0.1. **Conventions.** In this note, manifolds are C^∞ and finite-dimensional, vector bundles are of finite rank, and smooth maps are C^∞.

If $I = (i_1, \dots, i_m) \in \mathbf{N}^m$ is a multi-index, put

$$|I| = i_1 + \cdots + i_m, \qquad I! = i_1! \cdots i_m!. \tag{0.1}$$

If $(x_1, \dots, x_m) \in \mathbf{R}^m$, we use the notation

$$x^I = x_1^{i_1} \cdots x_m^{i_m}, \qquad \left(\frac{\partial}{\partial x}\right)^I = \frac{\partial^{|I|}}{\partial x_1^{i_1} \cdots \partial x_m^{i_m}}. \tag{0.2}$$

1. Manifolds, vector bundles, and differential operators

I will assume the reader is familiar with the basic notions of differential geometry, such as smooth manifolds, vector bundles, de Rham complex, etc. We will recall briefly some of the basic definitions and properties.

1.1. **Smooth manifolds.**

Definition 1.1 (Manifolds)**.** 1) A topological space X is called a *smooth manifold* if

a) X is Hausdorff;

b) X is second-countable[1];

c) X is locally Euclidean, i.e., we have an open cover

$$X = \bigcup_{\alpha \in A} U_\alpha, \tag{1.1}$$

such that, for each $\alpha \in A$, there exist $m \in \mathbf{N}$, an open set $V_\alpha \subset \mathbf{R}^m$, and a homeomorphism

$$\psi_\alpha : U_\alpha \to V_\alpha, \tag{1.2}$$

such that if $U_\alpha \cap U_\beta \neq \varnothing$, then the following map is a diffeomorphism,

$$\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \psi_\beta(U_\alpha \cap U_\beta). \tag{1.3}$$

2) If m is independent of $\alpha \in A$, we say X has dimension m.

Remark 1.2. When one first encounters the definition of manifolds, the Hausdorff condition may seem quite natural, whereas the second-countability condition may appear somewhat less so. Indeed, these assumptions imply that the manifold is paracompact and metrizable. We recall the following definition and some standard facts from topology.

a) A topological space is called paracompact if every open cover admits a locally finite open refinement.

b) Michael's theorem: Hausdorff, second-countable, and locally compact $\implies$ paracompact.

c) Smirnov's theorem: Hausdorff, paracompact, and locally metrizable $\iff$ metrizable.

Remark 1.3. By definition, a discrete uncountable set is not a manifold.

[1]There exists a countable family $\mathcal{T}$ of open subsets such that any open set is a union of elements in $\mathcal{T}$.

In what follows, X will always denote a smooth manifold. The following proposition based on the paracompactness is very useful to construct global objects from local data.

Proposition 1.4 (Smooth partitions of unity)**.** *Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover on X.*

1) There exists a family $\{\varphi_\beta\}_{\beta\in B}$ of smooth functions with compact support in $C_c^\infty(X,[0,1])$ such that

a) for every $\beta\in B$, there exists $\alpha\in A$ such that $\operatorname{Supp}\varphi_\beta\subset U_\alpha$;

b) the following sum is locally finite,[2] and

$$\sum_{\beta\in B}\varphi_\beta = 1. \tag{1.4}$$

2) If each U_α is relatively compact, we can choose the index set B of our partition of unity to be equal to A, and in 1a) we can require $\alpha=\beta$.

Proof. See Exercises 1.2 and 1.3. □

Remark 1.5. Introduce another index set B in Proposition 1.4 is necessary. Indeed, if $X=\mathbf{R}$ which is covered by one open set $U=\mathbf{R}$, a function with compact support cannot be equal to 1 on $\mathbf{R}$.

Remark 1.6. There is rarely any need to delve into the details of general topology. It suffices to keep in mind that a manifold is locally Euclidean and metrizable, and that it admits smooth partitions of unity.

