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Some limiting properties of randomly forced 2D Navier–Stokes equations

Sergei Kuksin
Department of Mathematics, Heriot-Watt University,
Edinburgh EH14 4AS, UK
Steklov Institute of Mathematics, 8 Gubkina St., 117966 Moscow, Russia
E-mail: S.B.Kuksin@ma.hw.ac.uk

Armen Shirikyan
Department of Mathematics, Heriot-Watt University,
Edinburgh EH14 4AS, UK
New address: Laboratoire de Mathématiques, Université de
Paris-Sud XI, Bâtiment 425, 91405 Orsay Cedex, France
E-mail: Armen.Shirikyan@math.u-psud.fr

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We consider random perturbations of 2D Navier–Stokes equations. Under some natural conditions on random forces, we study asymptotic properties of solutions and stationary measures.

1. Introduction

We consider the 2D Navier–Stokes (NS) system perturbed by a random force:

$$\dot{u} - \nu\Delta u + (u,\nabla)u + \nabla p = \eta(t,x), \quad \operatorname{div} u = 0. \tag{1.1}$$

The space variable x belongs either to a bounded domain, and then the Dirichlet boundary condition is imposed, or to the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, and then we assume that $\int_{\mathbb{T}^2} u\,dx \equiv \int_{\mathbb{T}^2} \eta\,dx \equiv 0$. Let H be the corresponding L^2-space of divergence-free vector fields with the natural norm $|\cdot|$ and let Π be the orthogonal projection to H (see [4]). We are interested in time-evolution of the velocity-field u (not of the pressure p). Accordingly, we replace in (1.1) the force η by its divergence-free component $\Pi\eta$ (neglecting the gradient component), and assume below that

$$\operatorname{div}\eta = 0.$$

Applying Π to equation (1.1), we write it as a random system in H:

$$\dot{u}(t) + \nu L u(t) + B(u(t),u(t)) = \eta(t). \tag{1.2}$$

Here $L = -\Pi\Delta$ and $B(u,u) = \Pi(u,\nabla)u$. We denote by $\{e_j\}$ the Hilbert basis in H formed by the eigenfunctions of the operator L with eigenvalues $0 < \alpha_1 \le \alpha_2 \le \cdots$.

Concerning the force η, we assume that it is smooth in x, whereas as a function of time t it is either a kick-process or a white noise. Thus, we consider the following two cases:

- The function η models random kicks with some period $\varepsilon > 0$:

$$\eta = \eta_\varepsilon(t,x) = \sqrt{\varepsilon} \sum_{k=-\infty}^{\infty} \eta_k(x)\delta(t-k\varepsilon), \quad \eta_k(x) = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j(x), \tag{1.3}$$

  where $\{b_j\}$ is a sequence of nonnegative constants such that $\sum_j b_j^2 < \infty$ and $\{\xi_{jk}\}$ is a family of bounded independent scalar random variables with k-independent distributions that satisfy certain regularity assumptions (see condition (H) in §4).

- The function $\eta(t)$ is the time derivative of an H-valued Wiener process:

$$\eta = \eta(t,x) = \frac{\partial}{\partial t}\zeta(t,x), \quad \zeta(t,x) = \sum_j b_j \beta_j(t) e_j(x), \tag{1.4}$$

  where $\{b_j\}$ is a sequence of constants as above and $\{\beta_j\}$ is a family of independent standard Brownian motions.

Under the assumption that

$$b_j \neq 0 \quad \text{for} \quad 1 \le j \le N, \tag{1.5}$$

where $N = N_{\varepsilon\nu} \ge 0$ is a suitable integer, it was proved in [9] that the NS system (1.2), (1.3) has a unique stationary measure $\mu = \mu_{\varepsilon\nu}$ and that all solutions converge to μ in distribution:

$$\mathbb{E} f(u(t)) \to \int_H f(v)\mu(dv) \quad \text{as} \quad t \to \infty, \tag{1.6}$$

where f is any bounded continuous functional on H. In subsequent publications, we have shown that the convergence is exponentially fast provided that the functional f is Lipschitz continuous; see [8] for discussions and references.

The NS system (1.2), (1.4), (1.5) was considered later by E, Mattingly, Sinai [6] and Bricmont, Kupiainen, Lefevere [3]. Under the additional restriction that the sum in (1.4) is finite, they proved that the system has a unique stationary measure and studied convergence of solutions to this measure; see [8] for discussions and references. In the work [11] we applied to the white-forced NS system (1.2), (1.4), (1.5) the techniques developed for studying the kicked equations. It was proved that convergence (1.6) is exponentially fast for any bounded Lipschitz functional f and any solution $u(t)$ of the NS system such that $\mathbb{E}|u(0)|^2 < \infty$.

In this work, we specify convergence (1.6) in the case of the white-forced equation and study relations between the stationary measures for the kick- and white-forced equations. The paper is organised as follows.

In §2, we establish a priori estimates for moments of Sobolev norms of solutions for the white-forced NS system. In §3, we use these estimates to prove theorem 3.1, which claims that convergence (1.6) holds for functionals f that are defined and Hölder continuous on a Sobolev space H^k (with an arbitrary $k \geq 0$) and have a polynomial growth at infinity. This generalisation is important since functionals relevant for statistical hydrodynamics are unbounded and usually are not continuous on the space H. For example, the correlation tensor gives rise to functionals $f(u) = u_i(x)u_j(y)$, where x and y are points in the space domain and $i, j = 1, 2$.

Section 4 is devoted to a comparison of solutions and stationary measures for kick- and white-forced equations. Namely, let $u_\varepsilon(t)$ and $u(t)$ be solutions of the NS system (1.2) that correspond to a deterministic initial function $u_0 \in H$ and right-hand sides (1.3) and (1.4), respectively. Assuming that

$$\mathbb{E}\xi_{jk} \equiv 0, \quad \mathbb{E}\xi_{jk}^2 \equiv 1, \tag{1.7}$$

we first show that $u_\varepsilon(t)$ converges to $u(t)$ in distribution. Furthermore, if all b_j are nonzero, then the kick-forced equation (1.2), (1.3) has a unique stationary measure μ_ε for any $\varepsilon > 0$. The main results of this paper is theorem 4.1, according to which we have

$$\mu_\varepsilon \rightharpoonup \mu \quad \text{as} \quad \varepsilon \to 0, \tag{1.8}$$

where μ is the unique stationary measure for the white-forced equation. Since $\mathcal{D}(u_\varepsilon(t)) \rightharpoonup \mu_\varepsilon$ as $t \to \infty$ and $\mathcal{D}(u_\varepsilon(t)) \rightharpoonup \mathcal{D}(u(t))$ as $\varepsilon \to 0$, convergence (1.8) implies that we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{D}\big(u_\varepsilon(t)\big) & \xrightarrow{t\to\infty} & \mu_\varepsilon \\ {\scriptstyle \varepsilon\to 0}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon\to 0} \\ \mathcal{D}\big(u(t)\big) & \xrightarrow{t\to\infty} & \mu \end{array}$$

Convergence (1.8) manifests the fact that the white forced equations occupy a special place among randomly forced equations. The results proved and discussed in this paper suggests the following conjecture:

Conjecture. Let us set

$$\eta_\varepsilon(t, x) = \sqrt{\varepsilon} \sum_{j=1}^{\infty} b_j \xi_j(t/\varepsilon) e_j(x),$$

where $0 < \varepsilon \leq 1$, $\{\xi_j\}$ is a sequence of i.i.d. stationary processes that are mixing (in an appropriate sense) and satisfy the conditions $\mathbb{E}\xi_j(t) \equiv 0$ and $\mathbb{E}\xi_j^2(t) \equiv 1$. Then the assertions below take place on condition that $b_j \neq 0$ for all $j \geq 1$:

(i) Equation (1.2) with $\eta = \eta_\varepsilon(t, x)$ has a unique stationary measure μ_ε.

