If $f : Y \to X$ is a smooth map, we can define the pullback connection ∇^{f^*E}, which is uniquely characterized by[8]

$$\left\{\nabla_v^{f^*E}(f^*u)\right\}(y) = \left\{\nabla^E_{df_y(v)}u\right\}(f(y)), \text{ where } u \in C^\infty(X,E), v \in T_yY, \tag{2.18}$$

which is locally given by

$$\nabla^{f^*E}|_{f^{-1}(U_\alpha)} = d + f^*\Gamma^E_\alpha. \tag{2.19}$$

Remark 2.8. If $Y \subset X$ is a submanifold, one cannot in general define the restriction of an arbitrary differential operator to Y. In contrast, a connection admits a well-defined restriction through the pullback connection.

Let h^E be a Hermitian metric on E which is locally defined by self-adjoint positive-definite matrix-valued functions $\{h^E_\alpha\}_{\alpha\in A}$ (see (1.14)).

Example 2.9 (Adjoint connection)**.** The adjoint connection $\nabla^{E\prime}$ of ∇^E with respect to h^E is uniquely characterized by

$$d^X\langle u_1,u_2\rangle_{h^E} = \left\langle \nabla^{E\prime}u_1,u_2\right\rangle_{h^E} + \left\langle u_1,\nabla^E u_2\right\rangle_{h^E} \text{ where } u_1,u_2 \in C^\infty(X,E), \tag{2.20}$$

which is locally given by

$$\nabla^{E\prime}_{|U_\alpha} = d + \left(h^E_\alpha\right)^{-1}\left(dh^E_\alpha\right) - \left(h^E_\alpha\right)^{-1}{}^t\overline{\Gamma}_\alpha h^E_\alpha. \tag{2.21}$$

We consider h^E as a section in $C^\infty\left(X,\mathrm{Hom}\left(E,\overline{E}^*\right)\right)$. Put

$$\omega^E = \left(h^E\right)^{-1}\left(\nabla^{\mathrm{Hom}\left(E,\overline{E}^*\right)}h^E\right). \tag{2.22}$$

Proposition 2.10. *The following identity holds,*

$$\nabla^{E\prime} = \left(h^E\right)^{-1}\circ\nabla^{\overline{E}^*}\circ h^E. \tag{2.23}$$

Moreover, ω^E *lies in* $\Omega^1\left(X,\mathrm{End}^{\mathrm{auto}}(E)\right)$ *such that*

$$\nabla^{E\prime} = \nabla^E + \omega^E. \tag{2.24}$$

Proof. We can rewrite (2.20) as

$$d^X\left\langle h^E u_1,\overline{u}_2\right\rangle = \left\langle h^E\nabla^{E\prime}u_1,\overline{u}_1\right\rangle + \left\langle h^E u_1,\overline{\nabla^E u_2}\right\rangle, \tag{2.25}$$

where $\langle\cdot,\cdot\rangle$ is the natural pairing between $\overline{E}^*$ and $\overline{E}$.

By (2.8) and (2.10), we have

$$d^X\left\langle h^E u_1,\overline{u}_2\right\rangle = \left\langle\nabla^{\overline{E}^*}\left(h^E u_1\right),\overline{u}_2\right\rangle + \left\langle h^E u_1,\nabla^{\overline{E}}\overline{u}_2\right\rangle = \left\langle\nabla^{\overline{E}^*}\left(h^E u_1\right),\overline{u}_2\right\rangle + \left\langle h^E u_1,\overline{\nabla^E u_2}\right\rangle. \tag{2.26}$$

By (2.25) and (2.26), we get (2.23).

Using

$$\nabla^{\overline{E}^*}\left(h^E u_1\right) = \left(\nabla^{\mathrm{Hom}\left(E,\overline{E}^*\right)}h^E\right)u_1 + h^E\left(\nabla^E u_1\right), \tag{2.27}$$

and (2.23), we get (2.24).

[8]Footnote 7 applies here.