1.2. **Smooth vector bundles.**

Definition 1.7 (Smooth vector bundle)**.** 1) A complex smooth vector bundle E on X consists

- a smooth manifold $\mathscr{E}$ (called the total space);
- a map $\pi:\mathscr{E}\to X$,

with the property that there exists an open cover $X=\bigcup_{\alpha\in A}U_\alpha$ such that

a) for each $\alpha\in A$, there exist $r_\alpha\in\mathbf{N}$ and a diffeomorphism

$$\psi_\alpha^E:\pi^{-1}(U_\alpha)\to U_\alpha\times\mathbf{C}^{r_\alpha} \tag{1.5}$$

commuting with the projections onto U_α ;

b) if $U_\alpha\cap U_\beta\neq\varnothing$, then $r_\alpha=r_\beta$, denoted by r, and there exists a transition function

$$g_{\beta\alpha}^E\in C^\infty(U_\alpha\cap U_\beta,\mathrm{GL}_r(\mathbf{C})), \tag{1.6}$$

such that if $x\in U_\alpha\cap U_\beta$ and $v\in\mathbf{C}^r$,

$$\psi_\beta^E\psi_\alpha^{E,-1}(x,v)=\left(x,g_{\beta\alpha}^E(x)v\right). \tag{1.7}$$

2) If $r_\alpha=r$ is independent of $\alpha\in A$, we say E has rank r.

3) If we replace $\mathbf{C}$ by $\mathbf{R}$ in (1.5) and (1.6), we obtain the notion of real smooth vector bundles.

[2] For any $x\in X$, there is an open neighbourhood U_x of x such that $\{\beta\in B:\varphi_\beta|_{U_x}\neq 0\}$ is finite.

Remark 1.8. a) If $x \in X$, we call $E_x = \pi^{-1}(x)$ the fibre at x. This is a complex vector space.

b) By abuse of notation, we use the same letter E to denote the typical fibre of the vector bundle.

c) We have an identity of sets,

$$\mathscr{E} = \bigsqcup_{x \in X} E_x. \tag{1.8}$$

Intuitively, a vector bundle is a smooth family of vector spaces parametrized by the base manifold X.

Remark 1.9. The transition functions satisfy[3]

$$g^E_{\alpha\beta} = g^{E,-1}_{\beta\alpha} \text{ on } U_\alpha \cap U_\beta, \qquad g^E_{\alpha\beta} g^E_{\beta\gamma} g^E_{\gamma\alpha} = \mathrm{id} \text{ on } U_\alpha \cap U_\beta \cap U_\gamma. \tag{1.9}$$

Conversely, given smooth transition functions $\{g_{\beta\alpha}\}_{\alpha,\beta\in A}$ satisfying (1.9), we can define an equivalence relation on the disjoint union $\coprod_{\alpha\in A} U_\alpha \times \mathbf{C}^r$ by

$$(x,v) \in U_\alpha \times \mathbf{C}^r \sim (x',v') \in U_\beta \times \mathbf{C}^r \iff x = x', v' = g_{\beta\alpha}(x)v. \tag{1.10}$$

The quotient space $\coprod_\alpha U_\alpha \times \mathbf{C}^r / \sim$ defines a smooth vector bundle with transition functions $\{g_{\beta\alpha}\}_{\alpha,\beta\in A}$ on X.

Applying algebraic operations pointwisely to vector bundles, we obtain a family of vector spaces. Using the smoothness of the transition functions, we can deduce that the obtained family are smooth vector bundles. This gives us the following examples.

Example 1.10. If E is a vector bundle with transition functions $\left\{g^E_{\beta\alpha}\right\}_{\alpha,\beta\in A}$, we can define

a) the dual $E^* = \coprod_{x\in X} E_x^*$, with transition function $\left\{{}^t g^{E,-1}_{\beta\alpha}\right\}_{\alpha,\beta\in A}$;

b) the conjugate $\overline{E} = \coprod_{x\in X} \overline{E}_x$ with transition function $\left\{\overline{g}^E_{\beta\alpha}\right\}_{\alpha,\beta\in A}$;

c) the tensor powers $E^{\otimes k}$, or symmetric and antisymmetric tensor powers $S^k(E)$ et $\Lambda^k(E)$.

