Remark 2.32. If ∇^E is metric, we have

$$\nabla^{\mathrm{Hom}\left(E,\overline{E}^*\right)}h^E = 0. \tag{2.68}$$

By (2.68), we get

$$R^{\mathrm{Hom}\left(E,\overline{E}^*\right)}h^E = 0. \tag{2.69}$$

Equivalently, we have

$$R^{\overline{E}^*}\circ h^E - h^E\circ R^E = 0, \tag{2.70}$$

Using $R^{\overline{E}^*} = -{}^tR^{\overline{E}}$, (2.70) is equivalent to

$$\left\langle \cdot, R^E\cdot\right\rangle_{h^E} + \left\langle R^E\cdot,\cdot\right\rangle_{h^E} = 0, \tag{2.71}$$

which gives another proof of Proposition 2.31.

2.4. **Parallel transport and curvature.** Let us study how the curvature controls the parallel transport.

Let $x_\cdot = (x_s)_{s\in[0,1]}$ be a smooth curve on X. We say a smooth map

$$(s,\epsilon)\in[0,1]\times]-\delta,\delta[\longrightarrow x_{s,\epsilon}\in X \tag{2.72}$$

is a variation of $x_\cdot$ if

$$x_{\cdot,0} = x_\cdot. \tag{2.73}$$

If $s_0\in[0,1]$, let $J_{s_0}\in T_{x_{s_0}}X$ be the vector field defined by the small curve $\left(x_{s_0,\epsilon}\right)_{\epsilon\in]-\delta,\delta[}$ passing through x_{s_0}. Then, $J_\cdot$ is a vector field along $x_\cdot$. Equivalently,

$$J_\cdot = x_{\cdot,\cdot *}\left(\frac{d}{d\epsilon}\right)\Big|_{\epsilon=0}. \tag{2.74}$$

Let $x,y\in X$. Assume now that $x_\cdot$ is a smooth curve connecting x and y, and that $x_{\cdot,\epsilon}$ is a variation of $x_\cdot$ with fixed endpoints x and y. Then,

$$J_0 = 0, \qquad J_1 = 0. \tag{2.75}$$

Let

$$\tau_1^0(\epsilon): E_x\to E_y \tag{2.76}$$

be the parallel transport along the smooth curve $\left(x_{s,\epsilon}\right)_{s\in[0,1]}$.

Proposition 2.33. *The following identity holds,*

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\tau_1^0(\epsilon) = \int_0^1 \tau_1^s R^E(\dot{x}_s, J_s)\tau_s^0 ds. \tag{2.77}$$

Proof. We pull back $\left(E,\nabla^E\right)$ to a vector bundle with connection defined on $[0,1]\times]-\delta,\delta[$. We work on $[0,1]\times]-\delta,\delta[$ in our proof.

Denote $\frac{D}{Ds}$ and $\frac{D}{D\epsilon}$ the corresponding covariant derivatives on $x_{\cdot,\cdot}^*E$. Since $\left[\frac{\partial}{\partial s},\frac{\partial}{\partial\epsilon}\right] = 0$, by (2.57), we have

$$\frac{D}{Ds}\frac{D}{D\epsilon} - \frac{D}{D\epsilon}\frac{D}{Ds} = R^E\left(\frac{\partial}{\partial s},\frac{\partial}{\partial\epsilon}\right). \tag{2.78}$$

If $u \in E_{x_0}$, then for each ϵ, $\tau_\cdot^0(\epsilon)u$ is a smooth parallel section of E along $x_{\cdot,\epsilon}$. By (2.78), we have

$$\frac{D}{Ds}\frac{D}{D\epsilon}\tau_s^0(\epsilon)u = R^E\left(\frac{\partial}{\partial s},\frac{\partial}{\partial \epsilon}\right)\tau_s^0(\epsilon)u. \tag{2.79}$$

Restricting (2.79) to $\epsilon = 0$, we get

$$\frac{D}{Ds}\left\{\frac{D}{D\epsilon}\tau_s^0(\epsilon)u\right\}_{\epsilon=0} = R^E(\dot{x}_s, J_s)\tau_s^0 u. \tag{2.80}$$

By (2.45), we can write (2.80) as

$$\tau_s^0\frac{d}{ds}\tau_0^s\left\{\frac{D}{D\epsilon}\tau_s^0(\epsilon)u\right\}_{\epsilon=0} = R^E(\dot{x}_s, J_s)\tau_s^0 u. \tag{2.81}$$

