Proposition 1.25. *The family* $\{\sigma(P_\alpha)\}_{\alpha\in A}$ *defines a global section* $\sigma(P)$ *contained in* $C^\infty\big(X,\mathrm{Pol}^k(T^*X)\otimes_{\mathbf{R}}\mathrm{Hom}(E,F)\big)$.

Proof. We can check directly that $\sigma(P_\alpha)$ satisfies a condition similar to (1.12). Here, we give a more intrinsic proof.

Given $f\in C^\infty(X,\mathbf{R})$, if $t\in\mathbf{R}$, then $e^{-itf}Pe^{itf}$ is a differential operator of order at most k on $\mathbf{R}\times X$, which locally on $\mathbf{R}\times U_\alpha$, is given by

$$e^{-itf}P_\alpha e^{itf}=\sum_{|I|\le k}a_I\left(\frac{\partial}{\partial x}+itdf\right)^I. \tag{1.42}$$

Then the k-th commutator

$$\frac{1}{k!}\left[\frac{d}{dt},\left[\frac{d}{dt},\ldots,\left[\frac{d}{dt},e^{-itf}P_\alpha e^{itf}\right]\cdots\right]\right] \tag{1.43}$$

is still a differential operator, and is locally given by

$$\sum_{|I|=k}a_I(x)(i\,df)^I=\sigma(P_\alpha)(df). \tag{1.44}$$

This is equivalent to say that the operator (1.43) is of order 0, and is given by a global section. This is just the global section $\sigma(P)(df)$. □

Definition 1.26. The smooth section $\sigma(P)$ is called the principal symbol of P.

Proposition 1.27. *We have an exact sequence*

(1.45)

$$0\longrightarrow\mathrm{Diff}_X^{k-1}(E,F)\xrightarrow{i}\mathrm{Diff}_X^{k}(E,F)\xrightarrow{\sigma}C^\infty\big(X,\mathrm{Pol}^k(T^*X)\otimes_{\mathbf{R}}\mathrm{Hom}(E,F)\big)\longrightarrow 0,$$

where i *is the obvious inclusion.*

Proof. Clearly, i is injective. Moreover, we have $\sigma\circ i=0$ and if $\sigma(P)=0$, then P has order $\le k-1$. Hence, $\mathrm{im}\,i=\ker\sigma$. It remains to show σ is surjective.

Take $a\in C^\infty\big(X,\mathrm{Pol}^k(T^*X)\otimes_{\mathbf{R}}\mathrm{Hom}(E,F)\big)$. We use the notation in the Step 3 of the proof of Theorem 1.20. Then, a is represented by a family $\{a_\alpha\}_{\alpha\in A}$ satisfying (1.12). Choose P_α such that $\sigma(P_\alpha)=a_\alpha$. Put

$$P=\sum_{\alpha\in A}\varphi_\alpha P_\alpha\psi_\alpha. \tag{1.46}$$

Then, P is a differential operator of order at most k such that

$$\sigma(P)=\sum_\alpha\varphi_\alpha\sigma(P_\alpha)\psi_\alpha=\sum_\alpha\varphi_\alpha a_\alpha=a, \tag{1.47}$$

which proves the surjectivity of σ. □

Remark 1.28. Proposition 1.27 means that the graduation of the filtrated algebra $\mathrm{Diff}_X(E,E)$ is given by

$$C^\infty\left(X,\mathrm{Pol}^\bullet\left(T^*X\right)\otimes_{\mathbf{R}}\mathrm{Hom}(E,E)\right). \tag{1.48}$$

Definition 1.29. A differential operator $P:C^\infty(X,E)\to C^\infty(X,F)$ is called elliptic if $\sigma(P)(x,\xi)\in\mathrm{Hom}(E_x,F_x)$ is invertible for every $(x,\xi)\in\mathcal{X}^*$ with $\xi\neq 0$.

Remark 1.30. If $P:C^\infty(X,E)\to C^\infty(X,F)$ is elliptic, then E and F has the same rank.

