9) equal as curves in H^{m-2} for a.a. ω.

Since H^M is embedded to the space of continuous functions if $M \ge 2$ (we recall that $n \le 3$), then a solution $u^\omega(t,x)$ is a continuous r.f..

We say that a r.f. u is a *(space-) smooth* solution for (1.1), (1.8) if it is a solution in each space H^m ($m \ge 2$).

Definitions of H^m-solutions and smooth solutions of the problem (1.1), (1.8) for $t \in [0,T]$ are quite similar. Obviously a r.f. $u(t,x), t \ge 0$, is a solution of (1.1), (1.8) if it is a solution for $t \in [0,T]$ for each $T > 0$.

Some elementary properties of solutions $u^\omega(t,x)$ are given in the following

Proposition 1. *1) Any two solutions* u_1, u_2 *for the problem (1.1), (1.8) coincide a.s.*

2) If S_T *is a semi group generated by the operator* $\sigma\triangle$ *under the odd periodic boundary conditions, then* $u^\omega(t,x)$ *is a solution for (1.1), (1.8) if and only if it satisfies the following integral equation:*

$$u(t,x) = S_t u_0(x) + \int_0^t S_{t-s} F(u)(s,x) ds + \int_0^t S_{t-s} \eta^\omega(s,x) dw(s). \tag{1.10}$$

Statement 2) of the proposition means that $u(t,x)$ is a *mild solution* for the problem (1.1), (1.8), see [PZ].

Proof. 1) The difference $\mu^\omega(t,x)$ of the solutions u_1 and u_2 a.s. satisfies the deterministic equation

$$\dot{\mu} - \sigma\triangle\mu + (F(u_1) - F(u_2)) = 0 \tag{1.11}$$

(with the odd periodic boundary conditions). By this equation, $\mu \in C([0,\infty), H^m) \cap C^1([0,\infty), H^{m-2})$. So μ vanishes due to the usual arguments based on the Granwall lemma.

2) Let $\{\varphi_j\}$ be an exponential basis of the L_2-space of odd periodic complex functions, formed by eigen functions of the operator $-\triangle$ with eigen values $\{\lambda_j\}$. Denoting by $u_j^\omega(t)$ coefficients of decomposition of the solution u^ω in this basis we write (1.9) as

$$u_j(t) - u_{j0} + \sigma\int_0^t \lambda_j u_j(s)ds = \int_0^t F(u(s,\cdot))_j ds + \int_0^t \eta_j(s)dw(s).$$

That is, $du_j = (-\sigma\lambda_j u_j + F(u)_j)dt + \eta_j dw(t)$. For the function $v_j = e^{\sigma\lambda_j t}u_j$ we have (e.g., using the Ito lemma) that

$$v_j(t) = v_{j0} + \int_0^t e^{\sigma\lambda_j s}F(u)_j ds + \int_0^t e^{\lambda_j s}\eta_j dw(s).$$

Hence,

$$u_j(t) = u_{j0} + \int_0^t e^{\sigma\lambda_j(s-t)}(F(u))_j ds + \int_0^t e^{\sigma\lambda_j(s-t)}\eta_j dw(s).$$

This is exactly the j-th component of the relation (1.10). $\square$

1.2 Existence of solutions for equations with uniformly Lipschitz nonlinearities.

The same classical arguments which prove solvability of a stochastic ODE with a Lipschitz nonlinearity are applicable to equation (1.1) with a uniformly Lipschitz nonlinearity:

Theorem 1. *If the nonlinearity $F(u)$ satisfies (1.4) and a random field $u_0^\omega(x)$ is such that $\mathbf{E}\|u_0\|_m^p \le C_p$ for some $p \ge 2$, then the problem (1.1), (1.8) has a unique solution $u^\omega(t,x)$ in the space H^m. Besides, $\mathbf{E}\|u^\omega(t,\cdot)\|_m^p \le C_p(t)$ for any $t \ge 0$.*

This is a well-known result, see [PZ, MaS].

In particular, the initial-value problem for equation (1.5) in Example 2 has a unique solution for any cut-off function $\varphi \in C_0^\infty$.

1.3 Stopping times and localisation

Let $u^\omega(t,x)$ be an H^m-solution of equation (1.1) and $\tau^\omega \geq 0$ be a stopping time with respect to the system of σ-algebras $\{\mathcal{F}_t\}$. We denote by $u_\tau(t,x)$ the stopped process: $u_\tau(t,x) = u(t\wedge\tau, x)$.

This process satisfies the stopped equation:

$$u_\tau(t,\cdot) = u_0(\cdot) + \int_0^t (\sigma\triangle u(s,\cdot) - F(u(s,\cdot))\chi_{s\leq\tau}ds + \int_0^t \eta^\omega(s,\cdot)\chi_{s\leq\tau}dw(s). \qquad (1.12)$$

(To deduce (1.12) from (1.9) one should repeat for the process $t \to u(t) \in H^m$ usual finite-dimensional arguments, see [Dyn], section 11.13)

Adapting Definition 1 to the equation (1.12) we say that a r.f. $u^\omega(t,x)$ as in Definition 1 is an H^m-solution of (1.12) if $u^\omega(t,x) = u^\omega(t\wedge\tau,x)$ and left and right hand sides of (1.12) with $u_\tau := u$ coincide a.s. as continuous curves in H^m.

The most important for us are the stopping times $\tau_M = \tau_{M,m}$ of the form

$$\tau_M = \tau_M(u) = \min\{t\geq 0 \mid \|u(t)\|_m \geq M\}. \qquad (1.13)$$

Lemma 2. *For $j=1,2$, let u^j be a solution of equation (1.12) with $\tau=\tau^j$. Then a.s. $u^1_\tau = u^2_\tau$, where $\tau = \tau^1\wedge\tau^2$. The result remains true if u^j is a solution of (1.12) with $F=F_j$ and $\tau=\tau^j$, provided that both stopping times τ^j have the form $\tau^j = \tau_{M_j}$ and $F_1(u)=F_2(u)$ if $\|u\|_m \leq M_1\wedge M_2$.*

Proof. In both cases for $t\leq\tau^\omega$ the difference $\mu = u_1 - u_2$ satisfies the deterministic equation (1.11) with zero initial conditions, so it vanishes. □

For any $M\in\mathbb{N}$ let us take a real function $\varphi_M \in C_0^\infty(\mathbb{R})$ such that $\varphi_M(r)=1$ for $0\leq r\leq M$. We cut out the nonlinearity F of equation (1.1), multiplying it by $\varphi_M(\|u\|_m)$ and consider the equation

$$\dot u - \sigma\triangle u + \varphi_M(\|u\|_m)F(u) = \zeta^\omega(t,x), \qquad (1.1_M)$$

(cf. Example 2). The nonlinearity is uniformly Lipschitz, so the problem (1.1_M), (1.8) has a unique H^m-solution $u=u^M$. If u^{M_1} and u^{M_2} are solutions for the problem (1.1_M), (1.8) with $M=M_1$ and $M=M_2$ respectively and $M_1\leq M_2$, then by Lemma 2 $\tau_{M_1}(u^{M_1}) = \tau_{M_1}(u^{M_2})$. Hence,

$$u^{M_1}_{\tau_N} = u^{M_2}_{\tau_N} \ \text{ if } \ N\leq M_1, M_2.$$

In particular, the stopping time τ_N does not depend on a solution u^{M_j} used for its construction, provided that $N\leq M_j$. In this way we obtain a well-defined r.f. $u^\omega_N(t,x) = (u^{M\omega})_{\tau_N}(t,x)$, $M\geq N$, and a stopping time τ_N. Moreover, any two solutions u_{N_1} and u_{N_2} agree in the sense that

$$(u_{N_1})_{\tau_N} = (u_{N_2})_{\tau_N} \ \text{ where } \ N\leq N_1, N_2. \qquad (1.14)$$

Let us fix any finite $T>0$ and define the sets $\Omega_N\in\mathcal{F}$:

$$\Omega_N = \{\omega \mid \tau_N \geq T\}.$$

By (1.14), the random fields u_{N_1} and u_{N_2} coincide for ω from Ω_N if $N_1,N_2\geq N$. Hence, the map $\omega \to u^\omega_N(t;\cdot) \in C(0,T;H^m)$ converges as $N\to\infty$ to a limiting map $u^\omega_\infty(t;\cdot)$ for each $\omega\in\Omega_\infty = \cup\Omega_N$.

We sum up information on the stopped solutions and on their convergence in the following:

Lemma 3. *Let* $u^{M\omega}(t,x)$ *be a solution for the problem* (1.1_M), *(1.8) and for* $N < M$ *let* $\tau = \tau_N(u^M)$ *be the stopping time defined as in (1.13). Then,*
1) the r.f. $u^\omega_N(t,x) := (u^{M\omega})_{\tau_N}(t,x)$ *is well defined — it does not depend on* $M \geq N$*;*
2) the r.f.'s u_{N_1} *and* u_{N_2} *coincide for* $\omega \in \Omega_N$ *if* $N_1, N_2 \leq N$. *Altogether they define a measurable map* $\Omega_\infty = \cup\Omega_N \to u^\omega_\infty(t,\cdot) \in C(0,T;H^m)$.
3) If $\mathbf{P}(\Omega_N) \to 1$ *as* $N \to \infty$, *then the r.f.* $u^\omega(t,x)$, *defined as* u^ω_∞ *for* $w \in \Omega_\infty$ *and as zero for* $w \notin \Omega_\infty$ *is a solution of the problem (1.1), (1.8) for* $0 \leq t \leq T$.