By (2.24), we can rewrite (2.20) as

$$d^X \langle u_1, u_2 \rangle_{h^E} = \left\langle \omega^E u_1, u_2 \right\rangle_{h^E} + \left\langle \nabla^E u_1, u_2 \right\rangle_{h^E} + \left\langle u_1, \nabla^E u_2 \right\rangle_{h^E}. \tag{2.28}$$

Using $\overline{\langle u_1, u_2 \rangle_{h^E}} = \langle u_2, u_1 \rangle_{h^E}$, and (2.28), we can deduce that

$$\left\langle \omega^E u_1, u_2 \right\rangle_{h^E} = \left\langle u_1, \omega^E u_2 \right\rangle_{h^E}, \tag{2.29}$$

which implies that $\omega^E \in \Omega^1\left(X, \mathrm{End}^{\mathrm{auto}}(E)\right)$. □

Definition 2.11 (Metric connection)**.** A connection ∇^E is called metric (or Hermitian) with respect to h^E if

$$\nabla^{E\prime} = \nabla^E. \tag{2.30}$$

Equivalently, for all $u_1, u_2 \in C^\infty(X, E)$,

$$d^X \langle u_1, u_2 \rangle_{h^E} = \left\langle \nabla^E u_1, u_2 \right\rangle_{h^E} + \left\langle u_1, \nabla^E u_2 \right\rangle_{h^E}. \tag{2.31}$$

Remark 2.12. If ∇^E and ∇^F are metric, then all the connections constructed in Example 2.7 are metric.

Proposition 2.13. *Every Hermitian vector bundle $\left(E, h^E\right)$ admits a metric connection. If ∇^E and $\nabla^{\tilde{E}}$ are two metric connections on $\left(E, h^E\right)$, then*

$$\nabla^E - \nabla^{\tilde{E}} \in \Omega^1\left(X, \mathrm{End}^{\mathrm{anti}}(E)\right), \tag{2.32}$$

where $\mathrm{End}^{\mathrm{anti}}(E)$ is the anti-self-adjoint endomorphisms bundle of E with respect to h^E.

Proof. By (2.28),

$$\nabla^{E,u} = \nabla^E + \frac{1}{2}\omega^E \tag{2.33}$$

is a metric connection. The second part of our proposition is an immediate consequence of (2.31). □

2.2. **Parallel transport.** We call a smooth map[9]

$$x_\cdot : s \in [0,1] \longrightarrow x_s \in X \tag{2.34}$$

a smooth curve on X.

A section $u \in C^\infty\left([0,1], x_\cdot^* E\right)$ will be called a section of E along the smooth curve $x_\cdot$. Intuitively, this is a family $\left(u_s \in E_{x_s}\right)_{s\in[0,1]}$ of vectors which various smoothly on $s \in [0,1]$.

Example 2.14. If $s_0 \in [0,1]$, the small curve $(x_s)_{s\in]-\delta+s_s, \delta+s_0[}$ passing through x_{s_0} defines a tangent vector denoted by $\dot{x}_{s_0} \in T_{x_{s_0}} X$. In this way, we obtain a vector field along $x_\cdot$, which will be denoted by $\dot{x}_\cdot$. Equivalently, we have

$$x_{\cdot *}\left(\frac{d}{ds}\right) = \dot{x}_\cdot. \tag{2.35}$$

[9] A smooth map defined on $[0,1]$ by definition means restriction of smooth map on some large open subset $]-\epsilon, 1+\epsilon[$.

Definition 2.15. If $u_\cdot \in C^\infty\left([0,1], x_\cdot^* E\right)$ is a section of E along $x_\cdot$, put

$$\dot{u}_\cdot = \nabla^{x_\cdot^* E}_{\frac{d}{ds}} u_\cdot. \tag{2.36}$$

Then, $\dot{u}_\cdot$ is still a section of E along $x_\cdot$. Sometimes by abuse of notation, we denote

$$\dot{u}_\cdot = \nabla^E_{\dot{x}_\cdot} u_\cdot. \tag{2.37}$$

Note that in the above expression $\nabla^E u_\cdot$ does not make sense.

Remark 2.16. If the curve $x_\cdot$ is contained in a local coordinate chart U_α, then

$$\dot{u}_s = \left(\frac{d}{ds} + \Gamma^E_\alpha(\dot{x}_s)\right) u_s. \tag{2.38}$$

Definition 2.17 (Parallel section)**.** A section $u_\cdot \in C^\infty([0,1], x_\cdot^* E)$ of E along the smooth curve $x_\cdot$ is called parallel with respect to ∇^E if

$$\dot{u}_\cdot = 0. \tag{2.39}$$

Proposition 2.18. *Given $u_0 \in E_{x_0}$, there exists a unique parallel section $u_\cdot$ with respect to ∇^E along $x_\cdot$ such that*

$$u_s|_{s=0} = u_0. \tag{2.40}$$

Proof. We can assume that the curve $x_\cdot$ is contained in a local coordinate chart U_α. Otherwise, we can decompose the curve $x_\cdot$ into a finite number of pieces, each of which is contained in a local coordinate chart.