1.5. **Differential forms.** The space of differential forms on X will be denoted by

$$\Omega^\bullet(X,\mathbf{R}) = C^\infty\left(X,\Lambda^\bullet\left(T^*X\right)\right). \tag{1.49}$$

Theorem 1.31. *1) There exists one and only one* $\mathbf{R}$*-linear operator*

$$d^X : \Omega^\bullet(X,\mathbf{R}) \longrightarrow \Omega^{\bullet+1}(X,\mathbf{R}) \tag{1.50}$$

such that

a) for every $f \in C^\infty(X,\mathbf{R})$, $d^X f \in \Omega^1(X,\mathbf{R})$ *is the usual differential;*

b) we have $\left(d^X\right)^2|_{\Omega^0(X,\mathbf{R})} = 0$ *;*

c) for $\alpha \in \Omega^k(X,\mathbf{R})$ *and* $\beta \in \Omega^\ell(X,\mathbf{R})$*, then*

$$d^X(\alpha\wedge\beta) = d^X\alpha\wedge\beta + (-1)^k\alpha\wedge d^X\beta. \tag{1.51}$$

2) The operator d^X *is a differential operator of order at most* 1 *with principal symbol* $i\xi$ *and satisfies* $\left(d^X\right)^2 = 0$ *on* $\Omega(X,\mathbf{R})$*.*

Proof. If $f \in C^\infty(X,\mathbf{R})$, by 1c), we have

$$\left[d^X, f\right] = d^X f. \tag{1.52}$$

By Theorem 1.20, d^X is a first order differential operator with principal symbol $i\xi$.

Unicity. We use the notation in the proof of Theorem 1.20. Namely, $X = \bigcup_{\alpha\in U} U_\alpha$ is covered by a locally finite family of open relatively compact coordinate charts, and $\{\phi_\alpha\}_{\alpha\in A}$ is a partition of unity subordinate to the cover $\{U_\alpha\}_{\alpha\in A}$, and $\psi_\alpha \in C_c^\infty(U_\alpha,\mathbf{R})$ is equal to 1 on some small open neighbourhood of $\operatorname{Supp}\phi_\alpha$.

Since d^X is a differential operator, as in (1.30), we have

$$d^X s = \sum_{\alpha\in A}\phi_\alpha d^X s = \sum_{\alpha\in A}\phi_\alpha d^X\psi_\alpha s. \tag{1.53}$$

In coordinates chart U_α, we can write[4]

$$\psi_\alpha s = \sum_k \sum_{1\le i_1<i_2<\cdots<i_k\le m} a_{i_1,\ldots i_k} dx^{i_1}\wedge\cdots\wedge dx^{i_k}, \tag{1.54}$$

where $a_{i_1,\ldots i_k} \in C_c^\infty(U_\alpha,\mathbf{R})$.

We can replace the function x^i by another function $\widetilde{x}^i \in C_c^\infty(U_\alpha,\mathbf{R})$ which coincides with x^i on the support of ψ_α, i.e.,

$$\psi_\alpha s = \sum a_{i_1,\ldots i_k} d^X\widetilde{x}^{i_1}\wedge\cdots\wedge d^X\widetilde{x}^{i_k}. \tag{1.55}$$

By (1.53) and (1.55), we see that $d^X s$ is determined by $d^X a_{i_1,\ldots i_k}$ and by $d^X d^X\widetilde{x}^i$. Those quantities are determined by the conditions 1a) and 1b).

Existence. On each U_α, $d^{U_\alpha} = \sum_i dx^i\frac{\partial}{\partial x^i}$ is an operator of form (1.17). It is elementary to see that d^{U_α} satisfies the conditions 1a)-1c) when X is replaced by U_α.

Appling the above unicity results to $U_\alpha\cap U_\beta$, we know that the operators d^{U_α} and d^{U_β} agree on $U_\alpha\cap U_\beta$. Equivalently, the family $\left(d^{U_\alpha}\right)_{\alpha\in A}$ satisfies the condition (1.18), and hence defines a differential operator d^X on X.

[4]Here, $x^i : \mathbf{R}^m \to \mathbf{R}$ is the i-th coordinate function and $d : \Omega^0(\mathbf{R}^m,\mathbf{R}) \to \Omega^1(\mathbf{R}^m,\mathbf{R})$ is the usual differential.

The three properties 1a)-1c) as well as the last statement in 2) are local. Therefore, d^X satisfies all these properties. □

Remark 1.32. The operator d^X is determined by its restrictions to $\Omega^0(X)$ et $\Omega^1(X)$.

The fact $\left(d^X\right)^2 = 0$ implies that $\left(\Omega(X,\mathbf{R}), d^X\right)$ is a complex. Moreover, d^X restricts to

$$d^X : \Omega_c^\bullet(X,\mathbf{R}) \longrightarrow \Omega_c^{\bullet+1}(X,\mathbf{R}), \tag{1.56}$$

so that $\left(\Omega_c(X,\mathbf{R}), d^X\right)$ forms another complex.