(ii) All solutions for (1.2) with deterministic initial data converge to μ_ε in distribution as $t \to \infty$.

(iii) We have $\mu_\varepsilon \rightharpoonup \mu$ as $\varepsilon \to 0$, where μ is the unique stationary measure for the white-forced equation.

2. A priori estimates for higher Sobolev norms

2.1. Periodic boundary conditions

Let $H^s(\mathbb{T}^2)$ be the Sobolev space of order s on $\mathbb{T}^2$ endowed with the norm $\|u\|_s = |L^{s/2}u|$ and let $H^s = H^s(\mathbb{T}^2) \cap H$. To simplify the notation, we set $\|\cdot\| = \|\cdot\|_1$. In this subsection, we derive some estimates for the expectation $\mathbb{E}\|u\|_s^m$, where $m \geq 1$ and $s \geq 0$ are arbitrary integers and $u(t)$ is a solution of the NS system (1.2), (1.4) under the periodic boundary conditions.

Let us set $b_{\max} = \sup_j b_j$ and $B_k = \sum_j \alpha_j^k b_j^2$. In what follows, we shall always assume that

$$B_0 < \infty, \quad \mathbb{E}\, e^{\sigma\nu|u_0|^2} < \infty, \tag{2.1}$$

where u_0 is the initial function and $\sigma > 0$ is so small that $2\sigma b_{\max}^2 \leq \alpha_1$. Condition (2.1) ensures that the Cauchy problem has a unique solution $u(t)$ in an appropriate functional space, and the following inequality holds for a second exponential moment of $u(t)$ (see [14, theorem 3.1], [2, lemma 1], and [13, remark 3.2]):

$$\mathbb{E}\, e^{\sigma\nu|u(t)|^2} \leq e^{-\sigma\nu t}\mathbb{E}\, e^{\sigma\nu|u_0|^2} + C, \quad t \geq 0, \tag{2.2}$$

where $C > 0$ depends only on σ and B_0. We emphasize that inequality (2.2) remains valid for the Dirichlet boundary condition.

For any $T > 0$ and any integer $k \geq 0$, we set

$$V_k(T) = \sup_{0 \leq t \leq T} \Bigl(t^k \|u(t)\|_k^2 + \nu \int_0^t s^k \|u(s)\|_{k+1}^2 ds \Bigr). \tag{2.3}$$

In the theorem below, we confine ourselves to the case $\nu = 1$. The dependence of the constants on ν will be specified later.

Theorem 2.1. *Suppose that $B_k < \infty$ for an integer $k \geq 0$. Then for any $T > 0$ and any integer $p \geq 1$ there is a constant $C_{kp} = C_{kp}(\sigma, T) > 0$ such that*

$$\mathbb{E}\, V_k(T)^p \leq C_{kp}\mathbb{E}\, e^{\sigma|u_0|^2}. \tag{2.4}$$

Inequality (2.4) immediately implies a priori estimates for moments of the random variables

$$U_k(t_0, T) = \sup_{t_0 \leq t \leq t_0 + T} \Bigl(\|u(t)\|_k^2 + \int_{t_0}^t \|u(s)\|_{k+1}^2 ds \Bigr). \tag{2.5}$$

Namely, we have the following corollary:

Corollary 2.2. *Under the conditions of theorem 2.1, for any k, p, $t_0 \geq 1$, and $T > 0$ we have*

$$\mathbb{E} U_k(t_0, T)^p \leq C'_{kp}(\sigma, T) \bigl(e^{-\sigma t_0}\mathbb{E}\, e^{\sigma|u_0|^2} + 1 \bigr), \tag{2.6}$$

where the positive constant $C'_{kp}(T)$ does not depend on t_0 and u_0.

To establish inequality (2.6), it suffices to apply an analogue of theorem 2.1 for the interval $[t_0 - 1, t_0 + T]$ and use (2.2) to estimate $\mathbb{E}\, e^{\sigma|u(t_0-1)|^2}$.

Proof of theorem 2.1. For $k=0$, inequality (2.4) follows from lemma 2.3 in [11]. We now assume that $k=m\geq 1$ and that for $k\leq m-1$ inequality (2.4) is already established. Application of Itô's formula to the functional $f_m(t)=t^m\|u(t)\|_m^2$ results in

$$df_m(t)=mt^{m-1}\|u\|_m^2dt-2t^m\left|L^{\frac{m+1}{2}}u\right|^2dt- \\ -2t^m\left(L^{\frac{m+1}{2}}u,L^{\frac{m-1}{2}}B(u,u)\right)dt+B_mt^mdt+2t^m\left(L^{\frac{m}{2}}u,L^{\frac{m}{2}}d\zeta\right). \quad (2.7)$$

To be precise, we shall assume that $m\geq 2$. The case $m=1$ is simpler since the third term on the right-hand side of (2.7) is zero. Integrating (2.7) with respect to t and using the inequality (see [10, Section 6.3])

$$\left|\left(L^{\frac{m+1}{2}}u,L^{\frac{m-1}{2}}B(u,u)\right)\right|\leq c_m'\|u\|_{m+1}^{\frac{4m-1}{2m}}\|u\|^{\frac{m+1}{2m}}|u|^{\frac{1}{2}} \\ \leq \tfrac14\|u\|_{m+1}^2+c_m\|u\|^{2(m+1)}|u|^{2m},$$

we derive

$$t^m\|u(t)\|_m^2+\int_0^t s^m\|u(s)\|_{m+1}^2ds\leq m\int_0^t s^{m-1}\|u\|_m^2ds+ \\ +\tfrac{B_mt^{m+1}}{m+1}+2c_m\int_0^t s^m\|u\|^{2(m+1)}|u|^{2m}ds+\xi_m(t), \quad (2.8)$$

where

$$\xi_m(t)=2\int_0^t s^m\left(L^{\frac{m}{2}}u,L^{\frac{m}{2}}d\zeta\right)-\frac12\int_0^t s^m\|u\|_{m+1}^2ds.$$

Since $s\|u(s)\|^2\leq V_1(T)$ and $|u(s)|^2\leq V_0(T)$ for $0\leq s\leq T$, it follows that

$$\int_0^t s^m\|u\|^{2(m+1)}|u|^{2m}ds\leq V_0(T)^mV_1(T)^m\int_0^T\|u\|^2ds\leq V_0(T)^{m+1}V_1(T)^m.$$

Combining this inequality with (2.8), we obtain

$$V_m(T)\leq mV_{m-1}(T)+\tfrac{B_mT^{m+1}}{m+1}+2c_mV_0(T)^{m+1}V_1(T)^m+\sup_{0\leq t\leq T}\xi_m(t). \quad (2.9)$$

We now repeat the argument used in the proof of lemma 2.3 in [11] to show that, if $\gamma>0$ is sufficiently small, then

$$\mathbb{P}\Big(\sup_{0\leq t\leq T}\xi_m(t)\geq\rho\Big)\leq e^{-\gamma\rho},\quad \rho\geq 0. \quad (2.10)$$