Proof. It remains to check the last assertion of the lemma. For $M \geq N$ the stopped solution $(u^M)_{\tau_N} = u_N$ satisfies the equation

$$u_N(t,x) = u_0(x) + \int_0^t (\sigma\triangle u_N(s,x) + \varphi_M F(u_N))\,\chi ds + \int_0^t \eta^\omega(s,\cdot)\chi dw(s),$$

where $0 \leq t \leq T$ and $\chi = \chi_{s\leq\tau_N}$. Let us compare this equation with (1.9). For $\omega \in \Omega_N$ we have $u = u_N, \varphi_M = 1$ and $\chi = 1$. Thus, the l.h.s.'s of (1.9) and of the last equation coincide for $\omega \in \Omega_N$, as well as the two first terms in the r.h.s.'s. Since $\eta = \eta\chi$ in Ω_N, then the stochastic integrals in the r.h.s.'s also are equal for a.a. $\omega \in \Omega_N$ due to a basic property of the Ito integral (see [Dyn], section 7.3).

We have seen that the function u satisfies (1.9) a.s. in Ω_N, for any N. It means that (1.9) holds a.s. and the lemma is proven. □

1.4 Ito Lemma.

We denote $V(u) = \sigma\triangle u - F(u)$ and abbreviate the stopped equation (1.12) as follows:

$$u_\tau(t) = u_0 + \int_0^t V(u_\tau(s))\chi ds + \int_0^t \eta(s)\chi dw(s), \tag{1.15}$$

where $\chi = \chi^\omega_{s\leq\tau}$. Let u_τ be a solution of (1.15). That is, the r.f. $u^\omega_\tau(t,x) = u^\omega_\tau(t\wedge\tau, x)$ defines a continuous process $u_\tau(t) \in H^m$ such that the r.h.s. and l.h.s. of (1.15) coincide as curves in H^{m-2}.

Let $G : H^{m-2} \to Z$ be a C^2-smooth map to a Hilbert space Z such that the maps $G(u), dG(u)$ and $d^2G(u)$ are uniformly bounded on bounded subsets of H^{m-2}.

Lemma 4. *The process* $g^\omega(t) = G(u^\omega_\tau(t)) \in Z$ *satisfies the following stochastic equation in* Z*:*

$$\begin{aligned} g(t) = g(0) + \int_0^t \Big(dG(u_\tau(s))V(u_\tau(s)) + \frac{1}{2}d^2G(u_\tau(s))(\eta(s),\eta(s))\Big)\chi ds \\ + \int_0^t dG(u_\tau(s))\eta(s)\chi dw(s), \end{aligned} \tag{1.16}$$

provided that for any finite T *we have:*

$$\mathbf{E}|dG(u_\tau(s))\eta(s)\chi|^2 \leq C_T < \infty \ \ \textit{for}\ \ 0 \leq s \leq T. \tag{1.17}$$

The lemma is proved in [PZ], section 4.5, without the extra restriction (1.17). We imposed it here to be able to treat the stochastic integral in (1.16) in the L_2-sense.

2 SNLS and stopped SNLS equations.

Now we pass to the stochastic nonlinear Schrödinger (SNLS) equations, which are our main goal in this work:

$$\dot{v}(t,x) - \delta\triangle v + i|v|^2 v = \zeta^\omega(t,x), \tag{2.1}$$

$$v|_{t=0} = \xi^\omega(x), \tag{2.2}$$

where $0 < \delta \leq 1$ and the r.f. ζ has the form (1.6), i.e.

$$\zeta^\omega(t,x) = \eta^\omega(t,x)\dot{w}(t).$$

The initial condition $\xi^\omega(x)$ is such that

$$\mathbf{E}|\xi^\omega|_\infty^p \leq C_p \delta^{-p/2}, \quad \mathbf{E}\|\xi^\omega\|_m^2 \leq C_m \delta^{-2m-1}, \tag{2.3}$$

for any $p \geq 1$ and any $m \in \mathbb{N}$. The r.f.'s ζ and ξ are odd periodic in x, as well as the solution v we are looking for.

Introducing the fast time $\tilde{t} = \delta t$ and denoting $v(\tilde{t}/\delta, x) = u(\tilde{t}, x)$ we rewrite the equation (2.1) in the form

$$\frac{\partial u}{\partial \tilde{t}} - \triangle u + i\delta^{-1}|u|^2 u = \delta^{-1}\eta\left(\tilde{t}/\delta, x\right)\dot{w}\left(\tilde{t}/\delta\right) = \delta^{-1/2}\eta\left(\tilde{t}/\delta, x\right)\frac{\partial}{\partial \tilde{t}}\left(\delta^{1/2}w\left(\tilde{t}/\delta\right)\right).$$

The random process $\tilde{w}^\omega(s) = \delta^{1/2}w^\omega(s/\delta)$ is Wiener and the random field $\tilde{\eta}^\omega(\tilde{t}, x) = \eta^\omega(\tilde{t}/\delta, x)$ satisfies the assumptions $(H0) - (H2)$ as soon as η does. Abusing notations we drop the tildes and rewrite the equation for u in the following form:

$$\begin{aligned} \dot{u} - \triangle u + iK^2|u|^2 u &= K\eta(t,x)\dot{w}(t), \quad K = \delta^{-1/2}, \\ u\,|_{t=0} &= \xi^\omega(x). \end{aligned} \tag{2.4}$$

Below we fix any $m \geq 2$ and study H^m-solutions for the problem (2.4) with large K (i.e., with small δ). A solution for this problem satisfies the integral equation:

$$u(t,x) = \xi^\omega(x) + \int_0^t (\triangle u(s,x) - iK^2|u|^2 u)ds + K\int_0^t \eta(s,x)dw(s), \quad t \geq 0. \tag{2.5}$$

Proceeding as in the section 1.3, we fix any $N \geq 1$ and modify the nonlinearity $-iK^2|u|^2 u$, multiplying it by $\varphi_N(\|u\|)_m$, where $\varphi_N \in C_0^\infty$ and $\varphi_N(r) = 1$ for $|r| \leq N$:

$$u(t,x) = \xi^\omega(x) + \int_0^t (\triangle u(s,x) - iK^2\varphi_N(\|u\|_m)|u|^2 u)ds + K\int_0^t \eta(s,x)dw(s). \tag{2.5_N}$$

By Theorem 1 the equation (2.5_N) has a unique smooth solution $u^{N\omega}(t,x)$. Let $\tau = \tau_M(u^N)$ be the stopping time (1.13), i.e.,

$$\tau_M = \min\{t \geq 0 \mid \ \|u^{N\omega}(t)\|_m \geq M\}. \tag{2.6}$$

By Lemma 2 the r.f. $u_\tau^\omega(t,x) = u^{N\omega}(t \wedge \tau_{M,} x)$ does not depend on $N \geq M$ and satisfies the stopped equation:

$$u_\tau(t,x) = \xi^\omega(x) + \int_0^t (\triangle u_\tau(s,x) - iK^2|u_\tau|^2 u_\tau)\chi_{s\le\tau}ds + K\int_0^t \eta(s,x)\chi_{s\le\tau}dw(s). \quad (2.5_\tau)$$

Below we omit the cut-off parameter N which was originally used to construct the stopped solution u_τ.

Since the process $t \to u^\omega(t,\cdot) \in H^m$ is continuous, then a.s. the deterministic integral in (2.5_τ) defines a Lipschitz curve in H^{m-2}. By Corollary 1, the stochastic integral in (2.5_τ) defines a continuous random process in H^m.

Lemma 5. *For any $L \in \mathbb{N}$ the process $t \to u_\tau(t,\cdot) \in H^L$ is continuous. For any $T < \infty$ and $p \ge 1$ it satisfies the estimate*

$$\mathbf{E} \sup_{0\le t\le T} (\|u_\tau(t,\cdot)\|_L^p \ \chi_{t\le\tau}) < \infty. \quad (2.7)$$

Proof. To prove (2.7) we shall compare u_τ with a solution for the linear equation

$$v(t,x) = \xi^\omega(x) + \int_0^t \triangle v(s,x)ds + K\int_0^t \eta(s,x)dw(s), \quad 0 \le t \le T. \quad (2.8)$$

This equation has a unique smooth solution which satisfies the estimate

$$\mathbf{E}\|v(t)\|_L^p \le C(L,p,T,K) \quad \forall\, p \ge 1,\ \forall\, 0 \le t \le T, \quad (2.9)$$

see [Roz, PZ]. The difference $h = u_\tau - v$ vanishes at $t = 0$ and solves the equation

$$\dot h - \triangle h = -iK^2|u_\tau|^2 u_\tau, \quad 0 \le t \le \tau^\omega \wedge T. \quad (2.10)$$

Since $\|u_\tau\|_m \le M$, then $\|iK^2|u_\tau|^2 u_\tau\|_m \le CK^2M^3$ and

$$\|h(t)\|_{m+1} \le C_T K^2 M^3. \quad (2.11)$$

This estimate is a consequence of the a priori bound

$$\sup_{0\le t\le T\wedge\tau} \|h(t)\|_{\ell+1} \le C\sqrt{T} \sup_{0\le t\le T\wedge\tau} \|iK^2|u_\tau|^2 u_\tau\|_\ell\,. \quad (2.12)$$

(It follows immediately after h and the r.h.s. of the equation are decomposed to Fourier series in x.)