Then, (2.39) and (2.40) become a linear ordinary differential equation,

$$\frac{d}{ds} u_s + \Gamma^E_{x_s}(\dot{x}_s) u_s = 0, \qquad u_s|_{s=0} = u_0. \tag{2.41}$$

The existence and uniqueness of the solution to (2.41) follow from the standard results of linear ordinary differential equations. □

Definition 2.19 (Parallel transport)**.** The parallel transport with respect to ∇^E along the smooth curve $x_\cdot$ is the linear map

$$\tau_1^0 : E_{x_0} \longrightarrow E_{x_1} \tag{2.42}$$

defined by

$$\tau_1^0 u_0 = u_1, \tag{2.43}$$

where $u_\cdot$ is a parallel section with respect to ∇^E along $x_\cdot$.

If $0 \le s, s' \le 1$, we denote by $\tau_{s'}^s$ the parallel transport along the smooth curve $x_\cdot$ from E_{x_s} to $E_{x_{s'}}$.

Remark 2.20. If $v_s \in C^\infty\left([0,1], E_{x_0}\right)$, then $u_\cdot = \left(\tau_s^0 v_s\right)_{s\in[0,1]}$ is a section along $x_\cdot$ such that

$$\dot{u}_s = \tau_s^0 \left(\frac{d}{ds} v_s\right). \tag{2.44}$$

Equivalently, we have

$$\nabla_{\dot{x}_s} \tau_s^0 = \tau_s^0 \frac{d}{ds}. \tag{2.45}$$

Intuitively, the parallel transport trivializes the covariant derivative along the curve $x_\cdot$.

Remark 2.21. If $x_\cdot$ is only C^1 or even piecewisely C^1, the parallel transport is still well-defined. However, if $x_\cdot$ is only continuous, the parallel transport is not well-defined. A fundamental result of Itô shows that parallel transport can be defined along almost every continuous path with respect to the Wiener measure. This is how stochastic analysis comes into play in differential geometry.

Remark 2.22. The parallel transport can be used to define a "canonical" trivialization of E on some local coordinate chart. More precisely, if $x_0 \in X$, choose a local coordinate near x_0, which identifies x_0 to the origin $0 \in \mathbf{R}^m$. Take a small open ball $B(\epsilon_0)$ of radius ϵ_0 in $\mathbf{R}^m$ and identify it with a small open neighbourhood of x_0.

If $Y \in B(\epsilon_0)$, then $(sY)_{0\leq s\leq 1}$ defines a smooth curve joining 0 and Y. Denote by $\tau_Y^0 : E_{x_0} \to E_Y$ the parallel transport along this curve. Then,

$$(Y,v) \in B(\epsilon) \times E_{x_0} \to \left(Y, \tau_Y^0 v\right) \in E_{|B(\epsilon)} \tag{2.46}$$

induces a trivialization of $E|_{B(\epsilon)}$. This trivialization plays an important and sometimes implicit role almost everywhere in differential geometry.

Proposition 2.23. *If ∇^E is a metric connection with respect to h^E, then the parallel transport $\tau_1^0 : E_{x_0} \to E_{x_1}$ along the smooth curve $x_\cdot$ is an isometry with respect to h^E.*

Proof. Let $u_\cdot$ and $v_\cdot$ be two parallel sections of E along $x_\cdot$. By (2.31), we have

$$\frac{d}{ds}\langle u_s, v_s\rangle_{h^E} = \langle \dot{u}_s, v_s\rangle_{h^E} + \langle u_s, \dot{v}_s\rangle_{h^E} = 0. \tag{2.47}$$

Therefore,

$$\langle u_1, v_1\rangle_{h^E} = \langle u_0, v_0\rangle_{h^E}, \tag{2.48}$$

or equivalently,

$$\left\langle \tau_1^0 u_0, \tau_1^0 v_0\right\rangle_{h^E} = \langle u_0, v_0\rangle_{h^E}. \tag{2.49}$$