Indeed, the quadratic variation of the integral $M_t:=\int_0^t s^m(L^{\frac{m}{2}}u,L^{\frac{m}{2}}d\zeta)$ has the form

$$\langle M\rangle_t=\sum_j\alpha_j^{2m}b_j^2\int_0^t s^{2m}u_j^2(s)\,ds\leq B_{m-1}\,T^m\int_0^t s^m\|u\|_{m+1}^2ds \\ \leq\frac{1}{4\gamma}\int_0^t s^m\|u\|_{m+1}^2ds,$$

where we set $\gamma = (4T^m B_{m-1})^{-1}$. It follows that

$$\xi_m(t) \le 2M_t - 2\gamma\langle M\rangle_t.$$

Since $\exp(2\gamma M_t - 2\gamma^2\langle M\rangle_t)$ is a supermartingale whose mean value does not exceed 1 (see [7, lemma IV.3.1]), the required estimate (2.10) follows from the classical supermartingale inequality (see theorem VI.T1 in [12] or theorem III.6.11 in [7]).

Combining inequalities (2.9) and (2.10) and the induction hypothesis, we see that all the moments of $V_m(T)$ are finite. This completes the proof of the theorem. □

We now study dependence of the constant C'_{kp} in (2.6) on ν. To simplify the presentation, we shall only consider the case of a stationary solution (as we show below, any stationary solution satisfies the second estimate in (2.1)).

Let us begin with the situation in which the right-hand side is analytic in x. More precisely, we assume that

$$\sum_{j=1}^{\infty} \exp(2\rho\sqrt{\alpha_j}\,)b_j^2 < \infty, \tag{2.11}$$

where $\rho > 0$ is a constant not depending on ν.

Proposition 2.3. *Let $u(t,x)$ be a stationary solution of* (1.2), (1.3). *Then for any integers $k, m \ge 1$ and any $\varepsilon > 0$ and $T > 0$ there is a constant $C_{km} = C_{km}(\varepsilon, T) > 0$ such that*

$$\mathbb{E}\Big(\sup_{t_0\le t\le t_0+T} \|u(t)\|_k^m\Big) \le C_{km}\nu^{-2km+\frac{3m}{2}-\varepsilon}, \quad t_0 \ge 0. \tag{2.12}$$

Proof. Since $u(t,x)$ is a stationary solution, it suffices to consider the case $t_0 = 0$. According to theorems 2.1 and 2.2 in [13], for any $\delta > 0$ and $T > 0$ there is a positive random variable r_ν such that for any integer $m \ge 1$ and an appropriate constant $K_m > 0$ we have

$$\mathbb{E}\Big(\sup_{0\le t\le T} \big\|\exp(r_\nu \nu^{2+\delta} L^{\frac12})u(t,\cdot)\big\|^m\Big) \le K_m \nu^{-\frac{m}{2}}, \quad \mathbb{E}\, r_\nu^{-m} \le K_m. \tag{2.13}$$

Let us fix an arbitrary integer $k \ge 1$. Direct verification shows that

$$\|u\|_k^2 = \sum_j \alpha_j^k |u_j|^2 \le C_k \nu^{-2(k-1)(2+\delta)}\, r_\nu^{-2(k-1)} \big\|\exp(r_\nu \nu^{2+\delta} L^{\frac12})u\big\|^2,$$

where $C_k > 0$ depends only on k. It follows that

$$\sup_{0\le t\le T} \|u\|_k^m \le C_k^{\frac{m}{2}}\, \nu^{-m(k-1)(2+\delta)}\, r_\nu^{-m(k-1)} \sup_{0\le t\le T} \big\|\exp(r_\nu \nu^{2+\delta} L^{\frac12})u\big\|^m. \tag{2.14}$$

Combining (2.13) and (2.14) and choosing a sufficiently small $\delta = \delta_{km}(T) > 0$, we obtain (2.12). □

We conclude this subsection with a short discussion of the case when the right-hand side η is smooth, but not analytic with respect to x.

Proposition 2.4. *Let $B_k<\infty$ for an integer $k\ge 0$ and let $u(t,x)$ be a stationary solution. Then for any integer $m\ge 1$ and any $T>0$ there is a constant $C_{km}(T)>0$ not depending on ν and $u(t,x)$ such that*

$$\mathbb{E}\Big(\sup_{t_0\le t\le t_0+T}\|u(t)\|_k^m\Big)\le C_{km}(T)\,\nu^{-3km-\frac{m}{2}},\quad t_0\ge 0. \tag{2.15}$$

To establish inequality (2.15), it suffices to repeat the proof of theorem 2.1, following the dependence of the constants on ν. We shall not dwell on it.

We note that, for $k=1$, a stronger assertion holds if we are interested in estimating the mean value at a fixed point t. Namely, if $B_1<\infty$, then there is a constant $C>0$ such that

$$\mathbb{E}\,\exp\big(\sigma\nu\|u(t,\cdot)\|^2\big)\le C,\quad t\ge 0. \tag{2.16}$$

In particular, since $y^m\le C_m\nu^{-\frac{m}{2}}e^{\sigma\nu y^2}$, for any $m\ge 1$ we have

$$\mathbb{E}\,\|u(t,\cdot)\|^m\le C_m\,\nu^{-\frac{m}{2}},\quad t\ge 0. \tag{2.17}$$

A proof of inequality (2.16) can be found in [13, theorem 2.2].

2.2. Dirichlet boundary condition

In this subsection, we briefly discuss the case in which equations (1.2), (1.3) is studied in a bounded domain $D\subset\mathbb{R}^2$ with smooth boundary and is supplemented with the Dirichlet boundary condition.

As in the case of the periodic boundary conditions, we assume that (2.1) holds. This implies that the Cauchy problem has a unique solution. Moreover, inequality (2.2) remains valid for $2\sigma b_{\max}^2\le\alpha_1$ (see [13, remark 3.2]). As a simple consequence, we obtain the following result:

Proposition 2.5. *Suppose that $B_0<\infty$. Then there are positive constants C and σ such that, for any $\nu\in(0,1]$ and any stationary solution $u(t,x)$, we have*

$$\mathbb{E}\,e^{\sigma\nu|u(t)|^2}\le C,\quad t\ge 0. \tag{2.18}$$

In particular, there are constants $C_m>0$ such that

$$\mathbb{E}\,\big|u(t)\big|^m\le C_m\nu^{-\frac{m}{2}},\quad t\ge 0,\quad m\ge 1. \tag{2.19}$$

The above assertions remain valid in the case of periodic boundary conditions.

Proof. We shall prove (2.18) under the additional assumption that the non-degeneracy condition (1.5) holds with a sufficiently large $N\ge 1$. The proof in the general case can be obtained by literal repetition of the arguments in [13, Section 3.2].