Definition 1.33. If $0 \leq k \leq m$, put

$$H^k(X,\mathbf{R}) = \frac{\ker d^X \cap \Omega^k(X,\mathbf{R})}{d^X\Omega^{k-1}(X,\mathbf{R})}, \qquad H_c^k(X,\mathbf{R}) = \frac{\ker d^X \cap \Omega_c^k(X,\mathbf{R})}{d^X\Omega_c^{k-1}(X,\mathbf{R})}. \tag{1.57}$$

The $\mathbf{Z}$-graded vector spaces $H(X,\mathbf{R}) = \bigoplus_{k=0}^m H^k(X,\mathbf{R})$ and $H_c(X,\mathbf{R}) = \bigoplus_{k=0}^m H_c^k(X,\mathbf{R})$ are called respectively the de Rham cohomology and the compactly supported de Rham cohomology of X.

Example 1.34. The following identities hold,

$$H^k\left(\mathbf{R}^m,\mathbf{R}\right) = \begin{cases} \mathbf{R}, & k=0, \\ 0, & k>0, \end{cases} \qquad H_c^k\left(\mathbf{R}^m,\mathbf{R}\right) = \begin{cases} 0, & k \leq m-1, \\ \mathbf{R}, & k=m. \end{cases} \tag{1.58}$$

1.6. **Orientation and integration.** Let V be a real vector space of dimension r. Two ordered basis $(e_1,\ldots,e_r)$ and $\left(e'_1,\ldots,e'_r\right)$ of V are said to define the same orientation if the determinant of the change of basis matrix is positive. This defines an equivalence relation on the set of ordered basis of V. The equivalence classes are called orientations of V. Clearly, V has exactly two orientations.

Let E be a real vector bundle of rank r. For any $x \in X$, we choose an orientation of E_x. The family of these orientations is called locally constant, if for any $x \in X$, there exists a small open neighborhood $U \subset X$ of x such that $E|_U = U \times \mathbf{R}^r$ and all the orientations of E_y, $y \in U$, induce on $\mathbf{R}^r$ the same orientation.

Definition 1.35. 1) A real vector bundle E is called *orientable* if there exists a locally constant family of orientations on E_x, $x \in X$.

2) An orientable real vector bundle equipped with a locally constant family of orientations is called an *oriented* vector bundle.

Proposition 1.36. *A real vector bundle E is orientable if and only if the family of transition functions can be chosen to have values in* $\mathrm{GL}_r^+(\mathbf{R})$.

Proof. The direction $\Longrightarrow$. We fix a locally constant family of orientations on E_x, $x \in X$. For any $x \in X$, we can choose a small open neighborhood $U_x \subset X$ of x such that

$$\psi_x^E : E_{|U_x} \simeq U_x \times \mathbf{R}^r. \tag{1.59}$$

Multiplying on the left by a matrix of determinant -1 if necessary, we can assume that for all $y \in U$, $\psi_x^E|_{E_y}$ sends the oriented basis of E_y to the oriented basis[5] of $\mathbf{R}^r$.

[5]Here, $\mathbf{R}^r$ is canonically oriented.

Therefore, if $y \in U_x \cap U_{x'}$, $\psi^E_{x'}\left(\psi^E_x\right)^{-1}(y,\cdot)$ sends the oriented basis of $\mathbf{R}^r$ to the oriented basis of $\mathbf{R}^r$. This means that the transition function $g^E_{x'x}$ takes values in $\mathrm{GL}^+_r(\mathbf{R})$.

The inverse direction $\Longleftarrow$. If the transition functions $\left(g^E_{\beta\alpha}\right)_{\alpha,\beta\in A}$ take values in $\mathrm{GL}^+_r(\mathbf{R})$, the canonical orientation of $\mathbf{R}^r$ induces an orientation on E_x for each $x \in U_\alpha$, which gives globally a locally constant family of orientations on E_x, $x \in X$. □

Definition 1.37. If TX is orientable (resp. oriented), then X is called orientable (resp. oriented).

Let X be an oriented manifold of dimension m. A coordinate chart

$$\psi_\alpha : U_\alpha \to V_\alpha \tag{1.60}$$

is called oriented, if ψ_α is orientation preserving, i.e., for all $x \in U_\alpha$, the differential $d\psi_\alpha(x)$ sends the oriented basis of T_xX to the oriented basis of $\mathbf{R}^m$.