By (2.9), (2.11) all momenta of the r.v.

$$\chi_{t\le\tau} \sup_{0\le t\le T} \|u_\tau(t)\|_{m+1} \quad (2.13)$$

are bounded. Due to (2.12) with $\ell = m+1$ this implies that the momenta of $\sup_{0\le t\le T\wedge\tau}$ $\|h(t)\|_{m+2}$ are bounded, as well as all momenta of the r.v. (2.13) with $m+1$ replaced by $m+2$. Hence, (2.7) holds true for any L and p. Due to (2.7), the solution h of (2.10) is a.s. H^L-continuous for $0 \le t \le \tau^\omega$, as well as $u_\tau = h + v$. Since u_τ is constant for $t \ge \tau$, then it is continuous for $0 \le t \le T$ and the lemma is proven. ☐

Corollary 2. *The r.f. u_τ is a smooth solution of (2.5_τ).*

3 L^∞-estimates for stopped solutions.

In this section we obtain estimates for momenta of the r.v. $\sup_x|u_\tau(t,x)|$, independent of the stopping level M.

3.1 An equation for $|u(t,x)|$.

Let us fix any smooth function $\zeta(r)$ equal r for $r\ge 1$ and vanishing for $r\le 1/2$. We denote by Z^ℓ the Sobolev space $Z^\ell = H^\ell_{op}(\mathbb{R}^n,\mathbb{R})$ formed by real-valued functions from H^ℓ and consider the map $G: H^\ell\to Z^\ell,\ \ u(x)\to\zeta(|u(x)|)$. This map is smooth if $\ell\ge 2$. Besides,

$$dG(u(x))v(x)=\zeta'(|u(x)|)\frac{u}{|u|}\cdot v$$

and

$$d^2G(u(x))(v(x),v(x))=\zeta''(|u(x)|)\left(\frac{u}{|u|}\cdot v\right)^2+\zeta'(|u|)\left(\frac{|v|^2}{|u|}-\frac{1}{|u|^3}(u\cdot v)^2\right),$$

where $\cdot$ stands for the scalar product in $\mathbb{R}^2\simeq\mathbb{C},\ u\cdot v=\operatorname{Re}\bar u v$.

Due to Lemma 5 and (H2), assumption (1.17) holds and Lemma 4 applies to the equation (2.5_τ). Before to write an equation for the process $g_\tau(t)=\zeta(|u_\tau(t,\cdot)|)$, we transform the term $dG(u)V(u)$. We have:

$$dG(u)(\triangle u-iK^2|u|^2u)=\zeta'(|u|)\frac{u}{|u|}\cdot(\triangle u-iK^2|u|^2u)=\zeta'(|u|)\frac{u}{|u|}\cdot\triangle u.$$

Writing $u_\tau=u_\tau(t,x)$ in the polar forms as $u_\tau=re^{i\varphi}$, where $r=|u_\tau|$, we have:

$$\triangle u_\tau=(\triangle r-r|\nabla\varphi|^2)e^{i\varphi}+i(2\nabla r\cdot\nabla\varphi+r\triangle\varphi)e^{i\varphi}.$$

Therefore,

$$u_\tau\cdot\triangle u_\tau=\ \operatorname{Re}\,(\bar u_\tau\triangle u_\tau)=r\triangle r-r^2|\nabla\varphi|^2.$$

Now Lemma 4 implies the following relation:

$$\zeta(r(t,x))=\int_0^t\Big(\zeta'(r)(\triangle r-r|\nabla\varphi|^2)+\frac12K^2\zeta''(r)(e^{i\varphi}\cdot\eta)^2$$
$$+\frac12K^2\zeta'(r)r^{-1}(|\eta|^2-(e^{i\varphi}\cdot\eta)^2)\Big)\chi\,ds+K\int_0^t\chi\zeta'(r)e^{i\varphi}\cdot\eta\,dw(s).$$

The idea to study the r.f. $r=|u_\tau|$ is to compare $\zeta(r)$ with a solution $v(t,x)$ for the following linear stochastic equation:

$$dv-\triangle v dt=K\tilde\eta^\omega(t,x)dw(t), \tag{3.1}$$
$$v(0,x)=|\xi^\omega(x)|=:v_0(x), \tag{3.2}$$

where $\tilde\eta=\zeta'(r)e^{i\varphi}\cdot\eta$. Obviously, $\tilde\eta$ is a continuous adapted r.f. which vanishes near ∂K^n and satisfies the estimate

$$|\tilde\eta(t,x)|\le C\quad\forall\,t,x,\omega.$$

We have to estimate $|u_\tau|$ and v, for $x \in K^n$. To do it we fix odd periodic extensions of $\tilde\eta$ and v_0 from K^n to the whole $\mathbb{R}^n$ and denote the extended r.f.'s also as $\tilde\eta$ and v_0. Now we specify v as an odd periodic solution for (3.1), (3.2). This solution satisfies certain estimates which play for the theory we develop in this work a role similar to the role which the maximum principle (see e.g. [La]) plays for deterministic equations:

Theorem 2. *The problem (3.1), (3.2) has a unique H^m-solution v. For any $J = 0, 1, \dots$ and any $q \geq 1$ it satisfies the estimate*

$$\mathbf{E}\left(\sup_{J\leq t\leq J+1}\ \sup_{x\in K^n} |v|\right)^q \leq C_q K^q. \tag{3.3}$$

Existence and uniqueness of a solution are obvious since $\|\tilde\eta(t)\|_m \leq C(M)$ for any t. To prove (3.3) we write v as $v = v_1 + v_2$, where v_1 is a solution of the problem (3.1), (3.2) with zero r.h.s. (i.e., $K\tilde\eta := 0$) and v_2 is a solution of the equation (3.1) with zero initial condition at $t = 0$. By the maximum principle, $0 \leq v_1^\omega(t,x) \leq |\xi^\omega(x)|_\infty$. So the estimate (3.3) for v_1 follows from (2.3). It remains to get the estimate (3.3) for $v_2(t,x)$ which is a solution of (3.1) subject zero initial conditions. We present its proof in the Appendix (in fact, we do more and prove there an estimate for a Hölder norm of the function $v_2|_{[J,J+1]\times K^n}$). We note that estimates, similar to (3.3), follow from a general theory developed in [Kry].

To compare $\zeta(r)$ with v we denote by h the difference $h = \zeta(r) - v$. For $0 \leq t \leq \tau$ the function h satisfies the deterministic equation, depending on the parameter ω:

$$\begin{aligned}\dot h(t,x) =&(\zeta'(r)\triangle r - \triangle v) - \zeta'(r)r|\nabla\varphi|^2\\ &+\frac12 K^2\zeta'(r)r^{-1}(|\eta|^2 - (e^{i\varphi}\cdot\eta)^2) + \frac12 K^2\zeta''(r)(e^{i\varphi}\cdot\eta)^2.\end{aligned} \tag{3.4}$$

Let us fix any finite $T > 0$ and denote $T^\omega = T \wedge \tau^\omega$. All estimates below are T-independent, unless T-dependence is stated explicitly.

We shall study the equation (3.4) in piece-wise cylindric sub-domains of the cylinder $Q_\omega = [0, T^\omega] \times K^n$, where

Definition 2. An open sub-domain $Q_R \subset Q_\omega$ is called *piece-wise cylindric* if there exist points $0 = t_0 < t_1 < t_2 < \cdots < t_R = T^\omega$ and C^1-smooth open domains $D_j \subset K^n, j = 0, \dots, R-1$ (some of them may be empty) such that Q_R equals to interior of the set

$$[t_0, t_1) \times \bar D_0 \cup [t_1, t_2) \times \bar D_1 \cup \cdots \cup [t_{R-1}, t_R) \times \bar D_{R-1}. \tag{3.5}$$

By $\partial_+ Q_R$ we denote a part of the boundary of Q_R where the external normal is not parallel to the time-axis, ie. $\partial_+ Q_R$ equals to the boundary of the set (3.5) (see Fig. 1 where $\partial_+ Q_R$ is drawn in bold). We also denote

$$\partial_0 Q_R = \{t_0, t_1\} \times \partial D_0 \cup \{t_1, t_2\} \times \partial D_1 \cup \cdots \cup \{t_{R-1}\} \times \partial D_{R-1}$$

and $Q_R^- = \overline{Q}_R \setminus \overline{\partial_+ Q_R}$.

We note that the set $\partial_0 Q_R$ contains all singularities of the boundary ∂Q_R minus the set $\{t_R\} \times \partial D_{R-1}$. The former set (i.e. $\partial_0 Q_R$) is bigger than the latter if some domains D_j coincide. We also note that the number R of pieces of a piece-wise cylindric domain is not uniquely defined since, for example, the cylinder Q_ω may be viewed as a domain Q_R with any $R \geq 1$ and with $D_0 = \cdots = D_{R-1} = K^n$.

Fig. 1.