We need to show that

$$\int_H e^{\sigma\nu|u|^2}\mu(du)\le C. \tag{2.20}$$

As was mentioned in the introduction, if (1.5) is satisfied, then (1.6) holds for any solution $u(t)$ with deterministic initial condition $u_0\in H$. Let us take $u_0=0$ and

$f_M(u)=M\wedge\exp(\sigma\nu|u|^2)$ in (1.6), where $\sigma>0$ is the constant in (2.2). Then, using inequality (2.2), we derive

$$\int_H f_M(u)\,\mu(du)=\lim_{t\to+\infty}\mathbb{E}\,f_M(u(t))\le\lim_{t\to+\infty}\mathbb{E}\,e^{\sigma\nu|u(t)|^2}\le C.$$

Passing to the limit as $M\to+\infty$ and applying the Levi theorem, we obtain (2.20). $\square$

3. Convergence to the stationary measure

In this section, we assume that $\nu=1$ and that $B_{k+1}<\infty$ for some $k\ge 0$. We denote by $C_b(H)$ the space of bounded continuous functionals on H and by $\mathcal{P}(H)$ the set of all probability Borel measures on H. Let $P_t(u,\Gamma)$, $u\in H$, $\Gamma\in\mathcal{B}(H)$, be the transition function associated with the NS system (1.2) and let $\mathfrak{P}_t\colon C_b(H)\to C_b(H)$ and $\mathfrak{P}_t^*:\mathcal{P}(H)\to\mathcal{P}(H)$ be the corresponding Markov semigroups:

$$\mathfrak{P}_t f(u)=\int_H P_t(u,dv)f(v),\quad \mathfrak{P}_t^*\mu(\Gamma)=\int_H P_t(u,\Gamma)\mu(du),$$

where $f\in C_b(H)$ and $\mu\in\mathcal{P}(H)$. A measure $\mu\in\mathcal{P}(H)$ is said to be *stationary* if $\mathfrak{P}_t^*\mu=\mu$ for all $t\ge 0$.

We denote by $u(t;u_0)$ a solution for (1.2) equal to a random variable u_0 at $t=0$. In [11], we prove that, if the non-degeneracy condition (1.5) is fulfilled with a sufficiently large N, then the NS equation has a unique stationary measure μ and

$$\Bigl|\mathbb{E}\,f\bigl(u(t;u_0)\bigr)-\int_H f(v)\mu(dv)\Bigr|\le C\left(1+\mathbb{E}\,|u_0|^2\right)e^{-\alpha\gamma t},\quad t\ge 0,\tag{3.1}$$

for any function $f\in C_b(H)$ satisfying the conditions

$$\sup_{u\in H}\bigl|f(u)\bigr|\le 1,\quad \sup_{u\ne v}\frac{|f(u)-f(v)|}{|u-v|^\alpha}\le 1.$$

Here α is any number such that $0<\alpha\le 1$, and the constants $C\ge 1$ and $\gamma>0$ are independent of α. Inequality (3.1) implies that

$$\mathcal{D}\bigl(u(t;u_0)\bigr)\rightharpoonup\mu\quad\text{as}\quad t\to\infty,\quad\text{for all}\quad u_0\in H.\tag{3.2}$$

Below we use the results from the previous section to show that estimates similar to (3.1) hold for a much larger class of functionals f.

For $\alpha\in(0,1]$ and integers $p\ge 1$ and $l\ge 0$, we denote by $\mathcal{O}_l^p(\alpha)$ the set of continuous functionals g on H^l such that

$$\bigl|g(u)\bigr|\le 1+\|u\|_l^p,\tag{3.3}$$

$$\bigl|g(u)-g(v)\bigr|\le\|u-v\|_l^\alpha\bigl(1+\|u\|_l^{p-1}+\|v\|_l^{p-1}\bigr),\tag{3.4}$$

where $u,v\in H^l$. Let us take any constant $R\ge 1$ and consider two solutions $\tilde u_1(t)$ and $\tilde u_2(t)$ that correspond to initial functions u_{10} and u_{20} satisfying the inequality

$\mathbb{E}\,e^{\sigma|u_{i0}|^2} \le R < \infty$ for some $\sigma > 0$. It is proved in [11] (see there relation (3.39)) that there are weak solutions u_1 and u_2 for (1.2) that are distributed as $\tilde{u}_1$ and $\tilde{u}_2$, respectively, and satisfy the inequality

$$\begin{aligned}\mathbb{P}\bigl\{|u_1(t)-u_2(t)| \le C_1 e^{-\gamma t}\bigr\} &\ge 1 - C_1\bigl(1+\mathbb{E}\,|u_{10}|^2 + E\,|u_{20}|^2\bigr)e^{-\gamma t}\\ &\ge 1 - C_2 R\,e^{-\gamma t}.\end{aligned} \tag{3.5}$$

Denoting by $\Omega_1(t)$ the event in the left-hand side of (3.5), we have

$$\mathbb{P}\bigl(\Omega_1(t)^c\bigr) \le C_2 R\,e^{-\gamma t}, \tag{3.6}$$

$$|u_1(t)-u_2(t)| \le C_1 e^{-\gamma t}, \quad \omega \in \Omega_1(t). \tag{3.7}$$

Let us fix any $p \ge 1$, $k \ge 0$, and $\alpha \in (0,1]$ and take a functional $g \in \mathcal{O}_k^p(\alpha)$. Assuming that $B_{k+1} < \infty$, we conclude from Corollary 2.2 that

$$\mathbb{E}\,\bigl\|u_i(t)\bigr\|_r^m \le K(r,m,\sigma)R, \quad t \ge 1, \tag{3.8}$$

where $i = 1,2$, $0 \le r \le k+1$, $m \ge 1$, and $K(r,m,\sigma) > 0$ is a constant not depending on the solutions. Using the interpolation inequality

$$\bigl\|u\bigr\|_k \le |u|^{\frac{1}{k+1}}\,\|u\|_{k+1}^{\frac{k}{k+1}},$$

we derive from (3.4) and (3.7) that

$$\begin{aligned}\bigl|g(u_1(t)) - g(u_2(t))\bigr| &\le \bigl\|u_1(t)-u_2(t)\bigr\|_k^\alpha\bigl(1+\|u_1(t)\|_k^{p-1} + \|u_2(t)\|_k^{p-1}\bigr)\\ &\le C_3\,e^{-\gamma_k\alpha t}\bigl(1+\|u_1(t)\|_{k+1}^p + \|u_2(t)\|_{k+1}^p\bigr),\end{aligned} \tag{3.9}$$

where $\omega \in \Omega_1(t)$ and $\gamma_k = \frac{\gamma}{k+1}$. It follows from (3.9) and (3.8) that

$$\mathbb{E}\,\Bigl(I_{\Omega_1(t)}\bigl|g(u_1(t)) - g(u_2(t))\bigr|\Bigr) \le C_4 K(k+1,p,\sigma)R\,e^{-\gamma_k\alpha t}, \quad t \ge 0. \tag{3.10}$$

Furthermore, in view of (3.3) and (3.6), we have

$$\begin{aligned}\mathbb{E}\,\Bigl(I_{\Omega_1(t)^c}\bigl|g(u_1(t)) - g(u_2(t))\bigr|\Bigr) &\le \mathbb{E}\,\Bigl(I_{\Omega_1(t)^c}\bigl(2+\|u_1(t)\|_k^p + \|u_2(t)\|_k^p\bigr)\Bigr)\\ &\le \mathbb{P}\bigl(I_{\Omega_1(t)^c}\bigr)^{\frac12}\Bigl(2+\bigl(\mathbb{E}\,\|u_1(t)\|_k^{2p}\bigr)^{\frac12} + \bigl(\mathbb{E}\,\|u_2(t)\|_k^{2p}\bigr)^{\frac12}\Bigr)\\ &\le C_5 K(k,2p,\sigma)^{\frac12} R\,e^{-\gamma t/2}.\end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), we obtain

$$\bigl|\mathbb{E}\,g(u_1(t)) - \mathbb{E}\,g(u_2(t))\bigr| \le C(k,p,\sigma)\,R\,e^{-\gamma_k'\alpha t}, \quad t \ge 1, \tag{3.12}$$

where $g \in \mathcal{O}_k^p(\alpha)$. In particular, assuming that $u_2(t)$ is a stationary solution, we arrive at the following result:

