“2D Euler equation as a Hamiltonian PDE [1]”

1. Hamiltonian PDE.
H - a functional space which consists of smooth functions. For $u \in H$ let $J_u : H \to H$ be an operator, anti-symmetric w.r.t. the L_2-scalar product, which will be denoted as $\langle \cdot, \cdot \rangle$. For a functional $F : H \to \mathbb{R}$ let $\overline{\nabla} F : H \to H$ be its gradient. That is,

$$dF(u)v = \langle \overline{\nabla} F(u), v \rangle \qquad \forall\, v \in H.$$

For functionals F, G define their bracket as

$$\{F, G\}(u) = \langle J_u \overline{\nabla} F(u), \overline{\nabla} G(u) \rangle.$$

It is skew-symmetric. *Assume* that it satisfies the Jacobi identity. Then this is a Poisson bracket.

For any Hamiltonian $h : H \to \mathbb{R}$ the corresponding Hamiltonian equation is

$$\dot{u} = J_u \overline{\nabla} h(u). \tag{1}$$

2. Euler equation on $\mathbb{T}^2$.

Now let $\mathcal{H}$ be the space of smooth divergence-free vector-fields on $\mathbb{T}^2$. Then for a functional h on $\mathcal{H}$ we have

$$\overline{\nabla} h(u) = \Pi\, \delta h / \delta u(x),$$

where Π is the Leray projection and $\delta h / \delta u(x)$ is the variational derivative of h. In particular, for $h^0(u) = \frac{1}{2}\langle u, u \rangle$ we have $\overline{\nabla} h^0(u) = u$.

The Euler equation can be written as

$$\dot{u}(t) = \Pi\,(u \cdot \nabla) u = \Pi\,(u \cdot \nabla) \nabla h^0(u), \quad u(t) \in \mathcal{H}. \tag{2}$$

QUESTION: How to write (2) in the form (1)?

3. Hamiltonian form for Euler equation.

First Try. Choose

$$J_u(v) = \Pi\,(u \cdot \nabla) v, \qquad J_u : \mathcal{H} \to \mathcal{H}.$$

[1] Rather a Poissonian PDE

This is a skew-symmetric operator and (2) takes the form (1) with $h = h_0$. But this J_u does not satisfy the Poisson identity. So we failed.

In the r.h.s. of eq. (2) we have two factors u. This time we interpreted the first one as a factor from the Poisson structure and the second - from the hamiltonian. We can do this other way around. This is our

Second Try. Choose $J_u = \Pi \circ J_u^0$, where

$$\left(J_u^0(v)\right)^k = v^l \left(\frac{\partial u^k}{\partial x_l} - \frac{\partial u^l}{\partial x_k}\right).$$

That is,

$$J_u^0(v) = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} v\,,$$

where $\omega = \operatorname{rot}(u) = \partial u^2/\partial x_1 - \partial u^1/\partial x_2$ is the vorticity. Obviously $\langle J_u(v), v\rangle = 0\ \forall v$. So J_u is skew-symmetric.

To check the Jacobi identity let us start with linear functionals. For any $f \in \mathcal{H}$ denote $h_f(u) = \langle f, u\rangle$. Then $\overline{\nabla} h_f = f$. So

$$\{h_f, h_g\}(u) = \int \left(\frac{\partial}{\partial x_j} u^k\right) f^j g^k\, dx - \int \left(\frac{\partial}{\partial x_j} u^k\right) g^j f^k\, dx = -h_{[f,g]}(u),$$

where $[f, g]$ is the commutator of vector-fields (note that $[f, g] \in \mathcal{H}$ if $f, g \in \mathcal{H}$). So for functionals of the form h_f the Jacobi identity follows from the one for commutators of vector-fields. Certainly the Jacobi identity also holds for arbitrary functionals on $\mathcal{H}$. That is, we have constructed a Poisson structure.

To arbitrary hamiltonian F this Poisson structure corresponds the Hamiltonian equation

$$\dot{u} = J_u \overline{\nabla} F(u) = \Pi \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \overline{\nabla} F(u)\,, \qquad u(t) \in \mathcal{H}, \tag{3}$$

and the Poisson bracket of two functionals f and g is

$$\{f, g\} = \left\langle \Pi \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \overline{\nabla} f(u), \overline{\nabla} g(u) \right\rangle = \left\langle \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \overline{\nabla} f(u), \overline{\nabla} g(u) \right\rangle, \tag{4}$$

where the scalar product in the r.h.s. is the product in the space $L_2(\mathbb{T}^2; \mathbb{R}^2)$.