This means τ_1^0 is an isometry. □

2.3. **Curvature.** The connection (2.1) extends to an operator

$$\nabla^E : \Omega^\bullet(X,E) \longrightarrow \Omega^{\bullet+1}(X,E) \tag{2.50}$$

such that[10]

$$\nabla^E(\alpha \wedge s) = d^X\alpha \wedge s + (-1)^{\deg\alpha}\alpha \wedge \nabla^E s. \tag{2.51}$$

The local expression of (2.50) is given by

$$\nabla^E|_{U_\alpha} = d + \Gamma_\alpha : \Omega^\bullet\left(U_\alpha, \mathbf{C}^r\right) \longrightarrow \Omega^{\bullet+1}\left(U_\alpha, \mathbf{C}^r\right). \tag{2.52}$$

Consider the composition

$$\left(\nabla^E\right)^2 : \Omega^\bullet(X,E) \longrightarrow \Omega^{\bullet+2}(X,E). \tag{2.53}$$

[10]Footnote 7 applies here. The local expression (2.52) and as well as the well-definedness of d implies $d + \Gamma_\alpha^E$ satisfies (1.18) in this extended case.

Remark 2.24. A priori, $\left(\nabla^E\right)^2$ is a differential operator of order at most 2. It is principal symbol is given by $i\xi \wedge i\xi = 0$. Therefore, $\left(\nabla^E\right)^2$ is a differential operator of order at most 1. The following proposition shows that $\left(\nabla^E\right)^2$ is in fact a differential operator of order 0.

Proposition 2.25. *The differential operator* $\left(\nabla^E\right)^2$ *is of order* 0.

Proof. If $f \in C^\infty(X,\mathbf{R})$ and $s \in \Omega(X,E)$, we have

$$\begin{aligned} \nabla^E\nabla^E(fs) &= \nabla^E\left(d^X f \wedge s + f\nabla^E s\right) \\ &= -d^X f \wedge \nabla^E s + d^X f \wedge \nabla^E s + f\nabla^E\nabla^E s \\ &= f\nabla^E\nabla^E s. \end{aligned} \tag{2.54}$$

Therefore,

$$\left[\left(\nabla^E\right)^2, f\right] = 0. \tag{2.55}$$

In particular, $\left(\nabla^E\right)^2$ is of order 0. □

Remark 2.26. Using (2.52), the local expression of $\left(\nabla^E\right)^2$ is given by

$$\left(\nabla^E\right)^2|_{U_\alpha} = \left(d + \Gamma^E_\alpha\right)^2 = d\Gamma^E_\alpha + \Gamma^E_\alpha \wedge \Gamma^E_\alpha. \tag{2.56}$$

By Proposition 2.25, $\left(\nabla^E\right)^2_{|C^\infty(X,E)}$ can be identified with a section in $\Omega^2(X,\mathrm{End}(E))$. By (2.56), if $k \geq 1$, $\left(\nabla^E\right)^2_{|\Omega^k(X,E)}$ is given by the same section.

Definition 2.27 (Curvature)**.** The above section denoted by $R^E \in \Omega^2(X,\mathrm{End}(E))$ is called the curvature of the connection ∇^E.

Proposition 2.28. *If* $v,w \in C^\infty(X,TX)$*, we have*

$$R^E(v,w) = \nabla^E_v\nabla^E_w - \nabla^E_w\nabla^E_v - \nabla^E_{[v,w]}. \tag{2.57}$$

Proof. Since both sides of (2.57) are different operators, it is enough to verify (2.57) in a local chart U_α.

By (1.79), we have

$$\begin{aligned} d\Gamma^E_\alpha(v,w) &= v\left(\Gamma^E_\alpha(w)\right) - w\left(\Gamma^E_\alpha(v)\right) - \Gamma^E_\alpha([v,w]), \\ \Gamma^E_\alpha \wedge \Gamma^E_\alpha(v,w) &= \Gamma^E_\alpha(v)\Gamma^E_\alpha(w) - \Gamma^E_\alpha(w)\Gamma^E_\alpha(v). \end{aligned} \tag{2.58}$$

By (2.56) and (2.58), we get

$$R^E(v,w) = v\left(\Gamma^E_\alpha(w)\right) - w\left(\Gamma^E_\alpha(v)\right) - \Gamma^E_\alpha([v,w]) + \left[\Gamma^E_\alpha(v), \Gamma^E_\alpha(w)\right]. \tag{2.59}$$

