1.7. **Exercises.**

Exercise 1.1. The aim of the exercise is to show that any smooth manifold X has an exhausted sequence, i.e., there exists a sequence $X_i, i \in \mathbf{N}$ of relatively compact open sets such that

$$\overline{X}_i \subset X_{i+1}, \qquad \bigcup_{i\in\mathbf{N}} X_i = X. \tag{1.70}$$

a) If $\mathcal{T}$ is a countable basis of X, show that any $x \in X$, there exists $U_x \in \mathcal{T}$ which is relatively compact.
b) Show that X is a countable union of relatively compact open sets.
c) Conclure.

Exercise 1.2. Let X be a smooth manifold with an open cover $X = \bigcup_{\alpha\in A} U_\alpha$.

a) Show that there exists an open cover $\{V_\beta\}_{\beta\in B}$ of X such that
   - for each $\beta \in B$, V_β is a Euclidean open ball in a coordinate chart of X, and there exists $\alpha \in A$ such that $V_\beta \subset U_\alpha$;
   - the index set B is countable;
   - the open cover $\{V_\beta\}_{\beta\in B}$ is locally finite.

   (Indication : use Exercise 1.1.)

   In particular, we have shown that X is paracompact.
b) For each $\beta \in B$, there exists a smaller Euclidean open ball $V_{\beta 0}$ such that $\overline{V}_{\beta 0} \subset V_\beta$ and $\{V_{\beta 0}\}_{\beta\in B}$ is still an open cover of X.
c) Construct a smooth partition of unity as in Proposition 1.4.

Exercise 1.3. Given any open cover $\{U_\alpha\}_{\alpha\in A}$ of X, there exists a family $\{\varphi_\alpha\}_{\alpha\in A}$ of smooth functions in $C^\infty(X,[0,1])$ such that $\operatorname{Supp}\varphi_\alpha \subset U_\alpha$, and the following sum is locally finite, and

$$\sum_{\alpha\in A} \varphi_\alpha = 1. \tag{1.71}$$

(Indiction : use Exercise 1.2.)

Exercise 1.4. Let $\left(X, g^{TX}\right)$ be a Riemannian manifold. If $x_\cdot : s \in [0,1] \to x_s \in X$ is a smooth curve, we define the length of $x_\cdot$ by

$$\ell(x_\cdot) = \int_0^1 |\dot{x}_t|\, dt. \tag{1.72}$$

If $x, y \in X$, put

$$d(x,y) = \inf \ell(x_\cdot), \tag{1.73}$$

where the infimum is taken over all smooth curves $x_\cdot$ joining x and y. Show that d is a distance on X which induces the original topology of X.

Exercise 1.5. If E, F, G are three vector bundles on X, if $P_1 \in \operatorname{Diff}^{k_1}_X(E,F)$ and $P_2 \in \operatorname{Diff}^{k_2}_X(F,G)$, show that

$$\sigma_{k_1+k_2}(P_2P_1) = \sigma_{k_2}(P_2)\sigma_{k_1}(P_1). \tag{1.74}$$

Exercise 1.6. If $P \in \mathrm{Diff}^k_X(E,F)$, if $f \in C^\infty(X,\mathbf{R})$, show

$$\sigma_k(P)\left(id^X f\right) = \frac{1}{k!}[f,[f,\ldots,[f,P]\ldots]], \tag{1.75}$$

where the right-hand side is the k-th iterated commutator of P with f.

Exercise 1.7. Let $U \in C^\infty(X,TX)$ be a vector field on X. The aim of the exercise is to show the Cartan homotopy formula,

$$L_U = d^X i_U + i_U d^X. \tag{1.76}$$

1) Show that there exists at most one $\mathbf{R}$-linear degree preserving operator $P : \Omega^\bullet(X,\mathbf{R}) \to \Omega^\bullet(X,\mathbf{R})$ with the following properties:
   a) if $f \in C^\infty(X,\mathbf{R})$, then $Pf = U \cdot f$ and $P\left(d^X f\right) = d^X(U \cdot f)$.
   b) if $\alpha,\beta \in \Omega(X,\mathbf{R})$, then

$$P\left(\alpha \wedge \beta\right) = (P\alpha) \wedge \beta + \alpha \wedge \left(P\beta\right). \tag{1.77}$$

2) Let $\left(\phi_t\right)_{t\in\mathbf{R}}$ be the flow associated to U. Recall that the Lie derivation is defined for $\alpha \in \Omega(X,\mathbf{R})$ by

$$L_U\alpha = \frac{d}{dt}\Big|_{t=0}\phi_t^*\alpha. \tag{1.78}$$

   Show that L_U is a $\mathbf{R}$-linear degree preserving operator with the above properties.
3) Show that $d^X i_U + i_U d^X$ is also a $\mathbf{R}$-linear degree preserving operator with the above properties.
4) Deduce (1.76).

Comment: The above proof is due to Chern.

Exercise 1.8. If $\alpha \in \Omega^1(X,\mathbf{R})$, if $U,V \in C^\infty(X,TX)$ are two vector fields on X, show

$$d^X\alpha(U,V) = U[\alpha(V)] - V[\alpha(U)] - \alpha([U,V]). \tag{1.79}$$

(Try to find a proof without using local coordinates. Indication : apply Cartan homotopy formula (1.76).)

Exercise 1.9. Let G be a Lie group. Let $\mathfrak{g}$ be the corresponding Lie algebra, i.e., the tangent space of G at the point $1 \in G$.

a) Show that G is orientable.
b) Show that a volume element on $\mathfrak{g}$ induces a left invariant volume element on G, which is unique up to a multiplicative factor. This volume form is called the Haar measure.
c) Show that if G is compact, then the left invariant volume form is also right invariant.

Exercise 1.10. Let E be a vector bundle of rank r on X. Let ∇^E be a connection on E which is locally given by $d + \Gamma^E_\alpha$.

1) Show that $\Gamma^E_\alpha = \left(g^E_{\beta\alpha}\right)^{-1} dg^E_{\beta\alpha} + \left(g^E_{\beta\alpha}\right)^{-1} \Gamma^E_\beta g^E_{\beta\alpha}$ on $U_\alpha \cap U_\beta$.
2) Explain why the family $\left(\Gamma^E_\alpha\right)_{\alpha\in A}$ does not define a global section in $\Omega^1(X,\mathrm{End}(E))$.

3) Assume now that all the $g^E_{\beta\alpha}$ lie in $C^\infty\left(U_\alpha \cap U_\beta, \mathrm{SL}_r(\mathbf{R})\right)$.
   a) Show that $\det(E) = \Lambda^r(E)$ is a trivial line bundle on X.
   b) Show that $\left(\mathrm{Tr}\left[g^E_\alpha\right]\right)_{\alpha\in A}$ defines a 1-form on X. Explain the geometric meaning of this result.