By (2.81), we get

$$\tau_0^1\left\{\frac{D}{D\epsilon}\tau_1^0(\epsilon)u\right\}_{\epsilon=0} = \int_0^1 \tau_0^s R^E(\dot{x}_s, J_s)\tau_s^0 u\, ds. \tag{2.82}$$

When $s = 1$, we have $\Gamma^E_{x_1}(J_1) = 0$, so that

$$\frac{D}{D\epsilon} = \frac{d}{d\epsilon}. \tag{2.83}$$

By (2.82) and (2.83), we get (2.77). □

Corollary 2.34. *If $R^E = 0$, then the parallel transport τ_1^0 along a smooth curve depends only on the its homotopy class with fixed endpoints.*

2.5. **Connection on TX and its torsion.** Let ∇^{TX} be a connection on the tangent bundle TX. Let us introduce the torsion of ∇^{TX}. The term torsion in mathematics generally refers to the deviation from some canonical object. The torsion of ∇^{TX} measures, in a certain sense, its deviation from d^X.

To compare with d^X, we consider the composition

$$\wedge \circ \nabla^{T^*X} : \Omega^1(X, \mathbf{R}) \to \Omega^2(X, \mathbf{R}) \tag{2.84}$$

of the induced connection $\nabla^{T^*X} : C^\infty(X, T^*X) \to C^\infty(X, T^*X \otimes T^*X)$ and the wedge product $\wedge : C^\infty(X, T^*X \otimes T^*X) \to C^\infty\left(X, \Lambda^2(T^*X)\right)$. If $e_1, \ldots, e_m$ is a basis of TX, and if $e^1, \ldots, e^m$ is the corresponding dual basis of T^*X, the operator in (2.84) can be rewritten as

$$\sum_i e^i \wedge \nabla^{T^*X}_{e_i} : \Omega^1(X, \mathbf{R}) \to \Omega^2(X, \mathbf{R}). \tag{2.85}$$

Then, $\sum_i e^i \wedge \nabla^{T^*X}_{e_i}$ is a first order differential operator with principal symbol $i\xi\wedge$. Therefore, the operator

$$d^X - \sum_i e^i \wedge \nabla^{T^*X}_{e_i} : \Omega^1(X, \mathbf{R}) \to \Omega^2(X, \mathbf{R}) \tag{2.86}$$

is of order 0 and is defined by a global section

$$T \in C^\infty\left(X, \mathrm{Hom}\left(T^*X, \Lambda^2\left(T^*X\right)\right)\right) = \Omega^2(X, TX). \tag{2.87}$$

By construction, if $i_T = \frac{1}{2}\sum_{i,j} e^i e^j i_{T(e_i, e_j)}$, then

$$d^X|_{\Omega^1(X,\mathbf{R})} = \sum_i e^i \nabla^{T^*X}_{e_i} + i_T. \tag{2.88}$$

Definition 2.35. The section $T \in \Omega^2(X,TX)$ is called the *torsion* of ∇^{TX}.

Let $\nabla^{\Lambda(T^*X)}$ be the induced connection on $\Lambda(T^*X)$.

Proposition 2.36 (Bismut)**.** *The following identity holds,*

$$d^X = e^i \wedge \nabla^{\Lambda(T^*X)}_{e_i} + i_T. \tag{2.89}$$

Proof. By construction, (2.89) holds on $\Omega^0(X,\mathbf{R})$ and $\Omega^1(X,\mathbf{R})$. Clearly, the right-hand side of (2.89) satisfies the Leibniz rule (1.51). By Theorem 1.31, we get our proposition. □

Proposition 2.37. *If $U,V \in C^\infty(X,TX)$, then*

$$T(U,V) = \nabla^{TX}_U V - \nabla^{TX}_V U - [U,V]. \tag{2.90}$$

Proof. Let $f \in C^\infty(X,\mathbf{R})$. By (2.88), we have

$$0 = \sum_i e^i \nabla^{T^*X}_{e_i} d^X f + \left\langle T, d^X f \right\rangle = \sum_{i,j} e^i e^j \left\langle \nabla^{T^*X}_{e_i} d^X f, e_j \right\rangle + \left\langle T, d^X f \right\rangle. \tag{2.91}$$