Theorem 3.1. *Suppose that $B_{k+1} < \infty$ for some integer $k \ge 0$ and that condition* (1.5) *is satisfied with a sufficiently large $N \ge 1$. Let μ be the stationary measure for the NS system* (1.2), (1.3) *with the periodic boundary conditions* *and let $u(t) = u(t;u_0)$ be a solution for (1.2) with initial function u_0 such that $R := \mathbb{E}\, e^{\sigma|u_0|^2} < \infty$, where $\sigma > 0$ is a constant. Then for any $p \geq 1$, $\alpha \in (0,1]$, and $g \in \mathcal{O}_k^p(\alpha)$ we have*

$$\big|\mathbb{E}\, g\big(u(t)\big) - (\mu, g)\big| \leq C(k,p,\sigma)\, R\, e^{-\gamma' t}, \quad t \geq 0, \tag{3.13}$$

where $\gamma' > 0$ depends on α and k.

The results from [11] given in (3.1) and (3.5) apply to solutions of the NS equations both under the Dirichlet and periodic boundary conditions. Since our proof of theorem 3.1 uses estimates (3.8) for $r = k+1 \geq 1$, it applies for the periodic case only. Still, if we take $k = 0$, then inequality (3.8) with $r = 0$ remains valid (see (2.2) and Subsection 2.2), while (3.8) with $r = 1$ was used only in the derivation of inequality (3.10). The latter is now implied by (3.6), (3.7), and (3.8) with $r = 0$. After (3.8) and (3.10) are established, the proof of inequality (3.12) goes as before. Therefore, we have the following result:

Proposition 3.2. *If $B_1 < \infty$, $g \in \mathcal{O}_0^p(\alpha)$, and (1.5) holds, then the assertion of theorem 3.1 remains true for solutions of the NS system under the Dirichlet boundary condition.*

In fact, the arguments used to prove (3.13) with $k = 0$ apply to functionals $g \in C_b(H)$ such that

$$\big|g(u)\big| \leq e^{\varepsilon|u|^2}, \quad \big|g(u) - g(v)\big| \leq |u - v|\,\big(e^{\varepsilon|u|^2} + e^{\varepsilon|u|^2}\big),$$

where $u, v \in H$, and $\varepsilon > 0$ is a sufficiently small constant.

4. Kick-forced equations

4.1. Statement of the results

In this section, we consider the NS system (1.2) under the Dirichlet boundary condition, where the right-hand side is a short-period random kick-force η_ε defined in (1.3):

$$\dot u_\varepsilon + \nu L u_\varepsilon + B(u_\varepsilon, u_\varepsilon) = \eta_\varepsilon(t,x). \tag{4.1}$$

We assume that $0 < \varepsilon \leq 1$, the constants $b_j \geq 0$ are the same as in (1.4), and the distributions $\mathcal{D}(\xi_{jk})$ of the independent random variables ξ_{jk} satisfy (1.7) and the following condition:

(H) $\mathcal{D}(\xi_{jk}) = p_j(s)\, ds$, where p_j's are functions of bounded total variation such that $\operatorname{supp} p_j \subset [-1,1]$ and $\int_{-\delta}^{\delta} p_j(s)\, ds > 0$ for all $j \geq 1$ and $\delta > 0$;

To simplify notations, in what follows it is assumed that $\nu = 1$. We normalize solutions of (4.1) to be continuous from the right and set $u_\varepsilon^k = u_\varepsilon(k\varepsilon)$. Denoting by S_ε the operator of time ε shift along trajectories of the free NS system, we see that

$$u_\varepsilon^k = S_\varepsilon\big(u_\varepsilon^{k-1}\big) + \eta_k. \tag{4.2}$$

The random dynamical system (4.2) defines a family Markov chains in the space H. Let $P_{\varepsilon t}(u,\cdot)$, $t \in \varepsilon\mathbb{Z}_+$, be its transition function. In [9], we prove that if condition (1.5) is satisfied for a sufficiently large integer $N = N_\varepsilon \geq 1$, then the family of Markov chains has a unique stationary measure μ_ε and[1]

$$P_{\varepsilon t}(u,\cdot) \rightharpoonup \mu_\varepsilon \quad \text{as} \quad t \to \infty. \tag{4.3}$$

¿From now on, we assume that $B_1 < \infty$. For $\varepsilon \in (0,1]$, let us denote by $\zeta_\varepsilon(t) \in H^1$ the piecewise constant process

$$\zeta_\varepsilon(t) = \int_{0+}^{t+0} \eta_\varepsilon(s)\,ds, \quad t \geq 0. \tag{4.4}$$

Note that $\zeta_\varepsilon(t)$ vanishes at zero and is right-continuous. Let us fix any $T > 0$ and restrict $\zeta_\varepsilon(t)$ to the time interval $[0,T]$. Then, by the Donsker theorem (see [1]),

$$\mathcal{D}(\zeta_\varepsilon) \rightharpoonup \mathcal{D}(\zeta) \quad \text{as} \quad \varepsilon \to 0, \tag{4.5}$$

where the convergence holds in the space of measures on the Skorokhod space $D([0,T];H^1)$; see [1, Chapter 3] for definitions. Using (4.5), it is not difficult to show that (see assertion (ii) of §4.2 for a stronger result)

$$\mathcal{D}\big(u_\varepsilon(t,u_0)\big) \rightharpoonup \mathcal{D}\big(u(t,u_0)\big) \quad \text{as} \quad \varepsilon \to 0 \tag{4.6}$$

for each $t \in [0,T]$ and each $u_0 \in H$, where $u_\varepsilon(t;u_0)$ (accordingly, $u(t;u_0)$) stands for a solution for (4.2) (accordingly, for (1.2)) equal to u_0 at $t = 0$.

The following theorem shows that a similar result holds for the stationary measures μ_ε.

Theorem 4.1. *Let $B_1 < \infty$, $b_j \neq 0$ for all $j \geq 1$, and μ be the stationary measure for (1.2). Then $\mu_\varepsilon \rightharpoonup \mu$ as $\varepsilon \to 0$.*

We note that theorem 4.1 remains valid with no changes for periodic boundary conditions.