Since the r.f. u_τ is Hölder, then a.s. we can find a piece-wise cylindric domain $Q_R \subset Q_\omega$ (possibly disconnected) such that

$$r = |u_\tau| \geq K - \frac{1}{2} \quad \text{inside} \quad Q_R \quad \text{and} \quad r \leq K + \frac{1}{2} \quad \text{outside} \quad Q_R. \tag{3.6}$$

Inside Q_R we have $\zeta(r) = r$ and equation (3.4) simplifies to

$$\dot{h} - \triangle h = \frac{K^2}{2r}|\eta|^2 - \left(r|\nabla\varphi|^2 + \frac{K^2}{2r}(e^{i\varphi} \cdot \eta)^2 \right) =: g(t,x),\ (t,x) \in Q_R, \tag{3.7}$$

$$h|_{\partial_+ Q_R} = (r - v)|_{\partial_+ Q_R} =: m(t,x). \tag{3.8}$$

Due to the initial condition (3.2), $m(0,x) \equiv 0$.

3.2 Heat equation in piece-wise cylindric domains.

In this subsection we consider the boundary value problem (3.7), (3.8), forgetting the specific form of the r.h.s. g and of the boundary function m.

Lemma 6. *If $g(t,x)$ is a Hölder function in Q_R and $m(t,x)$ is a bounded Borel function on $\partial_+ Q_R$, continuous outside $\partial_0 Q_R$, then (3.7), (3.8) has a unique solution $h(t,x)$ such that*
1) $h \in C^{1,2}(Q_R^-)$, is bounded in Q_R and satisfies there the equation (3.7);
2) h is continuous in $\overline{Q}_R \backslash \partial_0 Q_R$ and satisfies the boundary condition (3.8) in $\partial_+ Q_R \backslash \partial_0 Q_R$;
3) if $g \leq 0$, then h satisfies the maximum principle:

$$h(t,x) \leq \sup_{\tau \leq t} \sup_{(\tau,y) \in \partial_+ Q_R} m(\tau, y) \tag{3.9}$$

for any $(t,x) \in Q_R$.

Proof. i) Existence. For $j = 0, 1, \dots, R-1$ we denote $Q_j = (t_j, t_{j+1}] \times D_j$, define sets $\partial_+ Q_j, \partial_0 Q_j \subset \partial Q_j$ as in Definition 2 and set $\Gamma_j = [t_j, t_{j+1}] \times \partial D_j$. In the domain Q_0 we solve the first boundary value problem for the heat equation (3.7) and find a solution $h_0(t,x)$ such that

$$h_0|_{t=0} = m|_{t=0}, \quad h_0|_{\Gamma_0} \ = m|_{\Gamma_0}.$$

The function h_0 is as smooth as specify items 1), 2) of the lemma (see [LSU], Theorems 16.1, 16.2).

Next we find a function $h_1(t,x)$ in the cylinder Q_1 which satisfies (3.7) as well as the boundary conditions:

$$h_1(t_1,x) = h_0(t_1,x) \text{ for } x \in D_0, \quad h_1(t_1,x) = m(t_1,x) \text{ for } x \in D_1 \setminus D_0,$$
$$h_1 \,|_{\Gamma_1} = m \,|_{\Gamma_1} .$$

The function h_{01}, equal to h_0 in Q_0 and equal to h_1 in Q_1, is continuous in the domain $Q_{01} = Q_0 \cup Q_1$. In the vicinity of $\partial\overline{Q}_0 \cap \partial\overline{Q}_1$ in this domain h_{01} is a generalised solution of (3.7), so it is $C^{1,2}-$ smooth there (see [La, LSU]). That is, in the domain Q_{01} the function h_{01} satisfies 1) and 2).

Iterating this procedure we get a solution $h = h_{01\dots(R-1)}$ of (3.7) in Q_R, which meets 1) and 2).

ii) Maximum principle for $g = 0$. Now we shall show that any solution h of (3.7), (3.8) with $g = 0$ which satisfies 1) and 2), also satisfies the estimate (3.9).

First we prove the estimate for $h_0 = h|_{Q_0}$. Let O_ε be the ε-neighbourhood of $\partial_0 Q_0$ in Q_0 (see Fig. 2) and $Q_\varepsilon = Q_0 \setminus \overline{O_\varepsilon}$. The function $h_\varepsilon = h|_{Q_\varepsilon}$ is a classical solution for a boundary-value problem for (3.7) in Q_ε.

Fig. 2. ($\partial_+ Q_0 \cap \overline{Q_\varepsilon}$ is drawn in bold)

To estimate h_ε we extend the continuous function $m|_{\partial_+ Q_0 \cap \bar{Q}_\varepsilon}$ to a continuous function $m_{1\varepsilon}$ on $\partial\overline{Q_\varepsilon} \setminus \{t_1\} \times D_0$ having the same C^0-norm and denote by $h_{1\varepsilon}$ a classical solution for the corresponding boundary-value problem for (3.7) in Q_ε. By classical arguments [La] this function satisfies the maximum principle (3.9) with the function m replaced by $m_{1\varepsilon}$. The difference $h_{2\varepsilon} = h_\varepsilon - h_{1\varepsilon}$ solves (3.7) in Q_ε, vanishes at $\partial_+ Q_0 \cap \partial\bar{Q}_\varepsilon$ and at $\partial\bar{O}_\varepsilon \cap Q_\varepsilon$ it is bounded by $C_* := \sup h + \ \sup m < \infty$.

By classical arguments (see [La]),

$$\sup_{t \geq \delta} \ \sup_{(t,x) \in Q_\varepsilon} |h_{2\varepsilon}(t,x)| \leq C_* \cdot o(1) \qquad (\varepsilon \to 0) \tag{3.10}$$

for any fixed $\delta > 0$.

Since for $t \geq \varepsilon$ we have $h = h_\varepsilon = h_{2\varepsilon} + h_{1\varepsilon}$, then (3.8) is proven for $t \geq \varepsilon \vee \delta$ with m replaced by $m + C_* o(1)$. Sending to zero ε and δ, we recover (3.8).

iii) Uniqueness is now obvious since the difference of any two solutions solves the problem (3.7), (3.8) with $g = 0, m = 0$ and must vanish.

iv) Maximum principle for $g \leq 0$ follow from its counterpart with $g = 0$. Indeed, a solution for (3.7), (3.8) with a Hölder function $g \leq 0$ equals to the sum of a classical solution for the problem with $g := g, m := 0$ and a solution for the problem with $g := 0$, $m := m_0$. The former is ≤ 0 by the classical maximum principle while the latter satisfies (3.9) due to the step ii) of the proof. $\square$

By the lemma, the problem (3.7), (3.8) with $g = 0$ defines positive linear functionals

$$C^0(\partial_+ Q_R) \ni m(\cdot) \to u(t,x), \quad (t,x) \in Q_R.$$

Their norms are bounded by one due to (3.9). Hence, there exist a (t,x)-dependent Borel measure $G(t,x;\cdot)$ on $\partial_+ Q_R$ such that

$$u(t,x) = \int\limits_{\partial_+ Q_R} G(t,x;d\xi)\, m(\xi), \quad \xi = (t_\xi, x_\xi).$$

The measures $G(t,x;\cdot)$ are probabilistic since to the function $m \equiv 1$ they correspond the solution $u \equiv 1$. We call G the *Green measure* for the problem (3.7), (3.8) and treat it as a measure on Q_R, supported by $\partial_+ Q_R$.

For any $a \leq b$ let us denote by $Q_{[a,b]}$ the layer in Q_R,

$$Q_{[a,b]} = Q_R \cap [a,b] \times K^n.$$

The sets $Q_{(a,b]}$ and $Q_{[a,b)}$ are defined similar. The Green measure G is future independent:

$$G(t,x;Q_{(t,R]}) = 0. \tag{3.11}$$

Indeed, $G(t,x;Q_{(t+1/N,R]}) = 0$ for any $N \geq 1$ by (3.9) so (3.11) follows due to the continuity of the measure G. What is more important, the Green measure forgets the past exponentially fast:

Lemma 7. *For any* $0 \leq s \leq t' \leq T^\omega$ *we have:*

$$G(t',x';\;\; Q_{[0,t'-s]}) \leq 2^{n/2} e^{-n\pi^2 s/4} \quad \forall x'. \tag{3.12}$$

Proof. Let us denote the function in the l.h.s. of (3.12) by $f(t',x')$. This function solves (3.7), (3.8) with $g = 0$ and $m = m_s(t,x)$, where m_s equals one for $t \leq t' - s$ and equals zero otherwise. This solution suits Lemma 6 if we add to the piece-wise cylindric domain Q_R an artificial singularity at the point $\tilde{t} = t' - s$ and replace Q_R by the corresponding domain Q_{R+1} (i.e., we find a segment (t_j, t_{j+1}) which contains $\tilde{t}$ and replace in (3.5) the cylinder $[t_j, t_{j+1}] \times D_j$ by $[t_j, \tilde{t}) \times D_j \cup [\tilde{t}, t_{j+1}) \times D_j$). To estimate $f(t,x)$ from above we come back to the cylinder $\Pi_\omega = [0, T^\omega] \times K^n$. In the cube K^n we consider the function

$$\Psi(x) = 2^{n/2} \prod \cos\frac{\pi}{2}\left(x_j - \frac{1}{2}\right). \tag{3.13}$$

Obviously, $2^{n/2} \geq \Psi \geq 1$ everywhere in K^n and $-\triangle\Psi = \frac{n}{4}\pi^2\Psi$. The function $U(t,x) = e^{-n\pi^2(t-\tilde{t})/4}\Psi(x)$ solves (3.7) in the cylinder Π_ω. Let us compare $f(t,x)$ with $U\mid_{Q_{[\tilde{t},T^\omega]}}$. Since $U(\tilde{t},x) = \Psi(x) \geq 1 \geq f(\tilde{t},x)$ everywhere in $Q_R \cap \{t = \tilde{t}\}$, then $U \geq f$ in $\partial_+ Q_{[\tilde{t},T^\omega]}$. Hence, $U \geq f$ in $Q_{[\tilde{t},t^\omega]}$ by the maximum principle (3.9) and (3.12) follows.