If F is another vector bundle with transition functions $\left\{g^F_{\beta\alpha}\right\}_{\alpha,\beta\in A}$, we can define

a) the direct sum $E \oplus F = \coprod_{x\in X} E_x \oplus F_x$ with transition functions $\left\{g^E_{\beta\alpha} \oplus g^F_{\beta\alpha}\right\}_{\alpha,\beta\in A}$;

b) the tensor product $E \otimes F = \coprod_{x\in X} E_x \otimes F_x$ with transition functions $\left\{g^E_{\beta\alpha} \otimes g^F_{\beta\alpha}\right\}_{\alpha,\beta\in A}$;

c) the homomorphism $\mathrm{Hom}(E,F) = \coprod_{x\in X} \mathrm{Hom}(E_x,F_x)$ with transition functions $\left\{{}^t g^{E,-1}_{\beta\alpha} \otimes g^F_{\beta\alpha}\right\}_{\alpha,\beta\in A}$.

If $f : Y \to X$ is a smooth map, we can define the pullback bundle $f^*E = \coprod_{y\in Y} E_{f(y)}$ with the transition functions $\left\{f^* g^E_{\beta\alpha}\right\}_{\alpha,\beta\in A}$ associated to the covering $\{f^{-1}U_\alpha\}_{\alpha\in A}$ of Y.

[3] Taking $\alpha = \beta = \gamma$ in (1.9), we get $g_{\alpha\alpha} = \mathrm{id}$ on U_α.

Example 1.11. If X is a smooth manifold, then the tangent bundle TX is a real vector bundle defined by transition functions

$$d\left(\psi_\beta \circ \psi_\alpha^{-1}\right) \in C^\infty(U_\alpha \cap U_\beta, \mathrm{GL}_m(\mathbf{R})). \tag{1.11}$$

We can define the cotangent bundle T^*X, exterior algebra $\Lambda(T^*X) = \bigoplus_{0\le k\le m} \Lambda^k(T^*X)$ according to the previous example.

Definition 1.12. A smooth section of E is a smooth map $s : X \to \mathscr{E}$ such that $\pi \circ s = \mathrm{id}$. Denote by $C^\infty(X,E)$ the space of smooth sections of E.

Equivalently, s corresponds to a family $\{s_\alpha \in C^\infty(U_\alpha, \mathbf{C}^r)\}_{\alpha\in A}$ such that

$$g^E_{\beta\alpha}(x) s_\alpha(x) = s_\beta(x). \tag{1.12}$$

Intuitively, a smooth section s is a family $\{s_x \in E_x\}_{x\in X}$ of vectors that varies smoothly with x.

Example 1.13. A smooth section of TX is called a vector field. A smooth section of $\Lambda(T^*X)$ is called a differential form.

Definition 1.14. 1) A Hermitian metric h^E on E is an element

$$h^E \in C^\infty\left(X, E^* \otimes \overline{E}^*\right) \tag{1.13}$$

such that for each $x \in X$, h^E_x is a Hermitian metric on E_x.
2) Similarly, we can define a Euclidean metric on a real vector bundle.

Let $\langle \cdot, \cdot \rangle$ be the standard Hermitian metric on $\mathbf{C}^r$. A Hermitian metric h^E corresponds to a family of smooth function $x \in U_\alpha \to h^E_{\alpha,x}$ with valued in positive defined self-adjoint matrix such that we have the following identity on $U_\alpha \cap U_\beta$,

$$\left\langle h^E_\alpha \cdot, \cdot \right\rangle = \left\langle h^E_\beta g^E_{\beta\alpha} \cdot, g^E_{\beta\alpha} \cdot \right\rangle. \tag{1.14}$$

Intuitively, a Hermitian metric is a family of Hermitian metrics $\left\{h^E_x\right\}_{x\in X}$ on E_x that varies smoothly with x.