4.2. Proof of theorem 4.1

Let $\mathfrak{P}_t(\varepsilon)$ and $\mathfrak{P}_t^*(\varepsilon)$ be the Markov semigroups corresponding to the transition function $P_{\varepsilon t}(u,\Gamma)$. Then

$$\mathfrak{P}_t^*(\varepsilon)\mu_\varepsilon = \mu_\varepsilon \quad \text{for } t \in \varepsilon\mathbb{Z}_+; \qquad \mathfrak{P}_t^*\mu = \mu \quad \text{for } t \geq 0.$$

It follows that

$$\mu_\varepsilon - \mu = \big(\mathfrak{P}_t^*(\varepsilon)\mu_\varepsilon - \mathfrak{P}_t^*\mu_\varepsilon\big) + \big(\mathfrak{P}_t^*\mu_\varepsilon - \mathfrak{P}_t^*\mu\big), \quad t \in \varepsilon\mathbb{Z}_+. \tag{4.7}$$

Let us denote by $\mathcal{L}_0(H)$ the space of globally Lipschitz functionals $f : H \to \mathbb{R}$ endowed with the quasi-norm

$$\|f\|_{\mathcal{L}_0} = \sup_{u \neq v} \frac{|f(u) - f(v)|}{|u - v|}$$

[1]In [9], we prove the convergence (4.3) under some unessential additional restrictions which were removed in our subsequence publications, see in [8].

and by $\mathcal{L}(H)$ the space of bounded functionals $f \in \mathcal{L}_0(H)$ endowed with the norm

$$\|f\|_{\mathcal{L}} = \sup_{u\in H} |f(u)| + \|f\|_{\mathcal{L}_0}.$$

We wish to show that the right-hand side of (4.7) goes to zero in the weak* topology as $t \to +\infty$ and $\varepsilon \to 0+$. To this end, we recall that the function

$$\|\lambda_1 - \lambda_2\|_{\mathcal{L}}^* := \sup_{\|f\|_{\mathcal{L}}\le 1} \bigl|(\lambda_1, f) - (\lambda_2, f)\bigr|, \quad \lambda_1, \lambda_2 \in \mathcal{P}(H), \tag{4.8}$$

defines a distance on $\mathcal{P}(H)$ which metrizes the weak* convergence (see [5, Section 11.3] for details). Therefore, the required assertion will be established if we prove the following two statements:

(i) There are positive constants C and γ such that

$$\bigl|(\mathfrak{P}_t f, \mu_\varepsilon) - (\mathfrak{P}_t f, \mu)\bigr| \le C\, e^{-\gamma t}, \quad t \ge 0, \quad 0 < \varepsilon \le 1, \tag{4.9}$$

where $f \in \mathcal{L}(H)$ is an arbitrary functional such that $\|f\|_{\mathcal{L}} \le 1$.

(ii) For any $T > 0$ and any subset $\mathcal{K} \subset \mathcal{P}(H)$ satisfying the condition

$$\sup_{\lambda\in\mathcal{K}} \int_H |u|^2 \lambda(du) < \infty, \tag{4.10}$$

we have

$$\sup_{t,\lambda,f} \bigl|(\mathfrak{P}_t(\varepsilon) f, \lambda) - (\mathfrak{P}_t f, \lambda)\bigr| \to 0 \quad \text{as} \quad \varepsilon \to 0+, \tag{4.11}$$

where the supremum is taken over $t \in \varepsilon\mathbb{Z}_+$, $\lambda \in \mathcal{K}$ and $f \in \mathcal{L}(H)$ such that $0 \le t \le T$ and $\|f\|_{\mathcal{L}} \le 1$.

Indeed, if (i) and (ii) are proved, then for any $\delta > 0$ we first choose a sufficiently large $T > 0$ such that

$$\|\mathfrak{P}_t^* \mu_\varepsilon - \mathfrak{P}_t^* \mu\|_{\mathcal{L}}^* = \sup_{\|f\|_{\mathcal{L}}\le 1} \bigl|(\mathfrak{P}_t f, \mu_\varepsilon) - (\mathfrak{P}_t f, \mu)\bigr| \le \tfrac{\delta}{2} \quad \text{for } t \ge T - 1, \tag{4.12}$$

and then, using the fact that the family $\{\mu_\varepsilon\}$ satisfies condition (4.10) (see (4.14)), we find $\varepsilon_0 > 0$ such that

$$\sup_{t\in\varepsilon\mathbb{Z}_+\cap[0,T]} \|\mathfrak{P}_t^*(\varepsilon)\mu_\varepsilon - \mathfrak{P}_t^*\mu_\varepsilon\|_{\mathcal{L}}^* \le \tfrac{\delta}{2} \quad \text{for } 0 < \varepsilon \le \varepsilon_0. \tag{4.13}$$

Combining (4.7), (4.12) and (4.13), we see that $\|\mu_\varepsilon - \mu\|_{\mathcal{L}}^* \le \delta$ for $0 < \varepsilon \le \varepsilon_0$.

We now turn to the proof of assertions (i) and (ii).

Proof of (i). It follows from (3.1) that the function $\Delta_t(u) := \mathfrak{P}_t f(u) - (\mu, f)$ satisfies the inequality

$$\bigl|\Delta_t(u)\bigr| \le C(1 + |u|^2) e^{-\gamma t}, \quad t \ge 0.$$

Since μ_ε and μ are probability measures, we derive

$$\big|(\mu_\varepsilon,\mathfrak{P}_t f)-(\mu,\mathfrak{P}_t f)\big|=\big|(\mu_\varepsilon,\Delta_t)-(\mu,\Delta_t)\big|\le C\,e^{-\gamma t}\big(2+(\mu_\varepsilon,|u|^2)+(\mu,|u|^2)\big).$$

By proposition 2.5, we have $(\mu,|u|^2)<\infty$. Thus, inequality (4.9) will be established if we show that there is $C_0>0$ not depending on ε such that

$$\int_H |u|^2\,\mu_\varepsilon(du)\le C_0 \quad \text{for} \quad 0<\varepsilon\le 1. \tag{4.14}$$

To this end, we note that $|S_\varepsilon(u)|\le e^{-\varepsilon}|u|$ for $u\in H$. Since η_k and u_ε^k are independent, it follows that

$$\mathbb{E}\,\big|u_\varepsilon^k\big|^2=\mathbb{E}\,\big|S_\varepsilon(u_\varepsilon^{k-1})\big|^2+\varepsilon\mathbb{E}\,|\eta_k|^2\le e^{-2\varepsilon}\mathbb{E}\,\big|u_\varepsilon^{k-1}\big|^2+\varepsilon B_0,$$

where we used the two relations below (see (1.7)):

$$\mathbb{E}\,\eta_k=0,\quad \mathbb{E}\,|\eta_k|^2=\sum_j b_j^2\,\mathbb{E}\,\xi_{jk}^2=\sum_j b_j^2=B_0. \tag{4.15}$$

In particular, if $u_\varepsilon(0)=0$, then

$$\mathbb{E}\,\big|u_\varepsilon^k\big|^2\le\frac{\varepsilon B_0}{1-e^{-2\varepsilon}}\le\frac{3B_0}{2}\quad\text{for each}\quad k\ge 0.$$

Combining this inequality with (4.3), we derive $\int_H\big(|u|^2\wedge M\big)\,\mu_\varepsilon(du)\le\frac32 B_0$, where $M>0$ is an arbitrary constant. Application of the Levi theorem results in (4.14) with $C_0=\frac32 B_0$.

Proof of (ii). We first explain the main idea. Let us set

$$g_\varepsilon(t,v)=f\big(u_\varepsilon(t,v)\big)-f\big(u(t,v)\big).$$

We wish to show that the function

$$I_\varepsilon:=\int_H \mathbb{E}\,g_\varepsilon(t,v)\,\lambda(dv)$$

goes to zero with ε uniformly with respect to t, f, and λ. Since $|g_\varepsilon|\le 2$, we have

$$|I_\varepsilon|\le 2\lambda\big\{|v|>R\big\}+\sup_{v\in B_H(R)}\big|I_\varepsilon(v)\big|,$$

where $I_\varepsilon(v):=\mathbb{E}\,g_\varepsilon(t,v)$ and $B_H(R)$ denotes the closed ball in H of radius R centred at zero. Hence, in view of (4.10), the required assertion will be established if we prove that, for any $R>0$,

$$I_\varepsilon(v)\to 0\quad\text{as }\varepsilon\to 0\text{ uniformly in }t,\ f,\text{ and }v\in B_H(R). \tag{4.16}$$

This property would be easy to prove if we knew that the difference $u_\varepsilon-u$ tends to zero for a.e. ω. However, this is not necessarily the case, since we only can assert that the distributions of u_ε and u are close for small ε (see (4.6)).