3.3 Estimate for $|u_\tau|$.

Now we can continue to study the function $h = \zeta(|u_\tau|) - v$ in the domain Q_R as in (3.6). Since the defined in (3.7) function g is $\le K^2|\eta|^2/2r$ and since $r \le K + \frac{1}{2} \le 2K$ on $\partial_+ Q_R$, then by the maximum principle (3.9) we have

$$h(t,x) \le h_1(t,x) + h_2(t,x) \text{ in } Q_R,$$

where the random fields h_1 and h_2 satisfy the following boundary value problems in the random domain Q_R:

$$\dot{h}_1 - \triangle h_1 = 0, \qquad h_1|_{\partial_+ Q_R} = 2K - v|_{\partial_+ Q_R}, \tag{3.14}$$

$$\dot{h}_2 - \triangle h_2 = \frac{K^2}{2r}|\eta|^2, \qquad h_2|_{\partial_+ Q_R} = 0. \tag{3.15}$$

It remains to estimate h_1 and h_2. We start with the easier problem (3.15) and consider the function $\Psi_1(t,x)$,

$$\Psi_1 = \frac{4K}{n\pi^2}\Psi(x),$$

where Ψ was defined in (3.13). Obviously, $\Psi_1 \ge h_2$ in $\partial_+ Q_R$. Besides,

$$\left(\frac{\partial}{\partial t} - \triangle\right)\Psi_1 = -\triangle\Psi_1 = K\Psi(x) \ge K \ge \frac{K^2}{2r}|\eta|^2$$

in Q_R, since there $r \ge K - \frac{1}{2} \ge K/2$ and $|\eta|^2 \le 1$ by (H0). Hence, $\Psi_1 \ge h_2$ in Q_R and

$$h_2(t,x) < 2^{n/2}K \text{ in } Q_R.$$

To estimate in Q_R the solution $h_1(t,x)$ of (3.14) we write it in terms of the Green measure:

$$h_1(t,x) = \int\limits_{\partial_+ Q_R} (2K - v(\xi))G(t,x;d\xi) = 2K - \int\limits_{\partial_+ Q_R} v(\xi)G(t,x;d\xi), \quad (t,x) \in Q_R.$$

Applying (3.11) and (3.12) we get an estimate which holds uniformly in $t \in [J, J+1]$ and $x \in K^n$:

$$\begin{aligned} h_1(t,x)\chi_{t\le\tau} &\le 2K + \sum_{j=0}^{J} \int\limits_{Q_{[J-j,J-j+1]}\cap\partial_+ Q_R} G(t,x;d\xi)|v(\xi)| \\ &\le 2K + 2^{\frac{n}{2}} \sum_{j=0}^{J} e^{-n\pi^2 j/4} \sup_{J-j\le\tau\le J-j+1} \sup_y |v(\tau,y)|. \end{aligned}$$

Since $|u_\tau| = r = \zeta(r)$ inside Q_R and $r \le 2K$ outside, then $r \le \max(2K, h_1 + h_2)$ and the r.v.

$$S_J := \sup_{J\le t\le J+1} \sup_x |u_\tau(t,x)|\chi_{t\le\tau}$$

satisfies the estimate

$$S_J \le 2^{n/2}K + 2K + 2^{n/2}\sum_{j=0}^{J} e^{-n\pi^2 j/4} \sup_{J-j\le\tau\le J-j+1} \sup_y |v(\tau,y)|.$$

By Theorem 2, the m-th moment of the sum in the r.h.s. is bounded by $C_m K^m$. Hence

$$\mathbf{E} S_J^m \le C_m K^m \;\; \forall m \ge 1. \tag{3.16}$$

Since the constants C_m are T-independent, then we have proved □

Theorem 3. *Let τ be any stopping time of the form (2.6) and $u_\tau(t,x)$ be a stopped solution for problem (2.4). Then for any natural number J and any $m \ge 1$ the random variable $S_J = \sup\limits_{J \le t \le J+1} \sup\limits_x |u_\tau(t,x)| \chi_{t \le \tau}$ satisfies the estimate (3.16). The constant C_m does not depend on J and on M from (2.6).*

4 Estimating of Sobolev norms of stopped solutions and passing to a limit

We continue to study a solution $u_\tau(t,x)$ for the stopped equation (2.5_τ). In this section we are interested in M-independent estimates for its Sobolev norms.

From the Corollary to Lemma 5 we know that for any $L \ge 2$ the function u_τ is an H^L-solution for the equation (2.5_τ) and satisfies the estimates (2.7). Hence, the Ito formula (Lemma 4) applies to the functional $G(u) = \|u_\tau\|_L^2$. Since $dG(u)\xi = 2\langle u, \xi \rangle_L$ and $d^2 G(u)(\xi,\xi) = 2\|\xi\|_L^2$, then taking the expectation of (1.16) and abbreviating $\chi_{s \le \tau}$ to χ we get:

$$\mathbf{E}\|u_\tau(t)\|_L^2 = \mathbf{E}\|\xi\|_L^2 + \mathbf{E}\int_0^t (2\langle u_\tau, \Delta u_\tau + 2iK^2 |u_\tau|^2 u_\tau \rangle_L + \|\eta(s)\|_L^2)\chi ds.$$

Let us denote $g_L(t) = \mathbf{E}\|u_\tau(t)\|_L^2$. Then the last equality and (H2) imply that

$$g_L(t) \le g_L(0) + 2\int_0^t (-g_{L+1}(s) + K^2 \mathbf{E}\langle u_\tau, |u_\tau|^2 u_\tau \rangle_L \,\chi + C) ds. \tag{4.1}$$

Lemma 8. *If $L \ge 2$, then*

$$|\langle u, |u|^2 u \rangle_L| \le C_L |u|_\infty^{2+\frac{2}{L+1}} \|u\|_{L+1}^{2-\frac{2}{L+1}}$$

and

$$\||u|^2 u\|_L \le C_L |u|_\infty^2 \|u\|_L.$$

The estimates follow by straight forward application of the Gagliardo-Nirenberg inequality, see e.g. [K2] (see there (6.5) for the first one and (7.6) for the second).

By the lemma, the Hölder inequality and Theorem 3,

$$\begin{aligned}|\mathbf{E}\langle u_\tau, |u_\tau|^2 u_\tau \rangle_L \chi| &\le C\mathbf{E}\Big(|u_\tau|_\infty^{\frac{2L+4}{L+1}} \|u_\tau\|_{L+1}^{\frac{2L}{L+1}} \chi\Big) \\ &\le C \left(\mathbf{E}|u_\tau|_\infty^{2L+4}\right)^{\frac{1}{L+1}} \left(\mathbf{E}\|u_\tau\|_{L+1}^2\right)^{\frac{L}{L+1}} \le C_1 K^{\frac{2L+4}{L+1}} g_{L+1}^{\frac{L}{L+1}}.\end{aligned}$$

Substituting this estimate to (4.1) we find that

$$g_L(t) \le g_L(0) + 2\int_0^t \Big(-g_{L+1}(s) + C_1 K^{\frac{2L+4}{L+1}+2} g_{L+1}^{\frac{L}{L+1}} + C\Big) ds. \tag{4.2}$$

Hence, the continuous function $g_L(t)$ decays in the vicinity of t if $g_{L+1}(t) > 2C$ and $g_{L+1}(t) > 2C_1 K^{\frac{4L+6}{L+1}} g_{L+1}^{\frac{L}{L+1}}$. The second inequality implies the first. It holds if

$$g_{L+1}(t) > CK^{4L+6}. \tag{4.3}$$

Since $\|u_\tau\|_L \le \|u_\tau\|_0^{1/(L+1)} \|u_\tau\|_{L+1}^{L/(L+1)}$ by the interpolation inequality, then

$$\begin{aligned} g_L \le \mathbf{E}\|u_\tau\|_0^{\frac{2}{L+1}} \|u_\tau\|_{L+1}^{\frac{2L}{L+1}} &\le \left(\mathbf{E}\|u_\tau\|_0^2\right)^{\frac{1}{L+1}} \left(\mathbf{E}\|u_\tau\|_{L+1}^2\right)^{\frac{L}{L+1}} \\ &\le g_0^{\frac{1}{L+1}} g_{L+1}^{\frac{L}{L+1}} \le CK^{\frac{2}{L+1}} g_{L+1}^{\frac{L}{L+1}} \end{aligned}$$

(we used (3.16)). Hence, $g_{L+1} \ge C_1 K^{-2/L}\, g_L^{(L+1)/L}$. This inequality and (4.3) show that the function $g_L(t)$ decays near t if $C_1 K^{-2/L}\, g_L^{(L+1)/L} > CK^{4L+6}$, i.e. if

$$g_L > CK^{4L+2}.$$

Since initially we have $g_L(0) = \mathbf{E}\|\xi\|_L^2 \le C_L \delta^{-2L-1}$ (see (2.3)), then $g_L(t) \le C_L K^{4L+2}$ with some new constant C_L. That is,

$$\mathbf{E}\|u_\tau(t)\|_L^2 \le C_L K^{4L+2} \tag{4.4}$$

for all $t \ge 0$.