By (2.91), we get

$$\left\langle \nabla^{T^*X}_U d^X f, V \right\rangle - \left\langle \nabla^{T^*X}_V d^X f, U \right\rangle + \left\langle T(U,V), d^X f \right\rangle = 0. \tag{2.92}$$

By the definition of induced connection (2.8), we have

$$\begin{aligned} \left\langle \nabla^{T^*X}_U d^X f, V \right\rangle - \left\langle \nabla^{T^*X}_V d^X f, U \right\rangle &= U \left\langle d^X f, V \right\rangle - \left\langle d^X f, \nabla^{TX}_U V \right\rangle - V \left\langle d^X f, U \right\rangle + \left\langle d^X f, \nabla^{TX}_V U \right\rangle \\ &= \left\langle [U,V] - \nabla^{TX}_U V + \nabla^{TX}_V U, d^X f \right\rangle. \end{aligned} \tag{2.93}$$

By (2.92) and (2.93), we get (2.90). □

Remark 2.38. We can rewrite (2.90) as

$$L_U = \nabla^{TX}_U - \nabla^{TX}_{\cdot} U - T(U,\cdot), \tag{2.94}$$

where $L_U = [U,\cdot]$ is the Lie derivative along U. Note that the last two terms on the right-hand side of (2.94) is a section in $C^\infty(X,\mathrm{End}(TX))$.

Theorem 2.39. *Given a Riemannian metric g^{TX} and a section $T \in \Omega^2(X,TX)$, there exists one and only one metric connection ∇^{TX} on TX with torsion T.*

Proof. Assume that such connection ∇^{TX} exists. Let us deduce a formula for ∇^{TX}. If $U,V,W \in C^\infty(X,TX)$, since ∇^{TX} is metric with torsion T, we have

$$\begin{aligned} \left\langle \nabla^{TX}_U V, W \right\rangle_{g^{TX}} &= U \langle V,W \rangle_{g^{TX}} - \left\langle V, \nabla^{TX}_U W \right\rangle_{g^{TX}} \\ &= U \langle V,W \rangle_{g^{TX}} - \langle V,[U,W] \rangle_{g^{TX}} - \langle V, T(U,W) \rangle_{g^{TX}} - \left\langle V, \nabla^{TX}_W U \right\rangle_{g^{TX}}. \end{aligned} \tag{2.95}$$

Repeating the same procedure for the last term of (2.95), we have

$$\begin{aligned} -\left\langle V, \nabla^{TX}_W U \right\rangle_{g^{TX}} &= -W \langle V,U \rangle_{g^{TX}} + \left\langle \nabla^{TX}_W V, U \right\rangle_{g^{TX}} \\ &= -W \langle V,U \rangle_{g^{TX}} + \langle [W,V],U \rangle_{g^{TX}} + \langle T(W,V),U \rangle_{g^{TX}} + \left\langle \nabla^{TX}_V W, U \right\rangle_{g^{TX}}. \end{aligned} \tag{2.96}$$

Similarly, the last term of (2.96) has the formula,

(2.97)
$$\begin{aligned}\left\langle \nabla_V^{TX} W, U\right\rangle_{g^{TX}} &= V\left\langle W,U\right\rangle_{g^{TX}} - \left\langle W, \nabla_V^{TX} U\right\rangle_{g^{TX}} \\ &= V\left\langle W,U\right\rangle_{g^{TX}} - \left\langle W,[V,U]\right\rangle_{g^{TX}} - \left\langle W, T(V,U)\right\rangle_{g^{TX}} - \left\langle W, \nabla_U^{TX} V\right\rangle_{g^{TX}}.\end{aligned}$$

Taking the sum of (2.95)-(2.97), we get

$$\begin{aligned}2\left\langle \nabla_U^{TX} V, W\right\rangle_{g^{TXX}} &= \left\langle [U,V],W\right\rangle_{g^{TX}} - \left\langle [V,W],U\right\rangle_{g^{TX}} + \left\langle [W,U],V\right\rangle_{g^{TX}} \\ &\quad + U\left\langle V,W\right\rangle_{g^{TX}} + V\left\langle W,U\right\rangle_{g^{TX}} - W\left\langle U,V\right\rangle_{g^{TX}} \\ &\quad + \left\langle T(U,V),W\right\rangle_{g^{TX}} - \left\langle T(V,W),U\right\rangle_{g^{TX}} + \left\langle T(W,U),V\right\rangle_{g^{TX}}.\end{aligned} \tag{2.98}$$

Equation (2.98) determinates uniquely $\nabla_U^{TX} V$ via g^{TX} and T, from which we get the uniqueness of ∇^{TX}.