Proposition 1.15. *A Hermitian (resp. Euclidean) metric on a complex (resp. real) vector bundle always exists.*

Proof. Choose a relatively compact open cover $X = \bigcup_\alpha U_\alpha$ that trivializes E. The standard Hermitian metric on $\mathbf{C}^r$ induces a metric h_α on $E|_{U_\alpha}$. If $\left\{\varphi_\alpha\right\}_{\alpha\in A}$ is a partition of unity subordinate to the cover $\{U_\alpha\}_{\alpha\in A}$, then

$$h^E = \sum_{\alpha\in A} \varphi_\alpha h_\alpha \tag{1.15}$$

is a Hermitian metric on E. □

Definition 1.16. A complex (resp. real) vector bundle equipped with a Hermitian (resp. Euclidean) metric is called a Hermitian (resp. Euclidean) vector bundle.

Definition 1.17. A Euclidean metric g^{TX} on TX is called a Riemannian metric on X. A manifold X equipped with a Riemannian metric g^{TX} is called a Riemannian manifold.

1.3. **Differential operators.** Let E,F be two complex vector bundles on X. Let

$$P : C^\infty(X,E) \to C^\infty(X,F) \tag{1.16}$$

be a $\mathbf{C}$-linear map.

Definition 1.18 (Differential Operator)**.** The operator P is called a differential operator of order at most $k \in \mathbf{N}$ if there exist

a) an open cover $X = \bigcup_{\alpha\in A} U_\alpha$ with the property that
   i) $U_\alpha \simeq V_\alpha$ is a coordinate chart in $\mathbf{R}^m$;
   ii) we have the trivializations $E|_{U_\alpha} = U_\alpha \times \mathbf{C}^{r_1}$ and $F|_{U_\alpha} = U_\alpha \times \mathbf{C}^{r_2}$;

b) an operator on U_α of the form

$$P_\alpha = \sum_{|I|\le k} a_I(x)\left(\frac{\partial}{\partial x}\right)^I, \text{ where } a_I \in C^\infty(U_\alpha, \mathrm{Hom}(\mathbf{C}^{r_1},\mathbf{C}^{r_2})), \tag{1.17}$$

with the property that if $U_\alpha \cap U_\beta \neq \varnothing$, then

$$P_\beta g^E_{\beta\alpha} = g^F_{\beta\alpha} P_\alpha \text{ on } U_\alpha \cap U_\beta, \tag{1.18}$$

such that if $s \in C^\infty(X,E)$ is defined by $\{s_\alpha\}_{\alpha\in A}$ satisfying (1.12), then Ps is defined by the family $\{P_\alpha s_\alpha\}_{\alpha\in A}$ satisfying (1.12).

Example 1.19. Let Q_α be an operator on U_α of the form (1.17) (without assume (1.18)).

a) If $\phi_\alpha, \psi_\alpha \in C^\infty_c(U_\alpha,\mathbf{R})$, then $\phi_\alpha Q_\alpha \psi_\alpha$ is a differential operator of order at most k on X.
b) If $\{U_\alpha\}_{\alpha\in A}$ is locally finite, then the locally finite sum $\sum_{\alpha\in A}\phi_\alpha Q_\alpha\psi_\alpha$ is still a differential operator of order at most k on X.

Theorem 1.20 (Characterization of differential operators by commutators)**.** *A $\mathbf{C}$-linear operator $P : C^\infty(X,E) \to C^\infty(X,F)$ is a differential operator of order at most k, if and only if for any $f_0,\dots,f_k \in C^\infty(X,\mathbf{R})$, the $k+1$-commutator vanishes,*

$$[f_0,[f_1,\dots,[f_k,P]\dots]] = 0. \tag{1.19}$$

Proof. The direct implication is trivial. Let us show the inverse implication.

Step 1. Study the model case : $X = \mathbf{R}^m$, $E = \mathbf{C}^{r_1}$, $F = \mathbf{C}^{r_2}$. Let us show that in this case P has the form (1.17).

If $s \in C^\infty(\mathbf{R}^m,\mathbf{C}^{r_1})$, if $x_0 \in \mathbf{R}^m$, we can write

$$s(x) = \sum_{|I|\le k}(x-x_0)^I \frac{1}{I!}\left(\frac{\partial}{\partial x}\right)^I s(x_0) + \sum_{|I|=k+1}(x-x_0)^I s_I(x), \tag{1.20}$$

where s_I are smooth functions on $\mathbf{R}^m$.