Now we have

$$J_u^0(u) = (u \cdot \nabla) u - \frac{1}{2} \nabla |u|^2.$$

So $J_u(u) = \Pi\,(u\cdot\nabla)u$ and the Hamiltonian equation (3) with $F = h^0$ coincides with (1). That is, we have found a Hamiltonian representation for the Euler equation.

4. Functionals of vorticity.
Let X be the space of smooth functions on $\mathbb{T}^2$ with zero mean-value, and let F be a smooth functional on X. Define $f(u) = F(\mathrm{rot}(u))$. This is a smooth functional on $\mathcal{H}$ and

$$\overline{\nabla} f(u) = \nabla_x^{\perp} \overline{\nabla} F(\omega), \qquad \omega = \mathrm{rot}\, u.$$

In particular, if $F = F^h = \int h(\omega(x))\, dx$, then

$$\overline{\nabla} f(u) = \nabla_x^{\perp} h'(\omega(x)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla_x h'(\omega(x)). \tag{5}$$

So

$$J_u^0 \overline{\nabla} f(u) = -\omega(x) \nabla_x h'(\omega(x)) = -\nabla_x \tilde{h}(\omega(x))\,,$$

where $\tilde{h}(\omega) = \int \omega h'(\omega)\, d\omega$. Hence,

$$J_u \overline{\nabla} f(u) = \Pi J_u^0 \overline{\nabla} f(u) = 0.$$

We saw that *the functionals of the form* $f^h(u) = \int h(\mathrm{rot}\, u(x))\, dx$ *belong to the centre of the Poisson algebra.*

So, the functionals f^h

i) define trivial Hamiltonian equations (3),

ii) they are integrals of motion for each equation (3) (including the Euler equation).

5. The Poisson algebra in terms of functionals of vorticity.
Consider two smooth functionals F_1, F_2 on X and the corresponding functionals $f_1 = F_1 \circ \mathrm{rot}$, $f_2 = F_2 \circ \mathrm{rot}$ on $\mathcal{H}$. Due to (5) and (4)

$$\{f_1, f_2\} = \int \omega \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 \nabla_x \overline{\nabla} F_1(\omega), \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla_x \overline{\nabla} F_2(\omega) \right) dx\,,$$

where $\omega = \omega(x) = \mathrm{rot}\, u$. Integrating by parts we find that

$$\{f_1, f_2\} = \int \big((\nabla^{\perp}\omega \cdot \nabla) \overline{\nabla} F_1(\omega) \big) \overline{\nabla} F_2(\omega)\, dx = \langle J_\omega \overline{\nabla}_\omega F_1, \overline{\nabla}_\omega F_2 \rangle\,,$$

where $\overline{\nabla}_\omega F(\omega)$ is the usual L_2-gradient and

$$J_\omega = (\nabla^\perp \omega \cdot \nabla).$$

That is, the map

$$\text{rot} : \mathcal{H} \to X$$

transforms the Poisson bracket $\{\cdot,\cdot\}$ for functionals on $\mathcal{H}$ to the bracket $\{\cdot,\cdot\}^\omega$ for functionals on X, where

$$\{F_1, F_2\}^\omega(\omega) = \langle J_\omega \overline{\nabla}_\omega F_1, \overline{\nabla} F_2\rangle.$$

For this bracket the Hamiltonian equation with a hamiltonian $F(\omega)$ is

$$\dot{\omega}(t,x) = \nabla_x^\perp \omega(x) \cdot \nabla_x (\overline{\nabla} F\big(\omega)(x)\big)\,. \tag{6}$$

Examples. 1) If $F_1 = \int h(\omega(x))\,dx$, then $\overline{\nabla} F_1(\omega) = h'(\omega(x))$. Now

$$J_\omega \overline{\nabla} F_1(\omega) = (\nabla_x^\perp \omega \cdot \nabla_x) h'(\omega) = 0.$$

So $\{F_1, F\}^\omega = 0$ for any F, as it should be.

2) If $F(\omega) = h^0(u)$, where $u = (\text{rot})^{-1}\omega$ and $h^0(u) = \frac{1}{2}\langle u, u\rangle$, then (6) is the Euler equation in terms of vorticity:

$$\dot{\omega} = (u \cdot \nabla)\omega.$$