If $s \in \Omega^m(X,\mathbf{R})$, we can write s in an oriented coordinate chart U_α as

$$s_\alpha = f_\alpha\, dx^1 \wedge \cdots \wedge dx^m. \tag{1.61}$$

We can identify s_α to a signed measure on U_α,

$$\mu_\alpha = f_\alpha\, dx^1 \cdots dx^m. \tag{1.62}$$

Since the Jacobi matrix associated to the change of coordinates has positive determinant, the family of signed measure $\{\mu_\alpha\}_{\alpha\in A}$ defines a signed measure μ_s on X.

Definition 1.38. If X is oriented, for $s \in \Omega_c(X,\mathbf{R})$, define

$$\int_X s = \int_X \mu_{s^{[m]}},$$

where $s^{[m]}$ denotes the degree-m component of s.

Theorem 1.39 (Stokes)**.** *If X is oriented, for $s \in \Omega_c(X,\mathbf{R})$, we have*

$$\int_X d^X s = 0. \tag{1.63}$$

Proof. Using the partition of unite, we have[6]

$$\int_X d^X s = \sum_{\alpha\in A} \int_X d^X(\varphi_\alpha s) = \sum_{\alpha\in A} \int_{\mathbf{R}^m} d(\varphi_\alpha s). \tag{1.64}$$

In this way, we can reduce the proof of (1.63) to the case of $X = \mathbf{R}^m$.

If $f \in C^\infty_c(\mathbf{R}^m)$, we have

$$\int_{\mathbf{R}^m} d\left(f\, dx^2 \cdots dx^m\right) = \int_{\mathbf{R}^m} \frac{\partial f}{\partial x^1} dx^1 dx^2 \cdots dx^m = \int_{\mathbf{R}^{m-1}} \left(\int_{\mathbf{R}} \frac{\partial f}{\partial x^1} dx^1\right) dx^2 \cdots dx^m = 0. \tag{1.65}$$

By (1.64) and (1.65), we get (1.63). □

[6] The sum is finite since s has compact support.

Proposition 1.40. *(Poincaré pairing) If X is oriented and of dimension m, the bilinear map*

$$(s,s') \in \Omega(X,\mathbf{R}) \times \Omega_c(X,\mathbf{R}) \to \int_X s \wedge s' \in \mathbf{R}. \tag{1.66}$$

induces a well-defined bilinear map

$$H^\bullet(X,\mathbf{R}) \times H_c^{m-\bullet}(X,\mathbf{R}) \longrightarrow \mathbf{R}, \tag{1.67}$$

Proof. Since s, s' are closed, for any $t \in \Omega(X,\mathbf{R})$ and $t' \in \Omega_c(X,\mathbf{R})$, we have

$$\left(s + d^X t\right) \wedge \left(s' + d^X t'\right) = s \wedge s' + d^X \left(t \wedge s' + (-1)^{\deg s} s \wedge t' + t \wedge d^X t'\right). \tag{1.68}$$

By Stokes' formula (1.63) and by (1.68), we get

$$\int_X \left(s + d^X t\right) \wedge \left(s' + d^X t'\right) = \int_X s \wedge s', \tag{1.69}$$

which implies (1.67) is well-defined. □

Remark 1.41. The oriented condition on X will be removed latter (see (4.16)).

2. Connection and curvature

In this section, we introduce the notation of connection on a vector bundle. We study the associated geometric object such as the parallel transport and the curvature form.

2.1. **Connections.**

Let X be a smooth manifold and let E be a complex smooth complex vector bundle on X.

Definition 2.1 (Connection)**.** A connection on E is a $\mathbf{C}$-linear operator

$$\nabla^E : C^\infty(X,E) \longrightarrow \Omega^1(X,E) \tag{2.1}$$

such that for every $f \in C^\infty(X,\mathbf{R})$ and $u \in C^\infty(X,E)$,

$$\nabla^E(fu) = d^X f \wedge u + f\nabla^E u. \tag{2.2}$$

Similarly, we can define the connection on a real smooth vector bundle.

Remark 2.2. We can rewrite (2.2) as

$$\left[\nabla^E, f\right] = d^X f. \tag{2.3}$$

By Theorem 1.20, the operator (2.1) is a connection if and only if it is a differential operator of order 1, with principal symbol $i\xi$.

Remark 2.3. A connection ∇^E is locally represented by the following family with property (1.18),

$$\nabla^E\big|_{U_\alpha} = d + \Gamma^E_\alpha, \text{ where } \Gamma^E_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(\mathbf{C}^r)). \tag{2.4}$$

Here, the matrix Γ^E_α is called the local connection one-form on U_α.