By Lemma 8, Theorem 3 and (4.4),

$$\mathbf{E}\||u_\tau|^2 u_\tau\|_L \le C\,\mathbf{E}\left(|u_\tau|_\infty^2 \|u_\tau\|_L\right) \le C_1 K^{2L+3}.$$

Now we go back to the equation (2.5_τ) and denote by $I_1(t)$ and $I_2(t)$ the two integrals in its right hand side. By the last inequality, for any $T > 0$ we have

$$\mathbf{E} \sup_{0\le t\le T} \|I_1(t)\|_L \le \mathbf{E}\int_0^T \left(\|u_\tau\|_{L+2} + K^2 \||u_\tau|^2 u\|_L\right) ds \le C_L T K^{2L+5}.$$

To estimate the stochastic integral $I_2(t)$ we apply Lemma 1 with $q = 1$ to get that

$$\mathbf{E} \sup_{0\le t\le T} \|I_2(t)\|_L \le C_1 K \mathbf{E} \left(\int_0^T \|\eta(s)\|_L^2 ds \right)^{1/2} \le CKT.$$

We have proved that

$$\mathbf{E} \sup_{0\le t\le T} \|u_\tau(t)\|_L \le C_L T K^{2L+5}, \tag{4.5}$$

for any $T > 0$ and any stopping time $\tau = \tau_M$ as in (2.6). Abbreviating u_{τ_M} to u_M, we have for $V \ge 1$:

$$\mathbf{P}\left(\sup_{0\le t\le T} \|u_M(t)\|_L \ge V \right) \le C_L T K^{2L+5} V^{-1}.$$

It means that if we define a set $\Omega_V \in \mathcal{F}$ as $\Omega_V = \{\omega \mid \|u_M(t)\|_m \le V \text{ for } 0 \le t \le T\}$ with any $M \ge V$ (this set is M-independent by Lemma 3), then $\mathbf{P}\Omega_V \nearrow 1$ as $V \to \infty$. Hence, we have the convergence:

$$u_M(\cdot) \to u(\cdot) \text{ in } C\left([0,T]; H^m\right), \quad \text{a.s. as } M \to \infty,$$

where $u(t)$ is an H^m-solution of (2.4). In fact, the sequence $\{u_M(\cdot)\}$ stabilises to $u(\cdot)$, i.e. $u_M = u$ for $M \geq M_0(\omega)$, where M_0 is a random variable, which is a.s. finite (see Lemma 3).

Applying Fatout lemma to estimates (3.16) and (4.4), (4.5) we find that they remain valid for the limiting process u:

$$\mathbf{E}\left(\sup_{J\leq s\leq J+1}\ \sup_{x\in K^n} |u(s,x)|\right)^q \leq C_q K^q, \tag{4.6}$$

$$\mathbf{E}\|u(t)\|_L^2 \leq\ C_L K^{4L+2}, \tag{4.7}$$

$$\mathbf{E}\sup_{0\leq t\leq T}\ \|u(t)\|_L \leq\ C_L T K^{2L+5}, \tag{4.8}$$

for any $L \geq 2$, any $J \in \mathbb{N}$ and any $t \geq 0$.

Let us fix $t \geq 0$ and abbreviate $u(t)$ to u. Applying (4.6) with $q = 2p-2$ and (4.7) with $L = pm$ (p is any integer ≥ 2) we get:

$$\begin{aligned}\mathbf{E}\|u\|_m^p &\leq \mathbf{E}\|u\|_0^{p\frac{L-m}{L}}\|u\|_L^{p\frac{m}{L}} = \mathbf{E}\|u\|_0^{p-1}\|u\|_L\\ &\leq \left(\mathbf{E}|u|_\infty^{2p-2}\right)^{1/2}\left(\mathbf{E}\|u\|_{pm}^2\right)^{1/2} \leq CK^{p-1}K^{2pm+1} = CK^{p(2m+1)}\end{aligned}$$

Since L_p-norms satisfy the M. Riesz interpolation inequality, then this estimate remains true for any real $p \geq 2$.

Going back to the problem (2.1), (2.2) we arrive at the main result of this work:

Theorem 4. *The problem (2.1), (2.2) has a unique smooth solution $v^\omega(t,x), t \geq 0$. For any integer $m \geq 2$ and any real numbers $t \geq 0$, $q \geq 1$ this solution satisfies the estimates:*

$$\mathbf{E}\left(\sup_{t\leq s\leq t+\delta^{-1}}\ \sup_{x} |v^\omega(s,x)|\right)^q \leq C_q\delta^{-q/2}, \tag{4.9}$$

$$\mathbf{E}\|v^\omega(t)\|_m^q \leq C_{q,m}\delta^{-qm-q/2}. \tag{4.10}$$

We note that (4.9) follows from (4.6) with $J = [t]$ and $J = [t]+1$.

The theorem admits a less specific version for solutions of a single equation (2.1) with fixed $\delta \in (0,1]$:

Corollary 3. *If a r.f. $\xi^\omega(x)$ is such that for any $m \geq 0$ all momenta of the r.v. $\|\xi^\omega(\cdot)\|_m$ are finite, then the problem (2.1), (2.2) has a unique smooth solution $v^\omega(t,x), t > 0$. This solution is such that for any $m \geq 0$ and any $0 < T < \infty$ all momenta of the r.v. $\chi_m^\omega = \sup_{0\leq t\leq T} \|v^\omega(t)\|_m$ are finite.*

Proof. The r.v. χ_0 has finite momenta due to (4.9). For $m > 0$ the interpolation inequality implies that $\chi_m \leq \chi_0^{(L-m)/L}\chi_L^{m/L}$. Hence,

$$\mathbf{E}\chi_m \leq (\mathbf{E}\chi_0)^{(L-m)/L}(\mathbf{E}\chi_L)^{m/L}.$$

The first factor in the right hands side is finite by (4.9) and the second is finite by (4.8) (more specifically, by a version of this estimate for a solution for the problem (2.1), (2.2)). □

We shall also need a result which follows from the proof of Theorem 4 rather than from its assertions:

Proposition 2. *Let us fix any* $T > 0$. *Then solutions* $u^N(t,x)$ $(0 \le t \le T)$ *for the problems* (2.5_N) *a.s. converge as* $N \to \infty$ *to a solution* $u(t,x)$ *for the problem (2.4) in the norm of the space* $C([0,T], H^m)$.

Proof. Let $\Omega_V \in \mathcal{F}$ be the set defined as above in this section. Then $\mathbf{P}(\Omega_V) \ge 1 - CV^{-1}$ with some $C = C(\delta, T, m)$. The solution $u_M(t,x)$ was defined as a stopped solution of the equation (2.5_N), $u_M(t,x) \overset{\text{def}}{=} u^N_{\tau_M}(t,x)$, where $N \ge M \ge V$. For $\omega \in \Omega_V$ and $0 \le t \le T$ we clearly have $u^N_{\tau_M} = u^N$. Besides, for ω and t like that we have $u_M = u$. Hence, $u^N = u$ for $\omega \in \Omega_V$ and $N > V$. So the assertion follows. □

5 The Markov property and an invariant measure

Below we call a r.f. $\xi^\omega(x)$ *smooth* if $\mathbf{E}\|\xi\|^L_m < \infty$ for all m and L.

Let us consider the SNLS equation (2.1) with a stationary and non-random smooth function $\eta = \eta(x)$:

$$\dot{u}(t,x) - \delta\Delta u + i|u|^2 u = \eta(x)\dot{w}(t). \tag{5.1}$$

By the Corollary from Theorem 4, for any t_0 and any $\mathcal{F}_{t_0}$–measurable smooth r.f. $\xi^\omega(x)$ this equation has a unique smooth solution $u^\omega(t,x)$, $t \ge t_0$, such that

$$u^\omega(t_0, x) = \xi^\omega(x). \tag{5.2}$$

Denoting this solution as $u(t,x;t_0,\xi(x))$ and using the uniqueness we get that

$$u(t,x;t_0,\xi(x)) = u\left(t,x;t_1,u(t_1,x)\right),$$

for any $t_0 \le t_1 \le t$. Since for $t > t_1$, the right hand side of (5.1) is independent of $\mathcal{F}_{t_1}$, then the increment $u(t,\cdot) - u(t_1,\cdot)$ is $\mathcal{F}_{t_1}$-independent. (This property is well known for solutions of SPDEs with Lipschitz nonlinearities [PZ]. For solutions of the SNLS equation (5.1) it follows from Proposition 2).