We claim that if $|I| = k+1$, then

$$P\Big((x-x_0)^I s_I(x)\Big)(x_0) = 0. \tag{1.21}$$

It is enough to show that, if $f_0,\dots,f_k$ are $(k+1)$-functions vanishing at x_0, then

$$P(f_0\dots f_k s_I)(x_0) = 0. \tag{1.22}$$

Indeed, we have

$$P(f_0\dots f_k s_I) = [P,f_0]f_1\dots f_k s_I + f_0 P(f_1\dots f_k s_I). \tag{1.23}$$

Since $f_0(x_0)=0$, by (1.23), we get

$$P(f_0\dots f_k s_I)(x_0) = [P,f_0](f_1\dots f_k s_I)(x_0). \tag{1.24}$$

Iterating (1.24) and using (1.19), we get (1.21).

By (1.20) and by (1.21), we get

$$(Ps)(x_0) = \sum_{|I|\le k} \big(P(x-x_0)^I\big)(x_0)\frac{1}{I!}\left(\frac{\partial}{\partial x}\right)^I s(x_0). \tag{1.25}$$

Put

$$a_I(x_0) = \frac{1}{I!}\big(P(x-x_0)^I\big)(x_0). \tag{1.26}$$

By (1.25) and (1.26), we have

$$P = \sum_{|I|\le k} a_I(x)\left(\frac{\partial}{\partial x}\right)^I. \tag{1.27}$$

Step 2. P is localisable. Let us go back to general manifolds. Let us show that if $\phi,\psi\in C^\infty(X,\mathbf{R})$ has disjoint support, then

$$\phi P\psi = 0. \tag{1.28}$$

Indeed, we can choose $\widetilde{\phi}\in C^\infty(X,\mathbf{R})$ such that $\widetilde{\phi}=1$ on the support of ϕ and that the supports of $\widetilde{\phi}$ and ψ are disjoint. Using $\phi\widetilde{\phi}=\phi$ and $\widetilde{\phi}\psi=0$, we have

$$\phi P\psi = \phi\widetilde{\phi}P\psi = \phi\big[\widetilde{\phi},P\big]\psi. \tag{1.29}$$

Iterating (1.29) and using (1.19), we get (1.28).

Step 3. Globalisation on manifold. Choose a locally finite open cover $X=\bigcup_{\alpha\in A}U_\alpha$ such that for each $\alpha\in A$, U_α is relatively compact and is contained in a coordinate chart, and that $E|_{U_\alpha}=U_\alpha\times\mathbf{C}^{r_1}$, $F|_{U_\alpha}=U_\alpha\times\mathbf{C}^{r_2}$.

Let $\{\varphi_\alpha\}_{\alpha\in A}$ be a partition of unity subordinate to the cover $\{U_\alpha\}_{\alpha\in A}$, and choose $\psi_\alpha\in C_c^\infty(U_\alpha,\mathbf{R})$ such that $\psi_\alpha=1$ on $\mathrm{Supp}\varphi_\alpha$. By (1.28), we have

$$P = \sum_{\alpha\in A}\phi_\alpha P = \sum_{\alpha\in A}\phi_\alpha P\psi_\alpha. \tag{1.30}$$

Choose $\widetilde{\phi}_\alpha,\widetilde{\psi}_\alpha\in C_c^\infty(U_\alpha,\mathbf{R})$ which equal to 1 on the support of ϕ_α,ψ_α respectively. By (1.30), we can write

$$P = \sum_{\alpha\in A}\widetilde{\varphi}_\alpha\big(\varphi_\alpha P\psi_\alpha\big)\widetilde{\psi}_\alpha. \tag{1.31}$$

