2.7. **Exercices.**

Exercise 2.1. Let (E,h^E) be a Hermitian vector bundle on X with metric connection ∇^E. Fix $x_0 \in X$.

a) Show that there exist a small open neighbourhood U_{x_0} of x_0, and a smooth sections $e_i \in C^\infty\left(U_{x_0}, E_{|U_{x_0}}\right)$, $i = 1,\dots,r$, such that $\{e_i\}_{1\le i\le r}$ forms an orthonormal basis of E_x at every point $x \in U_{x_0}$.

b) Show that there exist an open cover $X = \bigcup_{\alpha\in A} U_\alpha$ and trivialisations $E|_{U_\alpha} = U_\alpha \times \mathbf{C}^r$ such that
- the local h^E_α is the identity matrix;
- the transition function $g^E_{\beta\alpha}$ lies in $C^\infty\left(U_\alpha \cap U_\beta, \mathrm{U}(r)\right)$;
- the local connection 1-form Γ^E_α lies in $\Omega^1(U_\alpha, \mathfrak{u}(r))$.

Exercise 2.2. Let E be a vector bundle on X with connection ∇^E. If $x_0 \in X$, we trivialise E near x_0 as described in Remark 2.22. If Γ^E is the associated connection 1-form, show that Γ^E vanishes at the point $x_0 \in X$.

Exercise 2.3. The aim of this exercise is to introduce the horizontal subbundle $T^H\mathscr{E} \subset T\mathscr{E}$. This is the infinitisimal approach for the parallel transport, which provides another description of a connection.

1) Let U be an open subset of $\mathbf{R}^m$. Let $U \times \mathbf{C}^r$ be the total space of a trivial vector bundle on U. Let $\sigma : x \in U \to (x, s(x)) \in U \times \mathbf{C}^r$ be a section with representative $s \in C^\infty(U, \mathbf{C}^r)$. Let ∇ be a connection with representative $d + \Gamma$, where $\Gamma \in \Omega^1(U, \mathrm{End}(\mathbf{C}^r))$.

a) If $(x_t)_{t\in[0,1]}$ is a smooth curve on U, let $(x_t, s_t)_{t\in[0,1]}$ be the corresponding parallel section le long $(x_t)_{t\in[0,1]}$. With the help of Γ, write down an explicit formula for the map

$$\dot{x}_0 \in T_{x_0}U \to (\dot{x}_0, \dot{s}_0) \in T_{(x_0,s_0)}\left(U \times \mathbf{C}^r\right). \tag{2.131}$$

b) Define $T^V_{(x_0,s_0)}(U \times \mathbf{C}^r) = \{(0,v) : v \in \mathbf{C}^r\}$ and define $T^H_{(x_0,s_0)}(U \times \mathbf{C}^r)$ to be the image of (2.131). Show that $T^H(U \times \mathbf{C}^r)$ and $T^V(U \times \mathbf{C}^r)$ are subbundles of $T(U \times \mathbf{C}^r)$ such that

$$T\left(U \times \mathbf{C}^r\right) = T^V\left(U \times \mathbf{C}^r\right) \oplus T^H\left(U \times \mathbf{C}^r\right). \tag{2.132}$$

c) Let $(D\sigma)_{x_0} : T_{x_0}U \to T_{(x_0,s_0)}(U \times \mathbf{C}^r)$ be the differential of the smooth map σ at x_0. If P^V is the vertical projection with respect to (2.132), show that

$$(d + \Gamma)s = P^V D\sigma, \tag{2.133}$$

where we identify the vertical tangent space $T^V_{(x,s)}(U \times \mathbf{C}^r)$ with $\mathbf{C}^r$.

2) a) Show that the vertical tangent spaces define a globally well-defined vector subbundle of $T\mathscr{E}$, denoted by $T^V\mathscr{E}$. Show that

$$T^V\mathscr{E} \simeq \pi^* E. \tag{2.134}$$

b) Show that we have the following short exact sequence of vector bundles on $\mathscr{E}$,

$$0 \longrightarrow T^V\mathscr{E} \xrightarrow{i} T\mathscr{E} \xrightarrow{D\pi} \pi^* TX \longrightarrow 0, \tag{2.135}$$

where i is the inclusion.

c) Assume that E is equipped with a connection ∇^E.

i) Show that the horizontal tangent spaces define a globally well-defined vector subbundle of $T\mathscr{E}$, denoted by $T^H\mathscr{E}$.