Remark 2.4. The de Rham operator defines a connection on the trivial bundle $X \times \mathbf{R}$.

Proposition 2.5. *Every smooth vector bundle admits a connection. If ∇^E and $\widetilde{\nabla}^E$ are two connections on E, then*

$$\nabla^E - \widetilde{\nabla}^E \in \Omega^1(X, \mathrm{End}(E)). \tag{2.5}$$

Proof. The exact sequence (2.6) specialise to

$$0 \to \mathrm{Diff}^0_X\left(E, T^*X \otimes_{\mathbf{R}} E\right) \xrightarrow{i} \mathrm{Diff}^1_X\left(E, T^*X \otimes_{\mathbf{R}} E\right) \xrightarrow{\sigma} C^\infty\left(X, \mathrm{Poly}^1\left(T^*X\right) \otimes_{\mathbf{R}} \mathrm{Hom}\left(E, T^*X \otimes_{\mathbf{R}} E\right)\right) \to 0. \tag{2.6}$$

Since σ is surjective, we know that there exists always a connection on E. Since $\ker\sigma = \mathrm{im}\ i$, we see that the difference of two connections is given by an element in

$$\mathrm{Diff}^0_X(E, T^*X \otimes E) = \Omega^1(X, \mathrm{End}(E)). \tag{2.7}$$

The proof of our proposition is complete. □

Remark 2.6. Proposition 2.5 means that the space of connections on E is affine and modelled on the vector space $\Omega^1(X, \mathrm{End}(E))$.

Example 2.7. If E is a smooth vector bundle equipped with a connection ∇^E, locally given by $d + \Gamma^E_\alpha$, we can define

a) the induced connection ∇^{E^*} on the dual bundle E^*, which is uniquely characterized by

$$d^X\langle u^*,u\rangle=\left\langle\nabla^{E^*}u^*,u\right\rangle+\left\langle u^*,\nabla^E u\right\rangle,\ \text{where } u^*\in C^\infty(X,E^*),u\in C^\infty(X,E). \tag{2.8}$$

and which is locally given by

$$\nabla^{E^*}|_{U_\alpha}=d-{}^t\Gamma^E_\alpha; \tag{2.9}$$

b) the induced connection $\nabla^{\overline{E}}$ on $\overline{E}$, which is uniquely characterized by

$$\nabla^{\overline{E}}\overline{u}=\overline{\nabla^E u},\ \text{where } u\in C^\infty(X,E), \tag{2.10}$$

which is locally given by

$$\nabla^{\overline{E}}|_{U_\alpha}=d+\overline{\Gamma}^E_\alpha; \tag{2.11}$$

c) the induced connection $\nabla^{E^{\otimes k}}$ on $E^{\otimes k}$, which is uniquely characterized by[7]

$$\nabla^{E^{\otimes k}}(u_1\otimes\ldots\otimes u_k)=\sum_{i=1}^k u_1\otimes\ldots\otimes\nabla^E u_i\otimes\ldots\otimes u_k,\ \text{where } u_i\in C^\infty(X,E), \tag{2.12}$$

which is locally given by

$$\nabla^{E^{\otimes k}}|_{U_\alpha}=d+\sum_{i=1}^k 1\otimes\ldots\otimes\Gamma^E_\alpha\otimes\ldots\otimes 1; \tag{2.13}$$

d) the induced connections $\nabla^{S^kE},\nabla^{\Lambda^kE}$ on S^kE,Λ^kE obtained by restricting $\nabla^{E^{\otimes k}}$ to S^kE,Λ^kE.

If F is another vector bundle equipped with a connection ∇^F, locally given by $d+\Gamma^F_\alpha$, we can define

a) the direct sum connection $\nabla^{E\oplus F}$ on $E\oplus F$, which is uniquely characterized by

$$\nabla^{E\oplus F}(u_1+u_2)=\nabla^E u_1+\nabla^F u_2,\ \text{where } u_1\in C^\infty(X,E),u_2\in C^\infty(X,F), \tag{2.14}$$

and which is locally given by

$$\nabla^{E\oplus F}|_{U_\alpha}=d+\begin{pmatrix}\Gamma^E_\alpha & 0\\ 0 & \Gamma^F_\alpha\end{pmatrix}; \tag{2.15}$$

b) the tensor product connection $\nabla^{E\otimes F}$ on $E\otimes F$, which is uniquely characterized by