We can view $\varphi_\alpha P\psi_\alpha$ as an operator on $\mathbf{R}^m$. We claim that $\varphi_\alpha P\psi_\alpha$ satisfies the commutator condition (1.19) on $\mathbf{R}^m$. Indeed, let f be a smooth function on $\mathbf{R}^m$. If $\widetilde{f}\in C_c^\infty(U_\alpha,\mathbf{R})$ such that $\widetilde{f}=f$ on a small open neighbourhood of $\mathrm{Supp}\,\psi_\alpha$, then

$$f\varphi_\alpha P\psi_\alpha - \varphi_\alpha P\psi_\alpha f = \widetilde{f}\varphi_\alpha P\psi_\alpha - \varphi_\alpha P\psi_\alpha\widetilde{f}. \tag{1.32}$$

We can rewrite (1.32) as

$$\big[f,\varphi_\alpha P\psi_\alpha\big] = \varphi_\alpha\big[\widetilde{f},P\big]\psi_\alpha. \tag{1.33}$$

Iterating (1.33) and using (1.19), we seee that $\varphi_\alpha P \psi_\alpha$ satisfies (1.19) on $\mathbf{R}^m$.

By Step 1, $\varphi_\alpha P \psi_\alpha$ has the form (1.27). By Example 1.19 and by (1.31), we see that P is a differential operator. □

The following corollaries are direct consequences of Theorem 1.20.

Corollary 1.21. *If P is a differential operator of order at most 0, or equivalently, for all $f \in C^\infty(X,\mathbf{R})$,*

$$[f,P] = 0, \tag{1.34}$$

then there exists $A \in C^\infty(X,\mathrm{Hom}(E,F))$ such that if $s \in C^\infty(X,E)$,

$$Ps(x) = A(x)\cdot s(x). \tag{1.35}$$

Corollary 1.22. *If $P : C^\infty(X,E) \to C^\infty(X,F)$ is an operator such that for all $f \in C^\infty(X,\mathbf{R})$, the operator*

$$[f,P] \tag{1.36}$$

is a differential operator of order at most $k-1$, then P is a differential operator of order at most k.

Corollary 1.23. *A differential operator $P : C^\infty(X,E) \to C^\infty(X,F)$ preservers the support, i.e., if $s \in C^\infty(X,E)$,*

$$\mathrm{Supp}(Ps) \subset \mathrm{Supp}(s). \tag{1.37}$$

In particular, P restricts to

$$P : C_c^\infty(X,E) \to C_c^\infty(X,F). \tag{1.38}$$

Proof. This is a direct consequence of (1.28). □

Definition 1.24. Let $\mathrm{Diff}_X(E,F)$ be the space of differential operators. If $k \in \mathbf{N}$, denote by $\mathrm{Diff}_X^k(E,F) \subset \mathrm{Diff}_X(E,F)$ the space of differential operators of order at most k.

Clearly, the composition of differential operators is still a differential operator. If $E = F$, then $\mathrm{Diff}_X(E,E)$ forms an algebra such that

$$\mathrm{Diff}_X^k(E,E)\cdot\mathrm{Diff}_X^\ell(E,E) \subset \mathrm{Diff}_X^{k+\ell}(E,E). \tag{1.39}$$

This means that $\mathrm{Diff}_X(E,E)$ is a filtered algebra.

1.4. **Principal symbols and elliptic operators.** Let $P \in \mathrm{Diff}_X^k(E,F)$ be a differential operator with local expression P_α given by (1.17).

We can define the total symbol of P_α for $x \in U_\alpha$ and $\xi \in T_x^*X$ by

$$\sigma_{\mathrm{tot}}(P_\alpha)(x,\xi) = \sum_{|I|\le k} a_I(x)(i\xi)^I. \tag{1.40}$$

Unfortunately, $\sigma_{\mathrm{tot}}(P_\alpha)$ is not globally well-defined on X.

Put

$$\sigma(P_\alpha)(x,\xi) = \sum_{|I|=k} a_I(x)(i\xi)^I. \tag{1.41}$$

We can view $\sigma(P_\alpha)$ as a function defined on U_α, with values in $\mathrm{Pol}^k(\mathbf{R}^m)\otimes_{\mathbf{R}}\mathrm{Hom}(\mathbf{C}^{r_1},\mathbf{C}^{r_2})$.