Let show (2.98) gives also the existence results. Indeed, we can define $\nabla_U^{TX} V$ by the right-hand side of (2.98). We can check directly that if $f \in C^\infty(X,\mathbf{R})$,

$$\nabla_U^{TX}(fV) = f\nabla_U^{TX} V + U(f)V. \tag{2.99}$$

Therefore, ∇^{TX} is a connection on TX.

Moreover, if we interchange the role of U and V in (2.98), and if we take their difference, we get

$$2\left\langle \nabla_U^{TX} V - \nabla_V^{TX} U, W\right\rangle_{g^{TX}} = 2\left\langle [U,V],W\right\rangle_{g^{TX}} + 2\left\langle T(U,V),W\right\rangle_{g^{TX}}, \tag{2.100}$$

which implies the torsion of ∇^{TX} is T.

Also, if we interchange the role of V and W in (2.98), and if we take their sum, we get

$$2\left\langle \nabla_U^{TX} V, W\right\rangle_{g^{TX}} + 2\left\langle \nabla_U^{TX} W, V\right\rangle_{g^{TX}} = 2U\left\langle V,W\right\rangle_{g^{TX}}, \tag{2.101}$$

which implies ∇^{TX} is metric. □

Remark 2.40. The formula (2.98) is called the Koszul formula.

2.6. **Connection and curvature on Riemannian manifolds.** In this section, we assume that X is a Riemannian manifold with metric g^{TX}.

Definition 2.41. The Levi-Civita connection $\nabla^{TX,\mathrm{LC}}$ is the unique metric connection on TX with vanishing torsion.

The Levi-Civita connection induces a connection $\nabla^{\Lambda(T^*X),\mathrm{LC}}$ on $\Lambda(T^*X)$.

Definition 2.42. If $f \in C^\infty(X,\mathbf{R})$, the gradient of f with respect to g^{TX} is the vector field $\nabla^{TX} f \in C^\infty(X,TX)$ defined by

$$\left\langle \nabla^{TX} f, \cdot\right\rangle_{g^{TX}} = d^X f. \tag{2.102}$$

The Hessian $\mathrm{Hess}(f) \in C^\infty(X, T^*X \otimes T^*X)$ of f with respect to g^{TX} is defined for $U,V \in TX$ by

$$\mathrm{Hess}(f)(U,V) = \left\langle \nabla_U^{T^*X,\mathrm{LC}} d^X f, V\right\rangle. \tag{2.103}$$

Proposition 2.43. *If $U,V \in C^\infty(X,TX)$, the following identity holds,*

$$\mathrm{Hess}(f)(U,V) = UVf - \left(\nabla^{TX,\mathrm{LC}}_U V\right) f. \tag{2.104}$$

In particular, the Hessian is a symmetric bilinear form on TX.

Proof. By definition, we have

$$\left\langle \nabla^{T^*X,\mathrm{LC}}_U d^X f, V \right\rangle = U\left\langle d^X f, V\right\rangle - \left\langle d^X f, \nabla^{TX,\mathrm{LC}}_U V\right\rangle, \tag{2.105}$$

which is just the right-hand side of (2.104).

The fact that $\mathrm{Hess}(f)$ is symmetric follows from (2.104) and the fact that $T = 0$. □

Proposition 2.44. *The following identity holds,*

$$d^X = \sum_i e^i \wedge \nabla^{\Lambda(T^*X),\mathrm{LC}}_{e_i}. \tag{2.106}$$

Proof. This is a consequence of Proposition 2.36. □

Definition 2.45. The curvature of the Levi-Civita connection $\nabla^{TX,\mathrm{LC}}$ is called Riemannian curvature $R^{TX,\mathrm{LC}}$.

We have already seen that $R^{TX,\mathrm{LC}}$ is a two form with values in $\mathrm{End}^{\mathrm{anti}}(TX)$. Therefore, if $U,V \in TX$, we have

$$R^{TX,\mathrm{LC}}(U,V) = -R^{TX,\mathrm{LC}}(V,U), \qquad \left\langle R^{TX,\mathrm{LC}}U,V\right\rangle_{g^{TX}} = -\left\langle R^{TX,\mathrm{LC}}V,U\right\rangle_{g^{TX}}. \tag{2.107}$$

Other symmetric relation is given by the following proposition.