$$\nabla^{E\otimes F}(u_1\otimes u_2)=\left(\nabla^E u_1\right)\otimes u_2+u_1\otimes\nabla^F u_2,\ \text{where } u_1\in C^\infty(X,E),u_2\in C^\infty(X,F), \tag{2.16}$$

and which is locally given by

$$\nabla^{E\otimes F}|_{U_\alpha}=d+\Gamma^E_\alpha\otimes 1+1\otimes\Gamma^F_\alpha. \tag{2.17}$$

c) the induced connection $\nabla^{\mathrm{Hom}(E,F)}$ on the bundle $\mathrm{Hom}(E,F)=E^*\otimes F$ obtained from the tensor product connection of ∇^{E^*} and ∇^F.

[7]This formula can not be considered as a definition, since nothing guarantees that $\nabla^{E^{\otimes k}}$ acts to a general section is well-defined. One way to show the existence of $\nabla^{E^{\otimes k}}$ is to use the local expression (2.13) satisfying the condition (1.18).

If $f : Y \to X$ is a smooth map, we can define the pullback connection ∇^{f^*E}, which is uniquely characterized by[8]

$$\left\{\nabla_v^{f^*E}(f^*u)\right\}(y) = \left\{\nabla^E_{df_y(v)}u\right\}(f(y)), \text{ where } u \in C^\infty(X,E), v \in T_yY, \tag{2.18}$$

which is locally given by

$$\nabla^{f^*E}|_{f^{-1}(U_\alpha)} = d + f^*\Gamma^E_\alpha. \tag{2.19}$$

Remark 2.8. If $Y \subset X$ is a submanifold, one cannot in general define the restriction of an arbitrary differential operator to Y. In contrast, a connection admits a well-defined restriction through the pullback connection.

Let h^E be a Hermitian metric on E which is locally defined by self-adjoint positive-definite matrix-valued functions $\{h^E_\alpha\}_{\alpha\in A}$ (see (1.14)).

Example 2.9 (Adjoint connection)**.** The adjoint connection $\nabla^{E\prime}$ of ∇^E with respect to h^E is uniquely characterized by

$$d^X\langle u_1,u_2\rangle_{h^E} = \left\langle \nabla^{E\prime}u_1,u_2\right\rangle_{h^E} + \left\langle u_1,\nabla^E u_2\right\rangle_{h^E} \text{ where } u_1,u_2 \in C^\infty(X,E), \tag{2.20}$$

which is locally given by

$$\nabla^{E\prime}_{|U_\alpha} = d + \left(h^E_\alpha\right)^{-1}\left(dh^E_\alpha\right) - \left(h^E_\alpha\right)^{-1}{}^t\overline{\Gamma}_\alpha h^E_\alpha. \tag{2.21}$$

Remark 2.10. If we consider h^E as a section in $C^\infty\left(X,\mathrm{Hom}\left(E,\overline{E}^*\right)\right)$, then

$$\nabla^{E\prime} = \left(h^E\right)^{-1}\circ\nabla^{\overline{E}^*}\circ h^E. \tag{2.22}$$

Remark 2.11. Put

$$\omega^E = \left(h^E\right)^{-1}\left(\nabla^{\mathrm{Hom}\left(E,\overline{E}^*\right)}h^E\right). \tag{2.23}$$

Then,

$$d^X\langle s_1,s_2\rangle_{h^E} = \left\langle \omega^E s_1,s_2\right\rangle_{h^E} + \left\langle \nabla^E s_1,s_2\right\rangle_{h^E} + \left\langle s_1,\nabla^E s_2\right\rangle_{h^E}. \tag{2.24}$$

By (2.24), we get

$$\omega^E \in \Omega^1\left(X,\mathrm{End}^{\mathrm{auto}}(E)\right), \tag{2.25}$$

and

$$\nabla^{E\prime} = \nabla^E + \omega^E. \tag{2.26}$$

Definition 2.12 (Metric connection)**.** A connection ∇^E is called metric (or Hermitian) with respect to h^E if

$$\nabla^{E\prime} = \nabla^E. \tag{2.27}$$

Equivalently, for all $u_1,u_2 \in C^\infty(X,E)$,

$$d^X\langle u_1,u_2\rangle_{h^E} = \left\langle \nabla^E u_1,u_2\right\rangle_{h^E} + \left\langle u_1,\nabla^E u_2\right\rangle_{h^E}. \tag{2.28}$$

[8]Footnote 7 applies here.