ii) Show that the short exact sequence (2.135) splitings, i.e.,

$$T\mathscr{E} = T^V\mathscr{E} \oplus T^H\mathscr{E}, \qquad T^H\mathscr{E} \simeq \pi^* TX. \tag{2.136}$$

iii) For $s \in C^\infty(X,E)$, if $x \in X, A \in T_xX$, show that

$$\nabla^E_A s(x) = P^V \{D_x s(A)\} \text{ in } E_x. \tag{2.137}$$

Exercise 2.4. Let $\left(X, g^{TX}\right)$ be a Riemannian manifold. Let $K \in C^\infty(X,TX)$ be a vector field.

a) Show that if $A,B \in TX$, we have

$$\left\langle \nabla^{TX,\mathrm{LC}}_A K, B \right\rangle_{g^{TX}} + \left\langle \nabla^{TX,\mathrm{LC}}_B K, A \right\rangle_{g^{TX}} = \left\langle \left(L_K g^{TX}\right) A, B \right\rangle. \tag{2.138}$$

b) Recall that K is called a Killing vector field, if the flow generated by K preserves the Riemannian metric g^{TX}, or equivalently, $L_K g^{TX} = 0$. Show that K is Killing if and only if the bilinear form $\left\langle \nabla^{TX,\mathrm{LC}}_\cdot K, \cdot \right\rangle_{g^{TX}}$ is antisymmetric.

Exercise 2.5. Let X be a submanifold of $\mathbf{R}^N$ equipped with the induced Riemannian metric. Show that if U,V are vector fields on X, then

$$\nabla^{TX,\mathrm{LC}}_U V = P^{TX} \nabla_U V, \tag{2.139}$$

where $P^{TX} \in C^\infty\left(X, \mathrm{Hom}\left(\mathbf{R}^N, TX\right)\right)$ is the orthogonal projection, and ∇ is the trivial connection on $\mathbf{R}^N$.

Exercise 2.6. Let $a > 0$, and replace g^{TX} with the rescaled Riemannian metric $a g^{TX}$. Determine how the corresponding Riemannian curvature tensor, sectional curvature, Ricci curvature, and scalar curvature change under this rescaling.

Exercise 2.7. Let G be a Lie group. Let $\mathfrak{g}$ be the Lie algebra G which is identified with the space of left invariant vector fields on G. Let $a \in \mathbf{R}$.

a) If $U,V \in \mathfrak{g}$, show that there exists one and only one connection on TG such that $\nabla^{TG,a}_U V = a[U,V]$.

b) Evaluate the torsion and curvature of $\nabla^{TG,a}$.

c) Assume that G is compact.

a) Show that there exists a left and right invariant Riemannian metric on G.

b) Show that $\nabla^{TG,1/2}$ is the corresponding Levi-Civita connection.

c) Evaluate the Ricci curvature and scalar curvature.

d) When a compact Lie group has strictly positive Ricci curvature ?

Exercise 2.8. Consider the upper half plan $X = \{(x,y) \in \mathbf{R}^2 : y > 0\}$ equipped with the Poincaré metric $g^{TX} = \frac{dx^2+dy^2}{y^2}$. Put

$$e_1 = y\frac{\partial}{\partial x}, \qquad e_2 = y\frac{\partial}{\partial y}. \tag{2.140}$$

a) Show that $\nabla^{TX,\mathrm{LC}}_{e_1} e_2 = -e_1$ and $\nabla^{TX,\mathrm{LC}}_{e_2} e_2 = 0$.
b) Show that $\langle R^{TX,\mathrm{LC}}(e_1,e_2)e_2, e_1\rangle_{g^{TX}} = -1$, and deduce that the sectional curvature of (X, g^{TX}) is -1.

Exercise 2.9. Consider the sphere

$$X = \{(\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta)) \in \mathbf{R}^3 : \phi \in [0,2\pi], \theta \in [0,\pi]\}$$

equipped with the Riemannian metric $g^{TX} = \sin(\theta)^2 d\phi^2 + d\theta^2$. Put

$$e_1 = \sin(\theta)^{-1}\frac{\partial}{\partial \phi}, \qquad e_2 = \frac{\partial}{\partial \theta}. \tag{2.141}$$

a) Show that $\nabla^{TX,\mathrm{LC}}_{e_1} e_2 = \cot(\theta) e_1$ and $\nabla^{TX,\mathrm{LC}}_{e_2} e_2 = 0$.
b) Show that $\langle R^{TX,\mathrm{LC}}(e_1,e_2)e_2, e_1\rangle_{g^{TX}} = 1$, and deduce that the sectional curvature of (X, g^{TX}) is 1.