Proposition 2.46. *[Bianchi identity]*

a) If $U,V,W \in TX$, then

$$R^{TX,\mathrm{LC}}(U,V)W + R^{TX,\mathrm{LC}}(V,W)U + R^{TX,\mathrm{LC}}(W,U)V = 0. \tag{2.108}$$

b) If $A,B,U,V \in TX$, we have

$$\left\langle R^{TX,\mathrm{LC}}(A,B)U,V\right\rangle_{g^{TX}} = \left\langle R^{TX,\mathrm{LC}}(U,V)A,B\right\rangle_{g^{TX}}. \tag{2.109}$$

Proof. To keep the notation simple, we denote $R^{TX,\mathrm{LC}}$ by R, $\nabla^{TX,\mathrm{LC}}$ by ∇, and $\langle,\rangle_{g^{TX}}$ by $\langle,\rangle$ in the proof.

By (2.57), we have

$$\begin{aligned} R(U,V)W &= \nabla_U\nabla_V W - \nabla_V\nabla_U W - \nabla_{[U,V]}W,\\ R(V,W)U &= \nabla_V\nabla_W U - \nabla_W\nabla_V U - \nabla_{[V,W]}U,\\ R(W,U)V &= \nabla_W\nabla_U V - \nabla_U\nabla_W V - \nabla_{[W,U]}V. \end{aligned} \tag{2.110}$$

The sum of the first two terms on the right-hand side of each line in (2.110) gives

$$\nabla_U[V,W] + \nabla_V[W,U] + \nabla_W[U,V]. \tag{2.111}$$

The sum of the last terms of each line in (2.110) with (2.111) is given by

$$[U,[V,W]] + [V,[W,U]] + [W,[U,V]] = 0, \tag{2.112}$$

which implies (2.108).

Let us show (2.109) as a consequence of (2.108). Proceeding with the aim that putting U,V before A,B, we have

(2.113)
$$\langle R(A,B)U,V\rangle = -\langle R(B,U)A,V\rangle - \langle R(U,A)B,V\rangle = \langle R(B,U)V,A\rangle + \langle R(U,A)V,B\rangle .$$

Proceeding similarly, we have

(2.114)
$$\langle R(B,U)V,A\rangle = -\langle R(U,V)B,A\rangle - \langle R(V,B)U,A\rangle = \langle R(U,V)A,B\rangle - \langle R(V,B)U,A\rangle ,$$
$$\langle R(U,A)V,B\rangle = -\langle R(A,V)U,B\rangle - \langle R(V,U)A,B\rangle .$$

The sum of the last term of the first line of (2.114) with the middle term on the second line is given by

$$-\langle R(V,B)U,A\rangle - \langle R(A,V)U,B\rangle = \langle R(V,B)A,U\rangle + \langle R(A,V)B,U\rangle = -\langle R(B,A)V,U\rangle = -\langle R(A,B)U,V\rangle . \tag{2.115}$$

By (2.113)-(2.115), we get

$$\langle R(A,B)U,V\rangle = -\langle R(A,B)U,V\rangle + 2\langle R(U,V)A,B\rangle , \tag{2.116}$$

which implies (2.109). □

Definition 2.47. The following is called the Riemannian curvature $(0,4)$-tensor of (X,g^{TX}),

$$\left\langle R^{TX,\mathrm{LC}}(\cdot,\cdot)\cdot,\cdot\right\rangle_{g^{TX}}. \tag{2.117}$$

Thanks to the symmetric properties on $R^{TX,\mathrm{LC}}$. "Partial" information on the Riemannian curvature $(0,4)$-tensor determines the whole tensor.

Proposition 2.48. *The maps*

$$U,V\in TX \to \left\langle R^{TX,\mathrm{LC}}(U,V)V,U\right\rangle \tag{2.118}$$

determines the Riemannian curvature $(0,4)$*-tensor.*

Proof. Given $U\in TX$, by (2.109), the $(0,2)$-tensor

$$\left\langle R^{TX,\mathrm{LC}}(U,\cdot)\cdot,U\right\rangle_{g^{TX}} \tag{2.119}$$

is symmetric, so it is uniquely determined by the associated quadratic form,

$$V\in TX \to \left\langle R^{TX,\mathrm{LC}}(U,V)V,U\right\rangle_{g^{TX}}. \tag{2.120}$$

This means for any $A,B,U\in TX$, the values of

$$\left\langle R^{TX,\mathrm{LC}}(U,A)B,U\right\rangle_{g^{TX}} \tag{2.121}$$

is uniquely determined by (2.118).