Now usual arguments (see [PZ], section 9.2, and [Roz]) show that the solution for (5.1), (5.2) is a Markov process in any space $H^m, m \ge 2$.

Let us denote by $\mathcal{L}(u(t))$ distribution of the r.f. $u(t,\cdot)$ (in some space H^m) and consider the measure $\tilde{\mu}_t$,

$$\tilde{\mu}_t = \frac{1}{t}\int_0^t \mathcal{L}(u(\tau))d\tau.$$

Using (4.10) and the Chebyshev inequality we get that $\tilde{\mu}_t\{\|u\|_m \ge L\} \le L^{-1}C_m$ for any $m \ge 2$. Hence, by the Prokhorov theorem the system of measures $\{\tilde{\mu}_t \mid t > 0\}$ is precompact in H^m for any m. So for $r = 2, 3, \dots$ there are sequences $t^r = \{t^r_1 < t^r_2 < t^r_3 \cdots \nearrow \infty\}$ such that $t^r \supset t^l$ for $l > r$ and

$$\tilde{\mu}_{t^r_j} \rightharpoonup \tilde{\mu}^r \text{ weakly in } H^r \text{ as } j \to \infty. \tag{5.3}$$

By the classical arguments due to Krylov-Bogoliubov (see [PZ]), this measure is invariant for the Markov process which (5.1) defines in H^r. Since the sequences t^r form a nested family, then $\tilde{\mu}^r$ is an r-independent measure $\tilde{\mu}$. By (5.3), $\tilde{\mu}(H^r) = 1$ for any r. Hence, $\tilde{\mu}\big(\bigcap H^r = C^\infty(K^n;\mathbb{C})\big) = 1$ and we get:

Theorem 5. *The SNLS equation (5.1) defines a Markov process in any space $H^m, m \geq 2$. This process has an invariant measure, supported by the space of smooth odd periodic functions.*

A Appendix. A linear SPDE with additive noise.

Here we consider a linear SPDE:

$$\dot{v}(t,x) - \Delta v(t,x) = f^\omega(t,x)\dot{w}(t), \tag{1}$$

$$v(0,x) = 0, \tag{2}$$

where $w(t)$ is a Wiener process with respect to the system of σ-algebras $\{\mathcal{F}_t\}$ as in the main text; f is a continuous r.f., odd periodic in x and such that:

i) $|f^\omega(t,x)| \leq 1$,

ii) f is adapted to the flow $\{\mathcal{F}_t\}$.

Let us fix any $\theta < 1$. By C^θ we denote the space of Hölder functions $u(y)$ with the norm:

$$\|u(y)\|_{C^\theta} = \max\left(|u|_\infty, \sup_{\substack{y_1 \neq y_2 \\ |y_1 - y_2| \leq 1}} \frac{|u(y_1) - u(y_2)|}{|y_1 - y_2|^\theta} \right)$$

and by $C^{\theta/2,\theta}$ – the space of Hölder functions $u(t,x)$ with the norm:

$$\|u(t,x)\|_{C^{\theta/2,\theta}} = \max\left(|u|_\infty, \sup_{\substack{(t_1,x_1) \neq (t_2,x_2) \\ |(t_1,x_1)-(t_2,x_2)| \leq 1}} \frac{|u(t_1,x_1) - u(t_2,x_2)|}{|t_1 - t_2|^{\theta/2} + |x_1 - x_2|^\theta} \right).$$

The constants in the theorem below and in its proof depend on θ.

Let $\{S_t\}$ be the semi-group, generated by the Laplacian in the space of odd periodic functions. The operators S_t extend by continuity to linear contractions in the L_2- and L_∞-spaces of odd periodic functions and can be written using the fundamental solution of the heat equation:

$$S_t u(x) = \int_{\mathbb{R}^n} V(t, x-y) u(y) dy, \qquad V(t,x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}. \tag{3}$$

Let $v^\omega(t,x)$ be a mild solution for (1), (2), ie

$$v^\omega(t,x) = \int_0^t S_{t-s} f^\omega(s,x) dw(s).$$

We recall that the mild solution coincide with a solution as defined in section 1.1 (see Proposition 1).

Theorem. *For any $T > 0$ and $q \geq 1$ the mild solution v satisfies the estimate:*

$$\mathbf{E} \| v \mid_{[T,T+1]\times\mathbb{T}^n} \|^q_{C^{\theta/2,\theta}} \leq C_q. \tag{4}$$

Below we present an elementary proof of the estimate (4). For a more general related result see [KNP].

Proof. **Step 1**. *Some estimates for the flow-maps S_t.*

Lemma A1. *Let $u(x)$ be any odd periodic function such that $|u|_\infty \le 1$ and $u(t,x) = S_t u(x)$. Then*

1) *if $t \ge 1$, then $\|u(t,\cdot)\|_{C^\theta} \le Ce^{-ct}$,*
2) *if $0 < t \le 1$, then $\|u(t,\cdot)\|_{C^\theta} \le C_1 t^{-\theta/2}$,*
3) *if $0 < t \le 1$ and $0 < \Delta \le 1$, then $|u(t+\Delta,x) - u(t,x)| \le C_2 \Delta^\theta t^{-\theta}$ for any x.*

The constants c and $C - C_2$ do not depend on u.

Proof. 1) The first estimate readily follows from decomposition of $u(x)$ and $S_t u(x)$ to Fourier series since the mean value of $u(x)$ vanishes.

2) Since $|\nabla_x V(t,x)| = |(4\pi t)^{-n/2}(x/2t)e^{-|x|^2/4t}| \le Ct^{-n/2-1}|x|e^{-|x|^2/4t}$, then

$$|\nabla_x u(t,x)| \le Ct^{-n/2-1}\int\limits_{\mathbb{R}^n} |x|e^{-|x|^2/4t}dx = Ct^{-1/2}\int\limits_{\mathbb{R}^n} |z|e^{-|z|^2/4}dz = C_1 t^{-1/2}.$$

By the maximum principle, $|u(t,x)| \le 1$. Using these two estimates we get that

$$|u(t,x+\Delta) - u(t,x)| \le C_1 t^{-1/2}\Delta, \quad |u(t,x+\Delta) - u(t,x)| \le 2.$$

Raising the first inequality to degree θ, the second to degree $1-\theta$ and multiplying the results we obtain the estimate $|u(t,x+\Delta) - u(t,x)| \le C_1^\theta 2^{1-\theta} t^{-\theta/2}\Delta^\theta$. The second assertion is proven.

3) Similarly, since $|\partial V(t,x)/\partial t| \le Ct^{-n/2}\left(t^{-1} + |x|^2 t^{-2}\right)e^{-|x|^2/4t}$, then

$$|\dot u(t,x)| \le Ct^{-n/2}\int\limits_{\mathbb{R}^n}\left(t^{-1} + \frac{|x|^2}{t^2}\right)e^{-|x|^2/4t}dx = C\int\limits_{\mathbb{R}^n} t^{-1}\left(1+|z|^2\right)e^{-|z|^2/4}dz = C_1 t^{-1},$$

and the estimate for the increment $|u(t+\Delta,x) - u(t,x)|$ follows in the same way as above. □

Step 2. *Space-time increments of v.* Let us fix any two points, $x_1, x_2 \in \mathbb{R}^n$ such that $|x_1 - x_2| \le 1$ and consider the random process $U^\omega(t) = v^\omega(t,x_1) - v^\omega(t,x_2)$. We write it as:

$$U^\omega(t) = \int\limits_0^t \left(S_{t-s} f^\omega(s)(x_1) - S_{t-s} f^\omega(s)(x_2)\right)dw(s) =: \int\limits_0^t g^\omega(s,t-s)dw(s).$$

Let us consider the integral $X^\omega(t) = \int_0^t g^\omega(s,t-s)^2 ds$. Using items 1), 2) of the lemma we get that the following estimate holds uniformly in ω: $X^\omega(t) \le C|x_1 - x_2|^{2\theta}\int_0^t s^{-\theta}e^{-cs}ds \le C_1|x_1 - x_2|^{2\theta}$. Now application of the B-D-G inequality (see Lemma 1) to the process U^ω yields that

$$\mathbf{E}|U(t)|^p \le C_p\left(\mathbf{E}X(t)\right)^{p/2} \le C_p|x_1 - x_2|^{p\theta}.$$

To estimate a time-increment we take any $0 < \Delta \leq 1$, $t \geq 0$, $x \in \mathbb{R}^n$, and write the increment as

$$W^\omega(t) := v(t+\Delta, x) - v(t,x) = \int\limits_t^{t+\Delta} S_{t+\Delta-s} f^\omega(s)(x)\,dw(s)$$
$$+ \int\limits_0^t \Big(S_{t+\Delta-s} f^\omega(s) - S_{t-s} f^\omega(s) \Big)(x)\,dw(s) =: W_1^\omega(t) + W_2^\omega(t).$$

Denoting by $h_1^\omega(s,x)$ the integrand in the first integral W_1 we get that $|h_1^\omega(s,x)| \leq \sup|f^\omega(\tau,y)| \leq 1$ by the maximum principle. Hence, by B-D-G we have:

$$\mathbf{E} W_1^p \leq C_p \mathbf{E} \left(\int\limits_t^{t+\Delta} h_1^2\,ds \right)^{p/2} \leq C_p \Delta^{p/2}.$$

Denoting by h_2^ω the integrand in the second integral W_2 and using items 1) and 3) of Lemma A1, we get that $|h_2|^2 \leq C t^{-2\tilde{\theta}} \Delta^{2\tilde{\theta}} e^{-ct}$ for any $\tilde{\theta} < 1/2$. Hence,

$$\int\limits_0^t |h_2|^2 ds \leq C \Delta^{2\tilde{\theta}} \int\limits_0^t s^{-2\tilde{\theta}} e^{-cs} ds \leq C^1 \Delta^{2\tilde{\theta}},$$

and

$$\mathbf{E} W_2^p \leq C_p \mathbf{E} \left(\int\limits_0^t |h_2|^2 ds \right)^{p/2} \leq C_p^1 \Delta^{p\tilde{\theta}}$$

for any $\tilde{\theta} < 1/2$. We have got an estimate for the time-increment W: $\mathbf{E} W^p \leq C_p \Delta^{p\theta/2}$.