To overcome this difficulty, we remark that $I_\varepsilon(v)$ depends only on the distributions of u_ε and u. Therefore, if we replace the right-hand sides η_ε and η of equations (4.1) and (1.2) by some other processes $\hat\eta_\varepsilon$ and $\hat\eta$ with the same distributions, the corresponding function $\hat I_\varepsilon(v)$ will coincide with $I_\varepsilon(v)$. We use below the Donsker principle and a theorem of Kantorovich–Rubinstein to construct some right-hand sides that are very close with high probability. As a results, the corresponding solutions will also be close for small ε.

The accurate proof is divided into several steps. We begin with a construction of right-hand sides with the above-mentioned properties.

Step 1. We first recall the Kantorovich–Rubinstein theorem. For any separable metric space X with a distance d and a distinguished point $\mathbf{0}$ and any $k \ge 1$, we denote by $\mathcal{P}_k(X)$ the set of probability measures $\lambda \in \mathcal{P}(X)$ such that

$$\mathfrak{m}_k(\lambda) := \int_X d(u,\mathbf{0})^k \lambda(du) < \infty.$$

It is easy to see that, for any $C > 0$ and $k > 1$, the restriction of the weak* convergence to the set

$$\mathcal{P}_k(X,C) := \{\lambda \in \mathcal{P}_k(X) : \mathfrak{m}_k(\lambda) \le C\}$$

is equivalent to the convergence in the metric (see [5, Section 11.8]) and cf. (4.8))

$$\|\lambda_1 - \lambda_2\|^*_{\mathcal{L}_0} := \sup_{\|f\|_{\mathcal{L}} \le 1} \bigl|(\lambda_1, f) - (\lambda_2, f)\bigr|.$$

Moreover, according to the Kantorovich–Rubinstein theorem (see [5, theorem 11.8.3]), for any $\lambda_1, \lambda_2 \in \mathcal{P}_1(X)$ there are X-valued random variables ξ_1 and ξ_2 defined on a common probability space $(\Omega, \mathbb{P})$ such that $\mathcal{D}(\xi_i) = \lambda_i$, $i = 1, 2$, and

$$\int_\Omega d\bigl(\xi_1(\omega), \xi_2(\omega)\bigr)\, \mathbb{P}(d\omega) = \|\lambda_1 - \lambda_2\|^*_{\mathcal{L}_0}.$$

In particular, applying the Chebyshev inequality, we see that, for any $\delta > 0$,

$$\mathbb{P}\bigl\{d\bigl(\xi_1(\omega), \xi_2(\omega)\bigr) > \delta\bigr\} \le \delta^{-1} \|\lambda_1 - \lambda_2\|^*_{\mathcal{L}_0}. \tag{4.17}$$

Step 2. We now apply this result to the processes ζ_ε and ζ regarded as random variables in the space $D_T := D([0,T], H^1)$ endowed with the Skorokhod metric d_T. By the Donsker theorem (see [1, Section 16]), convergence (4.5) holds. Moreover, using the independence of the variables η_k and relations (4.15) with $|\cdot|$ replaced by $\|\cdot\|$, we derive

$$\mathbb{E}\bigl\|\zeta_\varepsilon(\varepsilon k)\bigr\|^2 = \mathbb{E}\bigl\|\zeta_\varepsilon(\varepsilon(k-1))\bigr\|^2 + \varepsilon\,\mathbb{E}\|\eta_k\|^2 = k\varepsilon B_1.$$

Therefore, since ζ_ε is a right-continuous martingale on $[0,T]$, the Doob moment inequality implies that (see [7, 12])

$$\mathbb{E}\|\zeta_\varepsilon\|^2_\infty \le 4\mathbb{E}\bigl\|\zeta_\varepsilon(T)\bigr\|^2 \le TB_1, \tag{4.18}$$

where for any H^1-valued function on $[0,T]$ we set $\|\xi\|_\infty = \sup_{t\in[0,T]} \|\xi(t)\|$. This inequality remains valid for $\zeta(t)$. The definition of the Skorokhod metric implies

$$d_T(\xi, \mathbf{0}) = \|\xi\|_\infty \quad \text{for any } \xi \in D_T([0,T], H^1),$$

where $\mathbf{0}$ is the function identically equal to 0. Combining this with inequality (4.18) and its analog for $\zeta(t)$, we conclude that the distributions of ζ and ζ_ε belong to $\mathcal{P}_2(D_T, TB_1)$. Since the restriction of the weak* topology of $\mathcal{P}(D_T)$ to $\mathcal{P}_2(D_T, TB_1)$ is equivalent to the topology defined by $\|\cdot\|^*_{\mathcal{L}_0}$, we conclude from (4.5) that for any $\delta > 0$ there is $\varepsilon_0(\delta) > 0$ such that

$$\big\|\mathcal{D}(\zeta_\varepsilon) - \mathcal{D}(\zeta)\big\|^*_{\mathcal{L}_0} \le \delta^2 \quad \text{for} \quad 0 < \varepsilon \le \varepsilon_0(\delta). \tag{4.19}$$

Using now the Kantorovich–Rubinstein theorem (see above), for any $\varepsilon \in (0, \varepsilon_0]$ we can construct random variables $\hat\zeta_\varepsilon$ and $\hat\zeta$ in D_T defined on a common probability space $(\widehat\Omega, \widehat{\mathbb{P}})$ such that $\mathcal{D}(\hat\zeta_\varepsilon) = \mathcal{D}(\zeta_\varepsilon)$, $\mathcal{D}(\hat\zeta) = \mathcal{D}(\zeta)$, and

$$\widehat{\mathbb{P}}\big\{d_T(\hat\zeta_\varepsilon, \hat\zeta) > \delta\big\} \le \delta \quad \text{for} \quad 0 < \varepsilon \le \varepsilon_0(\delta), \tag{4.20}$$

where we used inequalities (4.17) and (4.19). We emphasize that the random variable $\hat\zeta$ and the probability space $(\widehat\Omega, \widehat{\mathbb{P}})$ also depend on ε. To simplify notation, we shall not indicate that dependence and, moreover, we shall drop the hat as well.

Step 3. We now prove (4.16). For any $K > 0$, let us set

$$\Omega_K = \big\{\omega \in \Omega : \|\zeta(t) - \zeta(s)\| \le K|t-s|^{\frac13} \text{ for any } t, s \in [0,T];\ \zeta(0) = 0\big\}.$$

Well-known properties of the modulus of continuity for the Wiener process imply that $\mathbb{P}(\Omega_K) \to 1$ as $K \to \infty$. Therefore, it suffices to show that

$$I_{\varepsilon K}(v) := \mathbb{E}\big(I_{\Omega_K} g_\varepsilon(t,v)\big) \to 0 \quad \text{as } \varepsilon \to 0 \text{ uniformly in } t,\ f, \text{ and } v \in B_H(R), \tag{4.21}$$

where I_{Ω_K} is the indicator function of the set Ω_K.