Similarly, given $A,B\in TX$, the $(0,2)$-tensor

$$\left\langle R^{TX,\mathrm{LC}}(\cdot,A)B,\cdot\right\rangle_{g^{TX}} + \left\langle R^{TX,\mathrm{LC}}(\cdot,B)A,\cdot\right\rangle_{g^{TX}} \tag{2.122}$$

is symmetric. Proceeding as before, for $A,B,U,V \in TX$, the value of

$$\left\langle R^{TX,\mathrm{LC}}(U,A)B + R^{TX,\mathrm{LC}}(U,B)A,V\right\rangle_{g^{TX}} \tag{2.123}$$

is uniquely determined by (2.121), and therefore by (2.118).

By (2.108) and (2.123), we see that

$$\left\langle -R^{TX,\mathrm{LC}}(A,B)U + 2R^{TX,\mathrm{LC}}(U,B)A,V\right\rangle_{g^{TX}} \tag{2.124}$$

is uniquely determined by (2.118).

Changing the role of U and A, we see that

$$\left\langle -R^{TX,\mathrm{LC}}(U,B)A + 2R^{TX,\mathrm{LC}}(A,B)U,V\right\rangle_{g^{TX}} \tag{2.125}$$

is uniquely determined by (2.118).

Taking the sum of (2.124) with the twice of (2.125), we see that

$$3\left\langle R^{TX,\mathrm{LC}}(A,B)U,V\right\rangle_{g^{TX}} \tag{2.126}$$

is uniquely determined by (2.118), which implies our proposition. □

Remark 2.49. See for [CE75, (1.10)] for a formula relating $\langle R^{TX,\mathrm{LC}}(\cdot,\cdot)\cdot,\cdot\rangle_{g^{TX}}$ to (2.118).

If $x \in X$, if $P_x \subset T_xX$ is a two dimensional plane, if $U,V \in P_x$ is an orthogonal normal basis of P, the quantity

$$\left\langle R^{TX,\mathrm{LC}}(U,V)V,U\right\rangle_{g^{TX}} \tag{2.127}$$

is independent of the choice of the orthogonal normal basis $\{U,V\}$ of P_x.

Definition 2.50. The above quantity denoted by $K(P_x)$ is called the sectional curvature at the plan $P_x \subset T_xX$.

Remark 2.51. For each $x \in X$, the Grassmannian $\mathrm{Gr}_2(T_xX)$ of two dimensional planes in T_xX is a smooth manifold. The family of Grassmannians $\{\mathrm{Gr}_2(T_xX)\}_{x\in X}$ forms a fibre bundle $\mathrm{Gr}_2(TX)$ over X. The sectional curvature is a function defined on the total space of the fiber bundle $\mathrm{Gr}_2(TX)$.

Remark 2.52. If P_x is spanned by two vectors $U,V \in P_x$, then

$$K(P_x) = \frac{\left\langle R^{TX,\mathrm{LC}}(U,V)V,U\right\rangle_{g^{TX}}}{\|U\wedge V\|^2_{g^{\Lambda^2(TX)}}}. \tag{2.128}$$

Remark 2.53. Because of Proposition 2.48, sometimes people say vaguely[12] that the sectional curvature determines the Riemannian curvature $(0,4)$-tensor.

Let $e_1,\ldots,e_m$ be an orthonormal basis of $\left(TX,g^{TX}\right)$.

Definition 2.54. 1) The Ricci curvature is a symmetric $(0,2)$-tensor defined by

$$\mathrm{Ric}(\cdot,\cdot) = \sum_{i=1}^{m}\left\langle R^{TX,\mathrm{LC}}(\cdot,e_i)e_i,\cdot\right\rangle_{g^{TX}}. \tag{2.129}$$

[12] This is not correct rigorously because of the denominator in (2.128).

2) The scalar curvature is a real smooth function on X defined by

$$s = \sum_{i=1}^{m} \mathrm{Ric}(e_i, e_i). \tag{2.130}$$