Finally, at this step we have proved that

$$\mathbf{E}|v(t_1,x_1) - v(t_2,x_2)|^p \leq C_p \left(|t_1 - t_2|^{\theta/2} + |x_1 - x_2|^\theta \right)^p, \qquad (5)$$

for any $p \geq 1$, if $(t_1 - t_2) \leq 1$ and $(x_1 - x_2) \leq 1$.

Step 3. *Continuity of the r.f. v and boundedness of its momenta.*

Due to (5), $\mathbf{E}|v(t_1,x_1) - v(t_2,x_2)|^p \leq C_{T,p} |(t_1,x_1) - (t_2,x_2)|^{p\theta/2}$ for any (t_1,x_1) and (t_2,x_2) in $[0,T] \times \mathbb{T}^n$. Choosing here $p > 4(n+1)\theta^{-1}$ we get that the r.f. v is a.s. Hölder-continuous in $[0,T] \times \mathbb{T}^n$ due to the Kolmogorov criterion (see [Ad], p.48). Hence, u is a.s. Hölder-continuous in the whole $[0,\infty) \times \mathbb{T}^n$. Below we present a “qualified” version of classical Kolmogorov’s arguments in order to estimate momenta of the random variables $|v|_{L_\infty}$ and $|v|_{C^{\theta/2,\theta}}$.

For any fixed $T \geq 0$, we denote $Q = [T, T+1] \times K^n \subset \mathbb{R}^{n+1}$ and consider the random variable $U = \sup|v\,|_Q\,|$.

For any $N \in \mathbb{N}$ we define a subset $\mathcal{K}_N \subset \mathbb{Z}^{N+1}$ as $\mathcal{K}_N = 2^N Q \bigcap \mathbb{Z}^{N+1}$. Now we shall construct some events and estimate their probabilities:

i) for any $s \in \mathcal{K}_N, k > 0$ and $q < 1$ we set

$$A_s^N = \{\omega \mid |v\left(\frac{s}{2^N}\right) - v\left(\frac{s'}{2^N}\right)| \geq kq^N \quad \text{for some neighbour } s' \text{ of } s \text{ in } K_N\},$$

where points $s, s' \in \mathcal{K}_N$ are called neighbours if $\max_j |s_j - s'_j| = 1$. By (5),

$$\mathbf{E}|v(y_1) - v(y_2)|^p \le C_p |y_1 - y_2|^{p\theta/2} \quad \forall\, y_1, y_2 \in Q.$$

Hence, $\mathbf{E}|v(2^{-N}s) - v(2^{-N}s')|^p \le C2^{-Np\theta/2}$ for any neighbours s, s', and $\mathbf{P}(A_s^N) \le C2^{-Np\theta/2}k^{-p}q^{-Np}$ by the Chebyshev inequality.

ii) Let A^N be the union of all sets A_s^k with $s \in \mathcal{K}_N$. Since $|\mathcal{K}_N| \le C2^{N(n+1)}$, then

$$\mathbf{P}(A^N) \le C2^{N(n+1)-Np\theta/2}k^{-p}q^{-Np} = Ck^{-p}\mu^N,$$

where $\mu = 2^{n+1-p\theta/2}q^{-p}$. Clearly $\mu < 1$ if

$$2^{\theta/2}q > 2^{(n+1)/p}. \tag{6}$$

This relation holds if $q > 2^{-\theta/2}$ and p is sufficiently large.

Assuming (6) we construct the last set:

iii) $A = \cup_{N \ge 1} A^N$. Since $\mu < 1$, then $\mathbf{P}(A) \le Ck^{-p}$, where C depends on p and q.

Now, when the set $A = \cup A_s^N$ is constructed and measured, we write $Q = \{y = (t, x)\}$ as the 1-cube $Q = \{0 \le y_j \le 1\}$ and write any $y \in Q$ as a binary expansion:

$$y = (y_1, \dots, y_{n+1}), \quad y_j = \sum_{r=1}^{\infty} x_{jr}2^{-r},$$

where each x_{jr} equals 0 or 1. Let us take any $\omega \notin A$ and consider $v(y) = v^\omega(y)$. Denoting $y^m = (y_1^m, \dots, y_{n+1}^m)$, where $y_j^m = \sum_{r=1}^m x_{jr}2^{-r}$, we have $v(y) = \lim v(y^m)$ and $v(y^0) = v(0) = 0$. Since $2^m y^{m-1}$ and $2^m y^m$ are neighbouring points of $\mathcal{K}_m$ and since $\omega \notin A^m$, then

$$|v(y^{m-1}) - v(y^m)| \le kq^m. \tag{7}$$

Hence,

$$|v(y^m)| \le k \sum_{l=1}^{m} q^l \le k/(1-q)$$

for any $m \ge 1$. It means that $|v(y)| \le k/(1-q)$ for $\omega \notin A$ for any $y \in Q$. Since $\mathbf{P}(A) \le Ck^{-p}$, then the r.v. $U = \sup |v\,|_Q\,|$ is such that

$$\mathbf{P}(U \ge R) \le C_p R^{-p} \quad \forall\, R \ge C_0$$

if p is sufficiently large. Therefore,

$$\mathbf{E}U^q \le \int_0^\infty x^q dF^U(x) \le C_0^q\Big(1 - \int_{C_0}^\infty x^q d\mathbf{P}\{U \ge x\}\Big) \le 2C_0^q + qC_p \int_{C_0}^\infty x^{q-1}x^{-p}dx.$$

Choosing p bigger than $q+1$ we get:

$$\mathbf{E}U^q \le C_q. \tag{8}$$

This proves (4) with the Hölder norm replaced by the L_∞-norm. Since (8) (not 5) is the estimate we use in the main part of the paper, our arguments at the last step are sketchy. Moreover, we shall prove (4) in a weaker form, with the norm of the space $C^{\theta/2,\theta}$ replaced by the norm of the homogeneous space $C^{\theta/2}$.

Step 4. *Hölder norm of* v.

Lemma A2. *If a function $u(y)$ on the cylinder Q is such that for any lattice $2^{-N}\mathbb{Z}^{n+1}$ and for any its cell J_N we have* $\operatorname{osc}\left(v\mid_{J_N\bigcap Q}\right)\le\gamma_N$, *then* $|v(y+\Delta)-v(y)|\le 2\gamma_{[\log_2|\Delta|^{-1}]}$ *for any* $y, y+\Delta\in Q$.

Proof. Let us note that y and $y+\Delta$ lie in the same cell or in adjacent cells of the lattice $2^{-N}\mathbb{Z}^{n+1}$, provided that $2^{-N-1}<|\Delta|\le 2^{-N}$. That is, if $N=[\log_2\ |\Delta|^{-1}]$. Hence, $|v(y+\Delta)-v(y)|$ is bounded by the double oscillation along a cell J_N and the result follows. □

If $\omega\notin A$, then the function $v=v^\omega$ is such that for any N and any cell J_N the oscillation of v along J_N is bounded by $2k\sum_{m=N+1}^{\infty}q^m=2kq^{N+2}/(1-q)$ (this follows from (7) since all points $y\in J_N$ have the same $y^{(N)}$). Applying Lemma A2 with $\gamma_N=2kq^{N+2}/(1-q)$ we get that

$$|v(y+\Delta)-v(y)|\le\frac{2k}{1-q}\,q^{\log_2\ |\Delta|^{-1}+1}\le\frac{2k}{1-q}|\Delta|^{\log_2\ q^{-1}}. \tag{9}$$

Our calculations hold provided that (6) is fulfilled, i.e. if $q=2^{-\theta_1/2}$, $\theta_1<\theta$, and p is sufficiently large. For this choice of q we get from (9) that for $\omega\notin A$ we have $|v(y+\Delta)-v(y)|\le 2k|\Delta|^{\theta_1/2}/(1-q)$ if $y,y+\Delta\in Q$. Hence,

$$\mathbf{P}(\|v_{|Q}\|_{C^{\theta_1/2}}\ge R)\le C_{\theta_1,p}R^{-p}$$

if p is sufficiently large. As at the Step 3 this implies that $\mathbf{E}\|v_{|Q}\|^q_{C^{\theta_1/2}}\le C_{\theta_1,q}$ for any $\theta_1<1$. The theorem is proven. □

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