On the other hand,

$$\nabla^E_v\nabla^E_w - \nabla^E_w\nabla^E_v - \nabla^E_{[v,w]} = \left[v + \Gamma^E(v), w + \Gamma^E(w)\right] - [v,w] - \Gamma^E([v,w]), \tag{2.60}$$

which coincides with the right-hand side of (2.59). □

Proposition 2.29 (Bianchi identity)**.** *The curvature satisfies*

$$\left[\nabla^E, R^E\right] = 0. \tag{2.61}$$

Proof. Our proposition is a trivial[11] consequence of the definition $R^E = \left(\nabla^E\right)^2$. □

Example 2.30. If E is a smooth vector bundle with connection ∇^E and curvature R^E, then

a) the curvature of the induced connection ∇^{E^*} is given by $R^{E^*} = -{}^tR^E$;
b) the curvature of the induced connection $\nabla^{\overline{E}}$ is given by $R^{\overline{E}} = \overline{R^E}$;
c) the curvature of the induced connection $\nabla^{E^{\otimes k}}$ is given by $R^{E^{\otimes k}} = R^E \otimes 1 \otimes \ldots + 1 \otimes R^E \otimes \ldots + \ldots$

If F is another smooth vector bundle with connection ∇^F and curvature R^F, then

a) the curvature of the induced connection $\nabla^{E\oplus F}$ is given by $R^{E\oplus F} = R^E \oplus R^F$;
b) the curvature of the induced connection $\nabla^{E\otimes F}$ is given by $R^{E\otimes F} = R^E \otimes 1 + 1 \otimes R^F$;
c) the curvature of the induced connection $\nabla^{\mathrm{Hom}(E,F)}$ is given by $R^{\mathrm{Hom}(E,F)} = -{}^tR^E \otimes 1 + 1 \otimes R^F$.

If $f : Y \to X$ is a smooth map, then the curvature of the pullback connection ∇^{f^*E} is given by $R^{f^*E} = f^*R^E$.

Proposition 2.31. *If ∇^E is a metric connection on $\left(E, h^E\right)$, then*

$$R^E \in \Omega^2(X, \mathrm{End}^{\mathrm{anti}}(E)). \tag{2.62}$$

Proof. By (2.31), we have

$$0 = d^X d^X \langle s_1, s_2\rangle_{h^E} = d^X\left(\left\langle \nabla^E s_1, s_2\right\rangle_{h^E} + \left\langle s_1, \nabla^E s_2\right\rangle_{h^E}\right). \tag{2.63}$$

We claim that

$$d^X\left\langle \nabla^E s_1, s_2\right\rangle_{h^E} = \left\langle R^E s_1, s_2\right\rangle_{h^E} - \left\langle \nabla^E s_1, \nabla^E s_2\right\rangle_{h^E}. \tag{2.64}$$

Indeed, $\langle \nabla^E s_1, s_2\rangle_{h^E}$ can be written as a linear combination of $\alpha\langle \nabla^E_v s_1, s_2\rangle_{h^E}$ where $\alpha \in \Omega^1(X, \mathbf{R})$ and $v \in C^\infty(X, TX)$. Moreover, we have

(2.65)
$$\begin{aligned} d^X\left(\alpha\left\langle \nabla^E_v s_1, s_2\right\rangle_{h^E}\right) &= d^X\alpha\left\langle \nabla^E_v s_1, s_2\right\rangle_{h^E} - \alpha\left\langle \nabla^E\nabla^E_v s_1, s_2\right\rangle_{h^E} - \alpha\left\langle \nabla^E_v s_1, \nabla^E s_2\right\rangle_{h^E} \\ &= \left\langle \nabla^E \alpha \nabla^E_v s_1, s_2\right\rangle_{h^E} - \alpha\left\langle \nabla^E_v s_1, \nabla^E s_2\right\rangle_{h^E}, \end{aligned}$$

which implies our claim (2.64).

Similarly, we have

$$d^X\left\langle s_1, \nabla^E s_2\right\rangle_{h^E} = \left\langle \nabla^E s_1, \nabla^E s_2\right\rangle_{h^E} + \left\langle s_1, R^E s_2\right\rangle_{h^E}. \tag{2.66}$$

By (2.63), (2.64), and (2.66), we get

$$\left\langle R^E s_1, s_2\right\rangle_{h^E} + \left\langle s_1, R^E s_2\right\rangle_{h^E} = 0, \tag{2.67}$$

which means R^E is a two form with values in the skew-Hermitian endomorphisms of E. □

[11] This might be the most advantage of our definition of curvatures.