Step 4. To prove (4.21), we need two auxiliary lemmas. The first of them is a straightforward consequence of the definition of the Skorokhod metric d_T, and its proof is omitted.

Lemma 4.2. *Let $\xi_1 \in C([0,T], H^1)$ and $\xi_2 \in D([0,T], H^1)$. Then*

$$\|\xi_1 - \xi_2\|_\infty \le d_T(\xi_1, \xi_2) + \rho(d_T(\xi_1, \xi_2)), \tag{4.22}$$

where $\rho(r)$ is the modulus of continuity for ξ_1.

Lemma 4.2 implies, in particular, that if $\omega \in \Omega_K$ and $d_T(\zeta_\varepsilon, \zeta) \le \delta$, then

$$\|\zeta_\varepsilon - \zeta\|_\infty \le \delta + K\delta^{\frac13}. \tag{4.23}$$

To formulate the second lemma, we consider equation (1.2) with a deterministic right-hand side:

$$\dot u + Lu + B(u,u) = \partial_t \zeta(t,x), \quad u(0) = u_0. \tag{4.24}$$

where $\zeta \in D([0,T];H^1)$ and $\zeta(0)=0$. Substituting $u=\zeta+v$ into (4.24), we get for v the problem

$$\dot v + Lv + B(v+\zeta, v+\zeta) = -L\zeta, \quad v(0)=u_0. \tag{4.25}$$

Due to the basic properties of the 2D NS system (see [4]), the problem (4.25) has a unique solution $v \in C([0,T];H)\cap L^2([0,T];H^1)$. We call the corresponding function $u=v+\zeta$ a *weak solution* for (4.24) and denote it by $u=U(\zeta)=U(\zeta,u_0)$. The proof of the following lemma is given at the end of this subsection:

Lemma 4.3. *Let $|u_0|\le R$, let $\|\zeta_i\|_\infty \le N$, $i=1,2$, and let $\|\zeta_1-\zeta_2\|_\infty\le\alpha\le 1$. We set $u_i=U(\zeta_i;u_0)$. Then*

$$|u_1(t)-u_2(t)| \le \alpha F(R,N,T) \quad \textit{for} \quad 0\le t\le T, \tag{4.26}$$

where $F\ge 1$ is a continuous function not depending on ζ_1, ζ_2, and u_0.

It is known (see [7]) that solutions for (1.2) and (4.1) 'can be treated pathwise,' that is,

$$u(t;u_0)=U(\zeta;u_0)(t), \quad u_\varepsilon(t;u_0)=U(\zeta_\varepsilon;u_0)(t). \tag{4.27}$$

Step 5. We can now prove (4.21) and thus to complete the proof of the theorem. Let us fix any $\delta>0$ and assume that $0<\varepsilon\le\varepsilon_0(\delta)$, where $\varepsilon_0(\delta)>0$ is the constant in (4.19). Since $|g_\varepsilon|\le 2$, we have

$$|I_{\varepsilon K}(v)| \le 2\mathbb{P}\{d_T(\zeta_\varepsilon,\zeta)>\delta\} + \sup_{t,\omega}|g_\varepsilon(t,v)|, \tag{4.28}$$

where the supremum is taken over $t\in[0,T]$ and $\omega\in\Omega_K\cap\{d_T(\zeta_\varepsilon,\zeta)\le\delta\}$. In view of (4.20), the first term in the right-hand side of (4.28) does not exceed 2δ. To estimate the second, we note that, if $\omega\in\Omega_K$ and $d_T\le\delta\le1$, then

$$\|\zeta\|_\infty \le KT^{\frac13}, \quad \|\zeta_\varepsilon\|_\infty \le N_K := K\bigl(T^{\frac13}+1\bigr)+1,$$

where we used (4.23) and the definition of the set Ω_K. Therefore, applying lemma 4.3 with $\alpha=\delta+K\delta^{\frac13}$, we conclude that

$$\sup_{t,\omega}|g_\varepsilon(t,v)| \le \sup_{t,\omega}\bigl|u_\varepsilon(t,v)-u(t,v)\bigr| \le F(R,N_K,T)(\delta+K\delta^{\frac13}). \tag{4.29}$$

Combining (4.28) and (4.29), we derive

$$\sup_{|v|\le R}|I_{\varepsilon K}| \le 2\delta + F(R,N_K,T)(\delta+K\delta^{\frac13}) \quad \text{for } 0<\varepsilon\le\varepsilon_0(\delta).$$

Since $\delta\in(0,1]$ was arbitrary, this inequality implies (4.21).

Proof of lemma 4.3. Let us write $u_i=\zeta_i+v_i$, $i=1,2$, and note that both v_1 and v_2 satisfy equations (4.25), where $\|\zeta\|_\infty\le N$. Multiplying the equation by v_i in H, we get

$$\frac12\frac{d}{dt}|v_i|^2 + \|v_i\|^2 + \bigl(B(v_i+\zeta_i,\zeta_i),v_i\bigr) \le \|\zeta_i\|\,\|v_i\|.$$

Since $\|\zeta_i\| \le N$, we have $(B(v_i+\zeta_i,\zeta_i),v_i)| \le CN^2\|v_i\|+CN|v_i|\,\|v_i\|$, and therefore

$$\frac{d}{dt}|v_i|^2 + \|v_i\|^2 \le C_1\bigl(N^2 + N^4 + N^2|v_i|^2\bigr). \tag{4.30}$$

This implies the inequality

$$|v_i(t)|^2 \le e^{C_1 N^2 t}\bigl(R^2 + C_2 t N^4\bigr), \quad 0 \le t \le T. \tag{4.31}$$

Besides, integrating (4.30) from 0 to T and using (4.31), we obtain

$$\int_0^T \|v_i\|^2 dt \le F_1(R,N,T), \tag{4.32}$$

where F_1 is a continuous function.

We now set $v = v_1 - v_2$ and $\zeta = \zeta_1 - \zeta_2$. Subtracting the equations for v_1 and v_2 from one another and multiplying the result by v in H, we derive

$$\frac{1}{2}\frac{d}{dt}|v|^2 + \|v\|^2 + \bigl(B(u_1,u_1) - B(u_2,u_2), v\bigr) = (Lv,\zeta).$$

Since $\|\zeta\| \le \alpha$, then evoking the usual estimates for the nonlinearity (see [4]), we get

$$\bigl|\bigl(B(u_1,u_1) - B(u_2,u_2), v\bigr)\bigr| \le C\,\alpha\,\|v\|\bigl(\|u_1\| + \|u_2\|\bigr) + C\,\|v\|\,|v|\,\|u_2\|.$$

It follows that

$$\frac{d}{dt}|v|^2 + \|v\|^2 \le C_1\bigl(\alpha^2(\|u_1\| + \|u_2\|)^2 + |v|^2\|u_2\|^2 + \alpha^2\bigr).$$

Since $v(0) = 0$, application of the Gronwall inequality results in

$$|v(t)|^2 \le C_1\alpha^2 \int_0^t \exp\Bigl(C_2\int_s^t \|u_2\|^2 d\theta\Bigr)\,\bigl(1 + (\|u_1\| + \|u_2\|)^2\bigr)\,ds.$$

Using (4.31) and (4.32), we obtain (4.26). □

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